ON $M_t/M_t/S$ TYPE QUEUE WITH GROUP SERVICES

Alexander Zeifman
Vologda State Pedagogical University, S.Orlova, 6, Vologda, Russia
Institute of Informatics Problems, Russian Academy of Sciences
Institute of Territories Socio-Economic Development, Russian Academy of Sciences

Yakov Satin and Galina Shilova
Vologda State Pedagogical University
S.Orlova, 6, Vologda, Russia

Victor Korolev and Vladimir Bening
Moscow State University, Leninskie Gory, Moscow, Russia
Institute of Informatics Problems, Russian Academy of Sciences

Sergey Shorgin
Institute of Informatics Problems, Russian Academy of Sciences
Vavilova str., 44-2, Moscow, Russia

KEYWORDS

Markovian queueing models; nonstationary Markov processes; group services; rate of convergence

ABSTRACT

We consider $M_t/M_t/S$-type queueing model with group services. Bounds on the rate of convergence for the queue-length process are obtained. Ordinary $M_t/M_t/S$ queue and $M_t/M_t/S$ type queueing model with group services are studied as examples.

INTRODUCTION

First investigations of non-stationary birth-death queueing models were published in 1970-s, see (Gnedenko and Makarov 1971, Gnedenko and Soloviev 1973, D. Gnedenko 1971). Namely, they studied qualitative properties of such models, firstly their ergodic properties. Some related problems were considered from the viewpoint of random summation in (Gnedenko and Korolev 1996, Korolev and Shevtsova 2012). Quantitative approach with estimation on the rate of convergence for non-stationary birth-death processes has been developed in our previous papers, see (Granovsky and Zeifman 2004, Zeifman 1995, Zeifman et al. 2006, Zeifman 2009, Zeifman and Korotysheva 2012).

The problem of construction of the limiting characteristics for the queue-length process for such models via truncations of birth-death processes was considered in (Zeifman et al. 2006).

A new class of Markovian non-stationary queueing models with batch arrivals and group services was introduced and studied in our recent papers, see (Satin et al. 2011, 2012).

The paper (Satin et al. 2011) dealt with finite state space models. First bounds on the rate of convergence and stability under perturbations of intensity matrix were obtained.

The respective countable model was investigated in (Satin et al. 2012). In this work general bounds on the rate of convergence were obtained. Moreover, the first truncation estimates were considered under some additional assumptions.

Erlang-type queueing model with group services was introduced and studied in (Zeifman et al 2013). Namely, in this paper criteria for weak ergodicity and bounds on the rate of convergence have been obtained.

Another popular and one of simplest queuing systems is $M/M/S$ queue. There is a large number of investigations for this model in stationary and non-stationary situations, see for instance, (Granovsky and Zeifman 2004; Zeifman 1995; Zeifman et al. 2006, 2008; Zeifman and Korotysheva 2012).

Here we introduce and study a natural generalization of this model for the queue with possible simultaneous services.

Namely, we suppose that there are S servers and infinitely many waiting rooms in the queueing system, an intensity of arrival of a customer to the queue is $\lambda(t)$, and an intensity of departure (servicing) of a group of k customers is $\mu_k(t) = \frac{\mu(t)}{k}$ for all $1 \le k \le S$.

Let $X = X(t)$, $t \ge 0$ be a queue-length process for the queue.

Let $p_{ij}(s,t) = Pr\{X(t) = j\,|X(s) = i\}$, $i,j \ge 0$, $0 \le s \le t$ be transition probabilities for $X = X(t)$, and $p_i(t) = Pr\{X(t) = i\}$ be its state probabilities.

We suppose that the intensities $\lambda(t)$ and $\mu(t)$ are nonnegative functions locally integrable on $[0,\infty)$.

