OPTIMAL CONTROL OF AN INCOME-CONTINGENT STUDENT LOAN SYSTEM

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ABSTRACT

A description and a stochastic model for income-contingent loan systems is presented. The creditor's risk is investigated in terms of the two basic control parameters of the model, the risk premium and the repayment quotient. Self-financing of the system is defined and visualized using simulation techniques. An optimal parameter setting is proposed to reach zero-profit operation.

INTRODUCTION

The Hungarian Student Loan system recently celebrated its 5th birthday. Such a short operation period is not enough to draw univocal conclusions but the first experiments seem to strengthen the original idea. The Hungarian student loan model is exceptional in the worldwide practice because access and conditions are universal, repayments are income-contingent and the system is designed to be self-sustaining in the sense that it has to operate without direct state subsidy and thus has to be independent of the state budget. Due to the changing environment the financial stability of the system can only be assured by periodic interventions. Our objective is to find the optimal intervention algorithm from the lender's point of view.

There are two key control-parameters: the interest rate and the repayment rate, and the main source of risk and uncertainty is the income process of the borrowers. In our previous paper presented on the ECMS 2006 Conference in Budapest we focused on the problem of the calculation of the risk premium if the repayment rate is given. The required risk premium is highly dependent on the income process. We have introduced a special micro-simulation method to analyze the effect of the changes of the relative income rankings within one generation. We have shown that the volatility of the rankings (i.e. social mobility) is negatively correlated to the risk premium.

In this paper we turn to the relationship between the two control parameters. We use a simple micro-simulation technique: borrowers' income follow the usual process for equity prices in a discrete world. This approach enables us to focus on the interference of the control parameters in the framework of this special non-linear loan-scheme, and determine the so-call zero-profit line: the set of those points where financial equilibrium is assured. We conclude with some remarks on the optimal parameter setting rule and the robustness of the optimum.

A DETERMINISTIC MODEL

First we present a model excluding random effects. For the sake of convenience a continuous time parameter is used. Let $t = 0$ represent the moment where a client graduates. Let H_0 and B_0 denote accumulated debt and income at this moment, respectively. The debt at time t, denoted by H_t, evolves following the ODE

$$\dot{H}_t = rH_t - \alpha B_t, \tag{1}$$

where α is the proportion of income paid as an installment, the *repayment rate*, B_t is the borrower's income at time t and r is the (continuous) interest rate compounded as $r = f + p$, with f representing the creditor's refinancing costs and p representing a marge. The income process of the borrower is assumed to follow

$$B_t = B_0 e^{\mu t},$$

with some growth rate μ. We may suppose that μ exceeds f and even r, at least for small p.

One also has to consider the rate of a bank loan h offered in the market. It is assumed that $h > r$, indicating that the student loan system is attractive from the borrowers' point of view. Typical values of these rates are $f = 0.07,\ h = 0.15,\ \mu = 0.1$.

A special feature of the Hungarian loan systems is that the debt is written off if the client reaches the age of retirement or dies prior to having fully repaid his/her debt. Consider a client who reaches his age of retirement. Let K denote the number of years until retirement. A typical value for a freshman is $K = 30$. Let us introduce

$$t^* := \inf\{t : H_t = 0\},$$

the moment of repaying all debts. It is easily seen that t^* is finite if and only if

$$\frac{H_0}{\alpha B_0}(\mu - r) > -1. \tag{2}$$

This work has been supported by the "Cooperative Center for Communication Networks Data Analysis", a NAP project sponsored by the National Office of Research and Technology under grant No. KCKHA005.

In a generic situation μ should slightly exceed r, in particular, (2) holds true. We then have

$$t^* = \frac{1}{\mu - r} \ln \left[\frac{H_0}{\alpha B_0} (\mu - r) + 1 \right], \text{ if } \mu \neq \text{r}, \quad (3)$$

For $\mu = r$ we have

$$t^* = T^* := \frac{H_0}{\alpha B_0}, \quad (4)$$

where T^* is dimensionless quantity denoting the measure of debt relative to the installment of the first year after graduation. Assuming $f = p = \mu = 0$ it would take time T^* to repay all debts. The same amount of time is needed if $\mu = r$, as (4) shows.