Then the probabilistic dynamics of the process is represented by the forward Kolmogorov system:

$$\frac{d\mathbf{p}}{dt} = A(t)\mathbf{p}(t), \qquad (1)$$

Proceedings 27th European Conference on Modelling and Simulation ©ECMS Webjørn Rekdalsbakken, Robin T. Bye, Houxiang Zhang (Editors)
ISBN: 978-0-9564944-6-7 / ISBN: 978-0-9564944-7-4 (CD)

where $A(t)$ is transposed intensity matrix,

$$A(t) = \begin{pmatrix} a_{00}(t) & \mu_1(t) & \mu_2(t) & \mu_3(t) & \cdots & \mu_r(t) & \cdots \\ \lambda(t) & a_{11}(t) & \mu_1(t) & \mu_2(t) & \cdots & \mu_{r-1}(t) & \cdots \\ 0 & \lambda(t) & a_{22}(t) & \mu_1(t) & \cdots & \mu_{r-2}(t) & \cdots \\ \cdots & & & & & & \\ 0 & 0 & \cdots & 0 & \lambda(t) & a_{rr}(t) & \cdots \\ \cdots & & & & & & \end{pmatrix},$$

where $\mu_k(t) = \mu(t)/k$, for $k \le S$, $\mu_k(t) = 0,\ k > S$ and $a_{ii}(t)$ are such that all column sums in $A(t)$ equal zero for any $t \ge 0$.

Throughout the paper by $\|\bullet\|$ we denote the l_1-norm, i. e. $\|\mathbf{x}\| = \sum |x_i|$, and $\|B\| = \sup_j \sum_i |b_{ij}|$ for $B = (b_{ij})_{i,j=0}^{\infty}$.

Let Ω be a set of all stochastic vectors, i. e. l_1 vectors with nonnegative coordinates and unit norm.

Then we have $\|A(t)\| = 2\left(\lambda(t) + \sum_{k=1}^{S} \mu_k(t)\right) \le 2\left(\lambda(t) + (1 + \log S)\,\mu(t)\right)$ for almost all $t \ge 0$. Hence operator function $A(t)$ from l_1 into itself is bounded for almost all $t \ge 0$ and locally integrable on $[0;\infty)$. Therefore we can consider (1) as a differential equation in the space l_1 with bounded operator.

It is well known (Daleckij and Krein 1974) that the Cauchy problem for differential equation (1) has unique solutions for arbitrary initial condition, and $\mathbf{p}(s) \in \Omega$ implies $\mathbf{p}(t) \in \Omega$ for $t \ge s \ge 0$.

Denote by $E(t,k) = E\{X(t)\,|X(0) = k\}$ the mean (the mathematical expectation) of the process at the moment t under initial condition $X(0) = k$.

Recall the basic definitions.

The process $X(t)$ is called weakly ergodic, if $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $\mathbf{p}^*(0), \mathbf{p}^{**}(0)$.

The process $X(t)$ is called *ergodic (or strongly ergodic)*, if there exists a vector $\pi \in \Omega$ such that $\lim_{t\to\infty} \|\mathbf{p}(t) - \pi\| = 0$ for any $\mathbf{p}(0) = \mathbf{p} \in \Omega$. The vector π is called the *stationary distribution* for Markov chain $X(t)$.

The process chain $X(t)$ has the limiting mean $\phi(t)$ if $|E(t,k) - \phi(t)| \to 0$ as $t \to \infty$, for any k.

A detailed discussion of these concepts is given in (Zeifman et al. 2008).

ERGODICITY BOUNDS

Theorem 1. Let there exist $\delta < 1$ such that

$$\int_0^{\infty} \alpha^*(t)\,dt = +\infty, \tag{2}$$

where

$$\alpha^*(t) = \sum_{k=1}^{S} \frac{\left(1-\delta^k\right)}{k}\mu(t) - \left(\frac{1}{\delta} - 1\right)\lambda(t). \tag{3}$$

Then queue-length process $X(t)$ is weakly ergodic.

Proof.

Put $d = \frac{1}{\delta} > 1$, and $d_{k+1} = d^k$, $i = 0, 1, \ldots$.

Let D be upper triangular matrix,

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots \\ 0 & d_2 & d_2 & \cdots \\ 0 & 0 & d_3 & \cdots \\ & \ddots & \ddots & \ddots \end{pmatrix}, \tag{4}$$

and let l_{1D} be the correspondent space of sequences $l_{1D} = \left\{\mathbf{z} = (p_1, p_2, \cdots)^T / \|\mathbf{z}\|_{1D} \equiv \|D\mathbf{z}\|_1 < \infty\right\}$.

Define

$$\alpha_i(t) = -a_{ii}(t) - \frac{\lambda(t)}{\delta} - \sum_{k=1}^{i-1}(\mu_{i-k}(t) - \mu_i(t))\delta^{i-k}, \tag{5}$$

and

$$\alpha^*(t) = \inf_{i\ge1} \alpha_i(t). \tag{6}$$

By introducing $p_0(t) = 1 - \sum_{i\ge1} p_i(t)$ we obtain from (1) the following equation, see detailed discussion in (Granovsky and Zeifman 2004; Zeifman 1995; Zeifman et al. 2006, 2008):

$$\frac{d\mathbf{z}}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t), \tag{7}$$

where $\mathbf{f}(t) = (\lambda(t), 0, \cdots)^T$,

$$B = \begin{pmatrix} a_{11} - \lambda & \mu_1 - \lambda & \cdots & \mu_{r-1} - \lambda & \cdots \\ \lambda & a_{22} & \cdots & \mu_{r-2} & \cdots \\ 0 & \lambda & \cdots & \mu_{r-3} & \cdots \\ \cdots & & & & \\ 0 & 0 & \cdots & a_{rr} & \cdots \\ \cdots\cdots\cdots & & & & \end{pmatrix}. \tag{8}$$

Consider equation (7) in the space l_{1D}, where $B(t)$ and $\mathbf{f}(t)$ are locally integrable on $[0, +\infty)$.

Then the following bound of logarithmic norm of operator function $B(t)$ holds, see details in (Satin et al. 2012):

$$\gamma(B(t))_{1D} = \gamma(DB(t)D^{-1}) = \sup_{i\ge1}\{-\alpha_i(t)\} = -\alpha(t), \tag{9}$$

where

$$DBD^{-1} = \begin{pmatrix} a_{11} & \delta(\mu_1 - \mu_2) & \cdots & \delta^{r-1}(\mu_{r-1} - \mu_r) & \cdots & \\ d\lambda & a_{22} & \cdots & \delta^{r-2}(\mu_{r-2} - \mu_r) & \cdots & \\ 0 & d\lambda & \cdots & \delta^{r-3}(\mu_{r-3} - \mu_r) & \cdots & \\ \cdots & & & & & \\ 0 & 0 & d\lambda & a_{rr} & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

We have:

$$\alpha_i(t) = \lambda(t)\left(1-\tfrac{1}{\delta}\right) + \mu(t) \times \left(\sum_{k=1}^{i} \tfrac{1}{k} - \sum_{k=1}^{i-1}\left(\tfrac{1}{i-k}-\tfrac{1}{i}\right)\delta^{i-k}\right) \geq \lambda(t)\left(1-\tfrac{1}{\delta}\right) + \mu(t)\left(\sum_{k=1}^{i}\tfrac{1}{k} - \sum_{k=1}^{i-1}\left(\tfrac{1}{i-k}-\tfrac{1}{i}\right)\right) = \lambda(t)\left(1-\tfrac{1}{\delta}\right) + \mu(t),$$

for $i \leq S$, and

$$\alpha_i(t) = \lambda(t)\left(1-\frac{1}{\delta}\right) + \mu(t)\sum_{k=1}^{S}\frac{1-\delta^k}{k} = \alpha^*(t), \quad (10)$$

for $i > S$. Moreover,

$$\alpha^*(t) = (1-\delta)\left(\mu(t)\left(1+\tfrac{1+\delta}{2}+\cdots+\tfrac{1+\cdots+\delta^{S-1}}{S}\right) - \tfrac{\lambda(t)}{\delta}\right) = (1-\delta)\,\alpha_*(t,\delta),$$

and assumption (2) is equivalent to $\int\limits_0^\infty \alpha_*(t,\delta)\,dt = +\infty$. On the other hand, $\alpha_*(t,\delta)$ is an increasing function of δ. Hence, if assumption (2) holds for some $\delta_0 \in (0;1)$, then it holds for any $\delta_1 \in (0;1)$ such that $\delta_1 > \delta_0$.