Introduce the notation $x \wedge y := \min\{x, y\}$. Assuming that the client reaches the age of retirement, payment is terminated either at t^* or at the time of retirement K. Up to this date, the net present value of the loss or gain of the creditor – evaluated using discount factor f, his/her cost of refinancing – equals

$$\chi := \int_0^{t^* \wedge K} \alpha B_t e^{-ft} dt - H_0.$$

Direct calculation gives that χ can be expressed for $\mu \neq r$ as

$$\chi = \frac{\alpha B_0}{\mu - r} \left[(T^*(\mu - r) + 1)^{\frac{\mu - f}{\mu - r}} \wedge e^{(\mu - f)K} - 1 \right] - H_0.$$

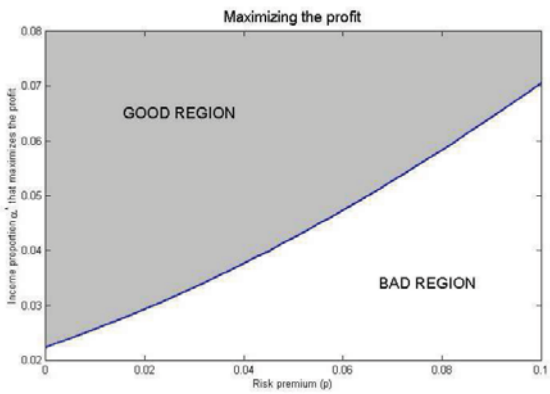

Fig. 1. The relevant region

Fig. 1 is divided by a line representing the pairs (α, p) for which repayment is completed precisely at time K. Only the region above this line is relevant for our investigations: these are the pairs of parameters for which the entire repayment of the debt is possible. We remark that even though χ takes nonnegative values also in the other region (due to the marge), this is not acceptable from a political point of view: one may not advertise and operate a system where no one pays back his or her debts.

The borrower's gain is defined by the total credit received minus future installments discounted by market rate h:

$$\theta = H_0 - \int_0^{t^* K} \alpha B_t e^{-ht} dt.$$

It can be written also as

$$\theta = H_0 - \frac{\alpha B_0}{\mu - r} \left[(T^*(\mu - r) + 1)^{\frac{\mu - h}{\mu - r}} \wedge e^{(\mu - f)K} - 1 \right].$$

Note that for $r < \mu < h$ we have $\theta > H_0$.

A CONTINUOUS-TIME STOCHASTIC MODEL

Next we incorporate two sources of uncertainty into the above simple deterministic model. First, the evolution of income is modelled as a geometric Brownian motion. The second source of uncertainty is in B_0, reflecting borrowers having different professions. We do not consider a third source of uncertainty due to premature quitting the system. This may occur when the client dies or repays his/her debt prematurely, which is allowed in the Hungarian system. First we consider a single client graduating at year $t = 0$. Let H_0 denote the accumulated debt at $t = 0$. The dynamics of (B_t) is modelled as

$$\dot{B}_t = \mu B_t + \sigma \dot{W}_t,$$

where $\dot{W}_t$ represents Gaussian white noise. More formally, we fix a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and define

$$B_t := B_0 e^{\mu t + \sigma W_t},$$

where (W_t) is a Wiener process. $B_0(\omega)$ is supposed to be a random variable, such that B_0 and (W_t) are independent. Thus (B_t) is a geometric Brownian motion with random initial value B_0, an object often used in financial mathematics for describing prices of risky assets.

We can now repeat the calculations given for a single deterministic trajectory as follows. As before, the dynamics of H_t is given by

$$\dot{H}_t = e^r H_t - \alpha B_t,$$

from which we get

$$H_t(\omega) = H_0 e^{rt} - \int_0^t \alpha B_u e^{r(t-u)} \, du.$$

The present value of the debt and income evaluated for the borrower is obtained by using a discount rate r:

$$\overline{H}_t = e^{-rt} H_t, \quad \overline{B}_t = e^{-rt} B_t,$$

and they follow the dynamics

$$\dot{\overline{H}}_t = -\alpha \overline{B}_t \quad \text{with} \quad \overline{H}_0 = H_0.$$

Note that $\overline{B}_t$ itself is a geometric Brownian motion with drift $\mu - r$.