Finally, we can choose δ so that $\sum_{k=1}^{S}\frac{1-\delta^k}{k} \leq 1$, and therefore, $\alpha_i(t) \geq \alpha^*(t)$ for $i \leq S$.

Hence,

$$\|V(t,s)\|_{1D} \leq e^{-\int_s^t \alpha^*(u)\,du}, \quad (11)$$

where $V(t,\tau)$ is the Cauchy operator of equation (7), and we obtain the weak ergodicity of $X(t)$ and the following bound:

$$\|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\|_{1D} \leq e^{-\int_s^t \alpha^*(u)\,du}\|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\|_{1D}, \quad (12)$$

for any initial conditions $\mathbf{p}^*(s)$, $\mathbf{p}^{**}(s)$ and any s, t, $0 \leq s \leq t$.

Corollary. Under the assumptions of Theorem 1 the following bounds on the rate of convergence hold: (12), and

$$\|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\|_1 \leq 4\,e^{-\int_s^t \alpha^*(u)\,du}\sum_{i\geq 1} g_i|p_i^*(s)-p_i^{**}(s)| \quad (13)$$

for any initial conditions $\mathbf{p}^*(s)$, $\mathbf{p}^{**}(s)$ and any s, t, $0 \leq s \leq t$, where $g_i = \sum_{n=0}^{i-1}\delta^{-n}$.

Proof. The statement follows from the inequalities:

$$2\|\mathbf{z}\|_{1D} = 2d_1\left|\sum_{i=1}^\infty p_i\right| + 2d_2\left|\sum_{i=2}^\infty p_i\right| + 2d_3\left|\sum_{i=3}^\infty p_i\right| + \cdots \geq 2\left(\left|\sum_{i=1}^\infty p_i\right| + \left|\sum_{i=2}^\infty p_i\right| + \left|\sum_{i=3}^\infty p_i\right| + \cdots\right) \geq \sum_{i=1}^\infty |p_i| = \|\mathbf{z}\|_1,$$

and

$$\|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\|_1 \leq 2\|\mathbf{z}^*(t)-\mathbf{z}^{**}(t)\|_1 \leq 4\,\|\mathbf{z}^*(t)-\mathbf{z}^{**}(t)\|_{1D} \leq 4\,e^{-\int_s^t \alpha^*(u)\,du}\|\mathbf{z}^*(s)-\mathbf{z}^{**}(s)\|_{1D} \leq 4\,e^{-\int_s^t \alpha^*(u)\,du}\sum_{i\geq 1} g_i|p_i^*(s)-p_i^{**}(s)|.$$

Put $W = \inf_{k\geq 1}\frac{1}{k\delta^k} > 0$.

Corollary. Under the assumptions of Theorem 1, the queue-length process $X(t)$ has the limiting mean $\phi(t)$ and, putting $\phi(t) = E(t,0)$, we obtain the following bound on the rate of convergence:

$$|E(t,k)-\phi(t)| \leq \frac{2g_k}{W}\,e^{-\int_0^t \alpha^*(u)\,du}. \quad (14)$$

for any k and any $t \geq 0$.

Proof. Estimate (14) follows from the inequalities

$$\|\mathbf{z}\|_{1D} \geq \frac{W}{2}\,\|\mathbf{z}\|_{1E}$$

where $\|\mathbf{z}\|_{1E} = \sum k|p_k|$, and

$$|E(t,k)-\phi(t)| \leq \|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\|_{1E} \leq \tfrac{2}{W}\,e^{-\int_0^t \alpha^*(u)\,du}\|\mathbf{e}_k-\mathbf{e}_0\|_{1D} = \tfrac{2g_k}{W}\,e^{-\int_0^t \alpha^*(u)\,du},$$

for respective $\mathbf{p}^*(t)$, $\mathbf{p}^{**}(t)$ and any $k \geq 0$.

Consider now the most important special cases.

1. Let $X(t)$ be a stationary Markov chain (i.e. let intensities λ and μ not depend on t).

Then assumption (2) is equivalent to the inequality

$$S\mu > \lambda. \quad (15)$$

In some cases we can find the best possible value for α^* in the following way.

Consider two functions: $h_1(\delta) = \lambda\left(1-\frac{1}{\delta}\right) + \mu\sum_{k=1}^{S}\frac{1-\delta^k}{k}$, and $h_2(\delta) = \lambda\left(1-\frac{1}{\delta}\right)+\mu$.

Then $h_1^{'}(\delta) = \frac{\lambda}{\delta^2} - \mu\sum_{k=1}^{S}\delta^{k-1}$. We have $h_1^{'}(\delta) > 0$ for $0 < \delta << 1$, and $h_1^{'}(1) = \lambda - S\mu < 0$. On the other hand, $h_1^{''}(\delta) = -\frac{2\lambda}{\delta^3} - \mu\sum_{k=1}^{S}(k-1)\delta^{k-2} < 0$ for any positive δ. Hence $h_1^{'}(\delta)$ is strictly decreasing, and there exists a unique $\delta^* \in (0,1)$ such that $h_1^{'}(\delta^*) = 0$ and $h_1(\delta^*) = \sup_{\delta\in(0,1)} h_1(\delta)$. If

$$h_1(\delta^*) < h_2(\delta^*), \quad (16)$$

then we obtain the respective best possible value of α^*.

Theorem 2. Let $X(t)$ be a stationary Markov chain, and let assumptions (15), (16) be satisfied. Then $X(t)$ is strongly ergodic (with stationary distribution π and the following bounds hold:

$$\|\mathbf{p}^*(t)-\pi\|_{1D} \leq e^{-\alpha^* t}\|\mathbf{p}^*(0)-\pi\|_{1D}, \quad (17)$$

$$\|\mathbf{p}^*(t) - \pi\|_1 \le 4\, e^{-\alpha^* t} \sum_{i \ge 1} g_i |p_i^*(0) - \pi_i|, \quad (18)$$

for any initial condition $\mathbf{p}^*(0)$ and any $t \ge 0$, and

$$|E(t,k) - \phi| \le \frac{2}{W}\, e^{-\alpha^* t} \|\mathbf{p}^*(0) - \pi\|_{1D}. \quad (19)$$

where now the limiting mean $\phi = \sum k\pi_k$, and $\alpha^* = \sum_{k=1}^{S} \frac{(1-\delta^{*k})}{k}\mu - \left(\frac{1}{\delta^*} - 1\right)\lambda$.

2. Let now the arrival and service rates $\lambda(t)$ and $\mu(t)$ be 1-periodic in t. Then assumption (2) is equivalent to the inequality

$$\int_0^1 \left(S\mu(t) - \lambda(t)\right)\, dt > 0. \quad (20)$$

Theorem 3. Let the arrival and service rates $\lambda(t)$ and $\mu(t)$ be 1-periodic in t, and let (20) be satisfied. Then the process $X(t)$ is weakly ergodic. In addition, the respective limiting regime $\mathbf{p}^{**}(t)$ and limiting mean $\phi(t)$ are also 1-periodic. Moreover, $X(t)$ is **exponentially** weakly ergodic, namely

$$e^{-\int_s^t \alpha^*(u)\, du} \le M e^{-a(t-s)u}, \quad (21)$$

where $M = \exp(\sup_{|t-s|\le 1} \int_s^t \alpha^*(u)\, du)$, and $a = \int_0^1 \alpha^*(t)\, dt$.