The time of full repayment is now defined as the stopping time

$$t^* := \inf\{t : H_t = 0\}.$$

The creditor's gain is obtained by using the more favourable discount rate f:

$$\chi = \int_0^{t^* \wedge K} \alpha B_u e^{-fu} \, du - H_0.$$

Proceedings 21st European Conference on Modelling and Simulation
Ivan Zelinka, Zuzana Oplatková, Alessandra Orsoni ©ECMS 2007
ISBN 978-0-9553018-2-7 / ISBN 978-0-9553018-3-4 (CD)

This can be written in terms of $\overline{B}_t$ as

$$\chi = \int_0^{t^* \wedge K} \alpha \overline{B}_u e^{pu} \, du - H_0,$$

reflecting the effect of the risk premium on the gain of the creditor.

This quantity can be directly controlled by the parameters p and α. The initial debt is partially controlled by the creditor: under the Hungarian system three levels of monthly credits are offered, but the final decision is with the client. The maximal one is about \$100/month, which is approximately 50% of the minimal monthly wage. Experience has shown that students generally opted for the highest credit level. Therefore, assuming a single client, we write

$$\chi = \chi(\alpha, p).$$

We formulate the following mathematical problem: let the expected gain be prescribed at a level κ_0, and solve

$$E\chi(\alpha, p) = \kappa_0,$$

either for a given p or for a given α. Here κ_0 denotes the operating costs plus safety reserve.

The Hungarian system is designed to be self-financing and accumulation of profit is not permitted. If a state subsidy is incorporated in the model construction as in the Australian system, for example, κ_0 could be set to a negative value. There exist also profit-oriented entreprises giving income-contingent loans. For such a creditor, a perspective is the maximization of $E\chi$ while the expected welfare of borrowers is kept at a fixed level.

After simple manipulations we arrive at the following problem: we are given a fixed geometric Brownian motion

$$\overline{B}_t^0 = e^{(\mu - r)t + \sigma W_t(\omega)},$$

together with an independent initial value $B_0(\omega)$. Set again

$$T^*(\alpha) := \frac{H_0}{\alpha B_0}. \tag{5}$$

Define the α-dependent stopping time

$$t^* = t^*(\alpha) := \inf\{t : \int_0^t \overline{B}_u^0 \, du = T^*(\alpha)\}.$$

Find α such that

$$\mathrm{E} \int_0^{t^*(\alpha) \wedge K} \overline{B}_u^0 e^{pu} \, du - \frac{H_0}{\alpha \mathrm{E} B_0} = \frac{\chi_0}{\alpha \mathrm{E} B_0}. \tag{6}$$

Dividing (6) by (5) we get

$$\frac{E \int_0^{t^*(\alpha) \wedge K} \overline{B}_u^0 e^{pu} du}{\int_0^{t^*(\alpha)} \overline{B}_u^0 du} = \frac{B_0}{H_0} \left(\frac{\chi_0}{EB_0} + \frac{H_0}{EB_0} \right).$$

Note that the right-hand side is independent of α.

Tackling (6) is quite hard even if we fix the value of the right-hand side at a value which is independent of α. There are a number of relevant results by Geman and Yor (Geman and Yor 1993; Yor 1992). One can obtain the distribution of

$$I_t = \int_0^t \overline{B}_u \, du \tag{7}$$

in terms of Bessel processes, see (Bingham and Kiesel 2000) and (Musiela and Rutkowski 1997). It can also be shown that evaluating the expected value of χ is related to the valuation of an Asian option, for which even explicit formulas are hard to handle. A natural approach would be to solve (6) by a stochastic approximation procedure, but even this is quite challenging since χ is defined via a Wiener process. Thus continuous-time modelling does not seem to have any advantage in stochastic modelling of the loan system.