EXAMPLES

We consider and compare the following two models: ordinary $M_t/M_t/S$ queue, and its analogue for a queue with group services with the same characteristics. Namely, we suppose that arrival and service intensities are $\lambda(t) = 1 + \sin 2\pi t$ and $\mu(t) = 3 + \cos 2\pi t$ respectively, and let for definiteness, $S = 10^{12}$.

We obtain and compare two limiting characteristics for the both models: the limiting probability of empty queue and the limiting mean.

Put $\delta = 0.5$, then we obtain $W = 1$, $g_k = 2^k - 1$, and $\alpha_k(t) \ge \mu(t) - \lambda(t)$, for any $k \ge 1$, any S, and any $t \ge 0$.

Therefore, we can assume that $\alpha^*(t) \ge 2 + \cos 2\pi t - \sin 2\pi t$, and moreover, inequality (21) holds for $a = 2$ and $M = 2$.

Therefore, we obtain for *both* models the following bounds on the rate of convergence to limiting 1-periodic characteristics $\mathbf{p}^{**}(s)$ and $\phi(t)$ respectively:

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_1 \le 4\, e^{-2t} \|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1D}, \quad (22)$$

for any initial condition $\mathbf{p}^*(0)$, and any $t \ge 0$,

$$|E(t,k) - \phi(t)| \le 4\, e^{-2t} \|\mathbf{e}_k - \mathbf{p}^{**}(0)\|_{1D}, \quad (23)$$

for any k and any $t \ge 0$. Putting $\mathbf{p}^*(0) = \mathbf{0}$, we have from (22) and (23) the inequalities

$$\begin{aligned} \|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_1 &\le 4\, e^{-2t} \|\mathbf{p}^{**}(0)\|_{1D}, \\ |E(t,0) - \phi(t)| &\le 4\, e^{-2t} \|\mathbf{p}^{**}(0)\|_{1D}, \end{aligned} \quad (24)$$

for obtaining the 1-periodic limiting probability of the empty queue and the 1-periodic limiting mean.

In order to be able to use bound (24), we need to estimate the initial condition for 1-periodic limiting regime.

We have $\|\mathbf{f}(t)\|_{1D} \le K = 2$ for any $t \ge 0$, and

$$\|\mathbf{p}^{**}(0)\|_{1D} \le \limsup_{t\to\infty} \|\mathbf{p}^{**}(t)\|_{1D}, \quad (25)$$

on the other hand,

$$\begin{aligned} &\|\mathbf{p}^{**}(t)\|_{1D} \le \|V(t)\|_{1D} \|\mathbf{z}^{**}(0)\|_{1D} + \\ &\int_0^t \|V(t,\tau)\|_{1D} \|\mathbf{f}(\tau)\|_{1D} d\tau \le \\ &\tfrac{KM}{a} + Me^{-at} \|\mathbf{z}^{**}(0)\|_{1D} \le 2 + o(1). \end{aligned}$$

Finally, we obtain for $X(0) = 0$:

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_1 \le 8\, e^{-2t}, \quad |E(t,0) - \phi(t)| \le 8\, e^{-2t}, \quad (26)$$

for obtaining the 1-periodic limiting probability of the empty queue $p_0^{**}(t)$ and the 1-periodic limiting mean $\phi(t)$.

Now we can use the approach of (Zeifman et al. 2006, Satin et al. 2012) for solving the Cauchy problem of forward Kolmogorov system with initial condition $\mathbf{e}_0$ for truncated processes $X_{N-1}(t)$. Namely, Theorem 2 (Zeifman et al. 2006) and Theorem 6 (Satin et al. 2012) give the following bounds on truncation errors for both models:

$$\begin{aligned} \|\mathbf{p}^*(t) - \mathbf{p}^*_{N-1}(t)\|_1 &\le \tfrac{25Nt}{2^{N-1}-1}, \\ |E(t,0) - E_{N-1}(t,0)| &\le \tfrac{25Nt}{2^{N-1}-1}, \end{aligned}$$

if $X(0) = X_{N-1}(0) = 0$.