A DISCRETE TIME STOCHASTIC MODEL

Let us now turn to a less elegant but more realistic model in discrete time. Thus from now on the time parameter is $t \in Z_+$. Let N_0 be a (finite) index set of customers, and let the income path of client i be

$$B_t^i := B_0^i e^{\mu t + \sum_{j=1}^t \varepsilon_j^i(\omega)}.$$

Here μ is the common growth rate, and (ε_t^i) is a sequence of independent, identically distributed random variables (e.g. normal or uniform on an interval) on a fixed probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

Let H_0^i denote the accumulated debt of client i at time 0 and let the debt of the ith person at time t be denoted by H_t^i. It satisfies the recursion

$$H_t^i := [e^r H_{t-1}^i - \alpha B_t^i]_+,$$

where $[x]_+$ denotes the positive part of x. From here we get

$$H_t^i = [e^{rt} H_0 - \sum_{j=1}^t \alpha B_j^i e^{r(t-j)}]_+.$$

The time of complete repayment for client i is the stopping time

$$t_i^* := \inf\{t : H_{t-1}^i \leq \alpha B_t^i\}.$$

The present value of the gain of the creditor after client i is

$$\chi_0^i = \sum_{t=1}^{t_i^* \wedge K^i} \alpha B_t^i e^{-ft} - H_0^i,$$

and the aggregate gain at time 0 equals

$$\chi_0 = \sum_{i \in N_0} \chi_0^i.$$

Denoting again by κ_0 the operating costs and safety reserves that we need to off-set the risk of falling below a deficit level prescribed by law, the mathematical problem mentioned in the previous section, i.e. the problem of solving

$$E\chi_0(\alpha, p) = \kappa_0 \tag{8}$$

now becomes a highly relevant problem for the efficient operation of the student loan system.

SIMULATION RESULTS

To get some insight into the highly non-linear structure of the above discrete-time stochastic model, several simulations were carried out in a MatLab environment. In all the simulations below, we took the following values (typical for the Hungarian loan system): $f = 0.07,\ h = 0.15,\ \mu = 0.1,\ B_0 = 1200$ (thousand Ft), $H_0 = 1000$ (thousand Ft). The driving noise ϵ (if otherwise stated) is uniformly distributed over the interval $(\ln 0.9, \ln 1.1)$.

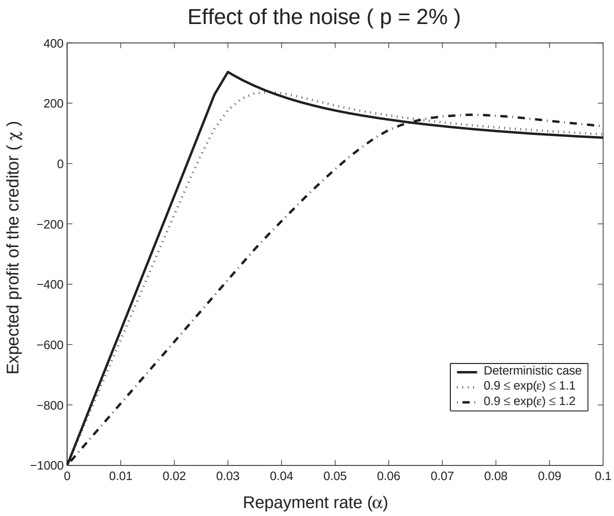

Fig. 2. Effect of the noise

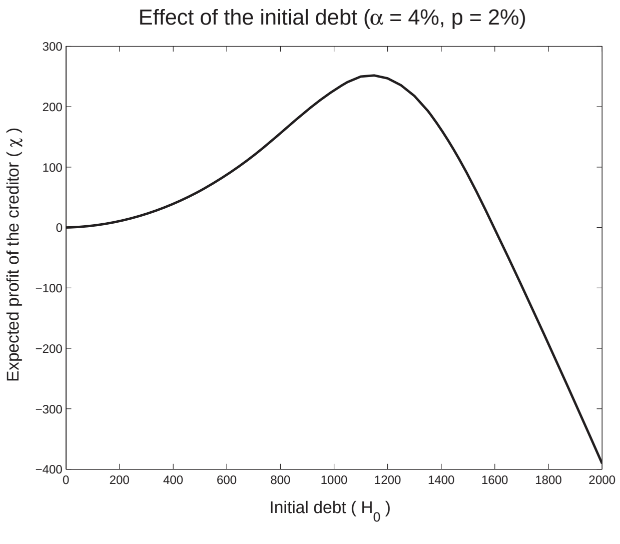

Fig. 3. Effect of initial debt

Fig. 2 shows the dependence of χ on α for three cases: the deterministic case and two stochastic settings. It is not surprising that there is a unique maximum, corresponding to that α for which the borrower repays his debt exactly at time $t = K$. In this case he causes no loss to the creditor, and he raises money by paying the interest premium the longest possible. It should, however, be taken into account that this optimal value is

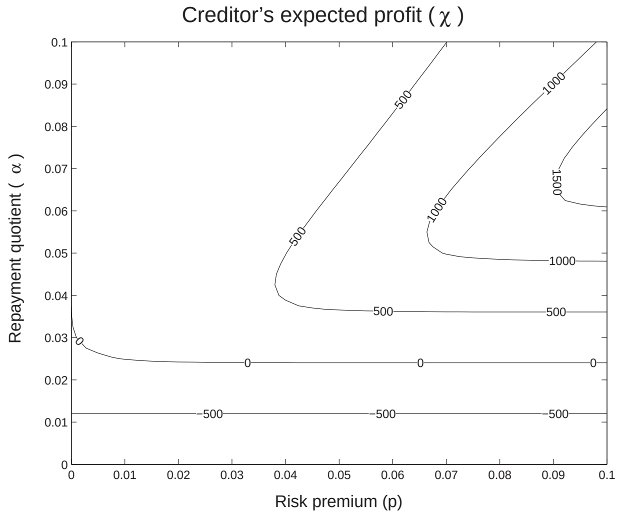

Fig. 4. Creditor's expected profit

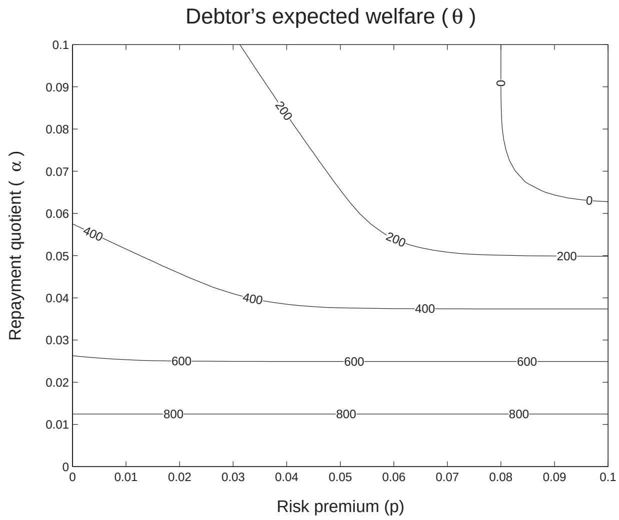

Fig. 5. Debtor's expected gain

not robust: small perturbations in market conditions (e.g. a decrease of μ, an increase of f, a change in the volatility of the noise etc.) may cause the assumed optimal α to fall to the left of the real optimal value thus causing a sharp drop in profit, reflecting a situation when the client is no longer able to repay his/her debt completely. It is therefore advisable to choose α slightly larger than the computed optimal value to increase the robustness of the performance. Smoothing effect of the noise is clearly seen. Large volatility decreases χ – as one would expect.

On Fig. 3 one can see how H_0 affects $E\chi_0$. Fig. 4 and Fig. 5 show the levels of the creditor's profit and the debtor's gain, respectively, as a function of the parameters α and p. Looking on the contour line map of the profit, some basic conclusions can be read off:

- For $\alpha = 0$ no profit can be achieved. This is evident: if there are no repayments, the credits are given for free; thus the system cannot be profitable.
- The profit of the creditor is negative for too small values of the repayment quotient no matter what the risk premium is. This warns us not to choose too low

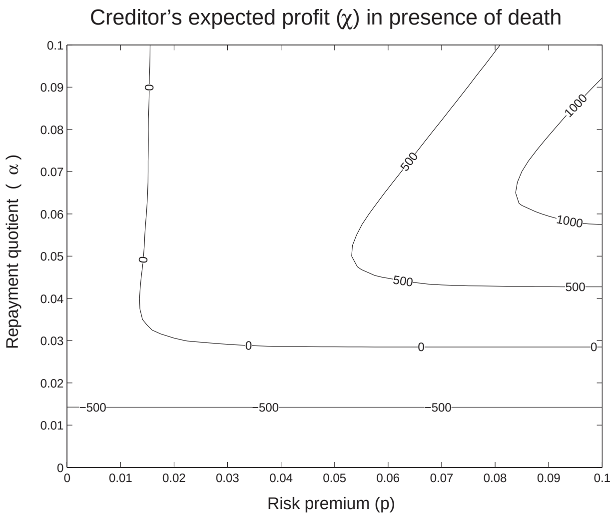

Fig. 6. Creditor's expected profit in presence of death

values of the repayment quotient.