Finally, putting $\varepsilon = 10^{-6}$, and $N = 30$, we find the limiting characteristics for both models approximately on interval $[10, 11]$ with error 10^{-3}.

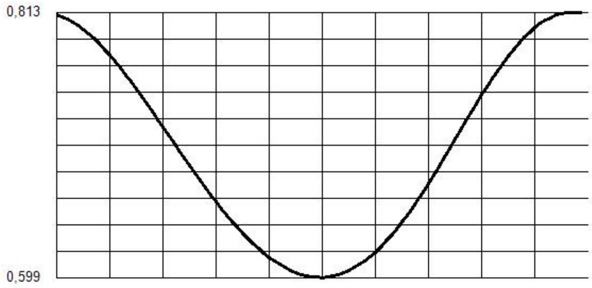

Figure 1: Approximation of the limiting probability of empty queue for ordinary $M_t/M/t/S$ queue on $[10, 11]$ with an error 10^{-3}.

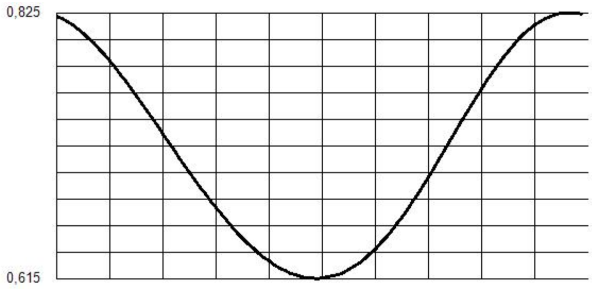

Figure 2: Approximation of the limiting probability of empty queue for analog of $M_t/M/t/S$ queue with group services on $[10, 11]$ with an error 10^{-3}.

Remark. One can see the interesting fact that the limiting mathematical expectations are the same for both examples.

This research was supported by the Russian Foundation for Basic Research, projects no. 11-07-00112a, 12-07-00115a, 12-07-00109a, 13-07-00223a.

REFERENCES

Daleckij, Ju.L. and M.G. Krein. 1974. *Stability of solutions of differential equations in Banach space.* Amer. Math. Soc. Transl. 43.

Gnedenko, B. V. and V. Yu. Korolev. 1996. *Random Summation: Limit Theorems and Applications.* Boca Raton: CRC Press.

Gnedenko, B.V. and I.P. Makarov. 1971. "Properties of a problem with losses in the case of periodic intensities." *Diff. equations.* 7, 1696–1698 (in Russian).

Gnedenko, B. and A. Soloviev. 1973. "On the conditions of the existence of final probabilities for a Markov process." *Math. Operationsforsh. Stat.* 4, 379–390.

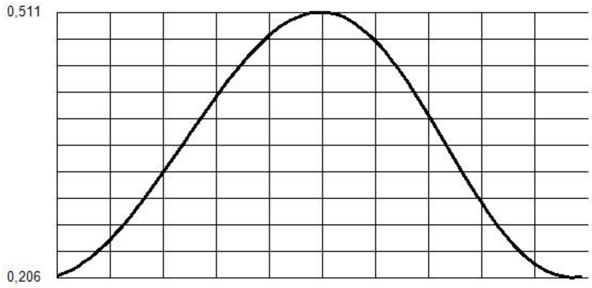

Figure 3: Approximation of the limiting mean for ordinary $M_t/M/t/S$ queue on $[10, 11]$ with an error 10^{-3}.

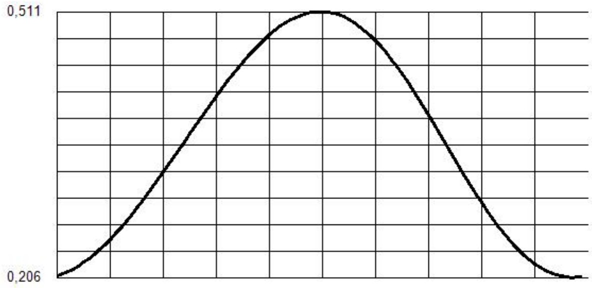

Figure 4: Approximation of the limiting mean for analog of $M_t/M/t/S$ queue with group services on $[10, 11]$ with an error 10^{-3}.