- For $p = 0$ no positive profit can be realized; what's more, losses occur if the repayment rate is too small.
- The risk premium, however, can be chosen arbitrarily close to zero and a positive profit may still be achieved provided the repayment rate is sufficiently large.
- The creditor gets the most profit if it chooses the repayment rate in such a way that the client keeps on paying until the age of retirement.
- It may seem striking that the maximal profit is *not* achieved at the right uppermost corner of the map: this is due to the observation that using high enough repayment rates, the debt is paid back quickly. On the contrary, for lower repayment quotients, the student loan center can achieve a higher profit since this way the client keeps on paying for much longer time periods.
- A useful observation is that if the current profit falls below zero, raising p is not the reasonable measure to take, one should increase α instead.
- In general, a rise in p increases the profit in an already favorable situation.
- Note that there are infinitely many pairs (α, p) that satisfy the zero-profit condition! Thus one needs further criteria to choose one of them: an obvious condition would be to choose the pair that maximizes the welfare of the borrower: evidently, this is the "corner point" of the zero-profit line (remember, however, that this point is not robust to market conditions). Note also that choosing this point minimizes both α and p subject to the zero-profit condition.

So which is the key control parameter: α or p? Increasing the repayment rate to a certain degree is acceptable for the borrower since this way he/she will repay the debt sooner. A rise in the risk premium increases both the charges and the duration time of the loan, thus it would probably generate strong protests. Therefore a politically sustainable regulating scheme should ensure the stability of the system by keeping the interest rate fixed and using the repayment rate as the key control parameter. This argument is also in line with our findings: the profit is much more sensitive to the repayment quotient α than to the risk premium p; furthermore, we have already noted that changing α is the right solution when the system suffers losses, i.e. the repayment rate is more sensitive to income risk. It seems that keeping both α and p fairly small is good for both parties: the repayment period gets longer, so the risk premium paid increases the creditor's profit, nevertheless the installments do not burden the borrowers too much.

Fig. 6 examines a case where a disappearance rate is included in the model: each client quits the system each year with a certain probability, 0.01 in the present case. The contour lines of the expected profit look similar to those of Fig. 4. but there is one drastic change that is obvious at first glance: the disappearance risk makes the zero-profit line "curl up" for small values of the risk premium. This has a very important consequence for policy makers: the risk premium can no longer be chosen arbitrarily small if the self-sustaining principle is to be obeyed. Thus a strictly positive risk premium (and of course a strictly positive repayment quotient) is needed for the long-run financial stability of the loan system. In the concrete numerical example the risk premium should be at least 1.5% and the repayment rate should take on a value of at least 3% in order to realize a non-negative expected profit. Let us emphasize our main assertion once again: disappearance risk does not allow too low values of the risk premium p.

CONCLUSION

One of our main findings is that there exist an infinity number of parameter settings satisfying the zero-profit condition. To visualize them we have drawn the zero-profit-line. It follows from the special shape of the line that the two control parameters can be by and large separated: the repayment rate has to be determined according to the income risk and risk premium is more sensitive to disappearance risk (death, disability, emigration etc.). In a risk-neutral world the lender's objective would be to "stay at the corner" which means keeping both the repayment rate and the risk premium as low as possible. In this case most of the borrowers would achieve full repayment just before the retirement point so repayment periods would be the longest possible. This strategy would however imply that the lender faces a significant risk of forecasting. This corner point would be very attractive for the borrowers, but it could be very risky. Periodic re-evaluation and parameter resetting can eliminate one part of the forecasting risk. The remaining risk could be further lowered by moving on the zero-profit-line from the "corner point" by increasing the repayment rate (and leaving the risk premium fairly unchanged). The risk of forecasting is not quantitatively embodied in our analysis, it can be the object of further investigations.

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