Gnedenko, D.B. 1973. "On a generalization of Erlang formulae." *Zastosow. Mat.* 12, 239–242.

Granovsky, B.L. and A.I. Zeifman. 2004. "Nonstationary Queues: Estimation of the Rate of Convergence." *Queueing Syst.* 46, 363–388.

Korolev V. and I. Shevtsova I. 2012. "An improvement of the Berry–Esseen inequality with applications to Poisson and mixed Poisson random sums." *Scandinavian Actuarial J.* 2, 81–105.

Satin, Ya.A.; A.I. Zeifman; A.V. Korotysheva and S.Ya. Shorgin. 2011. "On a class of Markovian queues." *Informatics and its applications.* 5. No. 4, 6–12 (in Russian).

Satin, Ya. A.; A.I. Zeifman and A.V. Korotysheva. 2012. "On the rate of convergence and truncations for a class of Markovian queueing systems." *Th. Prob. Appl.* 57, 611–621 (in Russian).

Zeifman, A. I. 1995 . "Upper and lower bounds on the rate of convergence for nonhomogeneous birth and death processes." *Stoch. Proc. Appl.* 59, 157–173.

Zeifman, A. I.; V.E. Bening and I.A. Sokolov. 2008. *Continuous-time Markov chains and models.* Elex-KM, Moscow (in Russian).

Zeifman, A.; S. Leorato; E. Orsingher; Ya. Satin and G. Shilova. 2006. "Some universal limits for nonhomogeneous birth and death processes." *Queueing Syst.* 52, 139–151.

Zeifman, A. I. 2009. "On nonstationary Erlang model." *Automation and Remote Control.* 70, 2003–2012.

Zeifman, A. and A. Korotysheva. 2012. "Perturbation bounds for $M_t/M_t/N$ queue with catastrophes." *Stochastic models.* 28, 49–62.

Zeifman, A.; A. Korotysheva; Ya. Satin; G. Shilova and T. Panfilova. 2013. "On a queueing model with group services," *Lecture Notes in Communications in Computer and Information Science," 356, 198–205.*

AUTHOR BIOGRAPHIES

ALEXANDER ZEIFMAN Doctor of Science in physics and mathematics; professor, Dean of the Faculty of Applied Mathematics and Computer Technologies, Vologda State Pedagogical University; senior scientist, Institute of Informatics Problems, Russian Academy of Sciences; leading scientist, Institute of Territories Socio-Economic Development, Russian Academy of Sciences. His email is a_zeifman@mail.ru and his personal webpage at http://uni-vologda.ac.ru/ zai/eng.html.

YAKOV SATIN is Candidate of Science (PhD) in physics and mathematics, associate professor, Vologda State Pedagogical University. His email is yacovi@email.ru.

GALINA SHILOVA is Candidate of Science (PhD) in physics and mathematics, associate professor, Vologda State Pedagogical University. Her email is shgn@email.ru.

VICTOR KOROLEV is Doctor of Science in physics and mathematics, professor, Department of Mathematical Statistics, Faculty of Computational Mathematics and Cybernetics, M.V. Lomonosov Moscow State University; leading scientist, Institute of Informatics Problems, Russian Academy of Sciences. His email is bruce27@yandex.ru.

VLADIMIR BENING is Doctor of Science in physics and mathematics; professor, Department of Mathematical Statistics, Faculty of Computational Mathematics and Cybernetics, M. V. Lomonosov Moscow State University; senior scientist, Institute of Informatics Problems, Russian Academy of Sciences. His email is bening@yandex.ru.

SERGEY YA. SHORGIN is Doctor of Science in physics and mathematics, professor, Deputy Director, Institute of Informatics Problems, Russian Academy of Sciences. His email is sshorgin@ipiran.ru.