ON CONVERGENCE OF RANDOM WALKS HAVING JUMPS WITH FINITE VARIANCES TO STABLE LÉVY PROCESSES

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ABSTRACT

A functional limit theorem is proved establishing weak convergence of random walks generated by compound doubly stochastic Poisson processes to Lévy processes in the Skorokhod space. As corollaries, theorems are proved on convergence of random walks with jumps having finite variances to Lévy processes with mixed normal distributions, in particular, to stable Lévy processes.

Statistical analysis of the traffic in information flows in modern computational and telecommunication systems sometimes shows that this characteristics possesses the property of self-similarity. In applied probability this property is usually modeled by Lévy processes. This communication gives some theoretical grounds to this convention.

In (Kashcheev 2000, 2001) some functional limit theorems were proved for compound Cox processes with square integrable leading random measures. However, the class of limit processes for compound Cox processes having jumps with finite variances and such leading random measures cannot contain any stable Lévy process besides the Wiener process. The aim of the present work is to fill this gap.

Let $D = D[0,1]$ be a space of real-valued right-continuous functions defined on $[0,1]$ and having left-side limits. Let $\mathcal{F}$ be the class of strictly increasing continuous mappings of the interval $[0,1]$ onto itself. Let f be a non-decreasing function on $[0,1]$, $f(0) = 0$, $f(1) = 1$. Let

$$\|f\| = \sup_{s\neq t}\left|\log\frac{f(t)-f(s)}{t-s}\right|.$$

If $\|f\| < \infty$, then the function f is continuous and strictly increasing, hence, it belongs to $\mathcal{F}$.

Define the metric $d_0(x,y)$ in $D[0,1]$ as the greatest upper bound of positive numbers ϵ for which $\mathcal{F}$ contains a function f such that $\|f\| \leq \epsilon$ and

$$\sup_t |x(t) - y(f(t))| \leq \epsilon.$$

It can be shown that $D[0,1]$ is complete with respect to the metric d_0. The metric space $(D[0,1], d_0)$ is referred to as the *Skorokhod space*.

We will consider stochastic processes as random elements in $\mathcal{D} \equiv (D[0,1],\, d_0)$ in the following sense. Let $\mathfrak{D}$ be the class of Borel sets of the space $\mathcal{D}$. The class $\mathfrak{D}$ is the σ-algebra generated by the open sets of $\mathcal{D}$. A mapping X of the basic probability space $(\Omega,\, \mathcal{A},\, \mathsf{P})$ to $\mathcal{D}$ is measurable if $\{\omega : X(\omega) \in B\} \in \mathcal{A}$ for any set $B \in \mathfrak{D}$. By a stochastic process we will mean a measurable mapping X of Ω to $\mathcal{D}$. By the distribution of a stochastic process we will mean the probability measure P^X on the measurable space $(\mathcal{D},\, \mathfrak{D})$ defined for any set $A \in \mathfrak{D}$ by the relation $\mathsf{P}^X(A) = \mathsf{P}\big(\{\omega : X(\omega) \in A\}\big) \equiv \mathsf{P}(X \in A)$. The symbol $\Longrightarrow$ will denote weak convergence: the sequence $\{X_n(t)\}_{n\geq 1}$ of stochastic processes weakly converges to a stochastic process $X(t)$, that is, $X_n(t) \Longrightarrow X(t)$, if

$$\int w(\omega)\mathsf{P}^{X_n}(d\omega) \longrightarrow \int w(\omega)\mathsf{P}^X(d\omega)$$

for any continuous bounded function w.

By a Lévy process, as usual, we will mean a homogeneous stochastically continuous stochastic process $X(t)$, $t \in [0,1]$, with independent increments such that

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$X(0)=0$ a.s. and the sample paths $X(t)\in D[0,1]$. As is easily seen, for each $t\in[0,1]$ the random variable $X(t)$ has an infinitely divisible distribution.

The strictly stable distribution function with the characteristic exponent $\alpha\in(0,2]$ and parameter θ ($|\theta|\le\theta_\alpha=\min\{1,\frac{2}{\alpha}-1\}$) determined by the characteristic function

$$\mathfrak{g}_{\alpha,\theta}(s)=\exp\Big\{-|s|^\alpha\exp\Big\{-\frac{i\pi\theta\alpha}{2}\mathrm{sign}s\Big\}\Big\},\quad s\in\mathbb{R},$$

will be denoted $G_{\alpha,\theta}(x)$. The value $\theta=0$ corresponds to symmetric strictly stable laws. The values $\theta=1$ and $0<\alpha\le1$ correspond to one-sided strictly stable distributions. As is known, if ξ is a random variable with the distribution function $G_{\alpha,\theta}(x)$, $0<\alpha<2$, then $\mathsf{E}|\xi|^\delta<\infty$ for any $\delta\in(0,\alpha)$, but the moments of orders greater or equal to α of the random variable ξ do not exist (see, e.g., (Zolotarev 1986)).

The distribution function of the standard normal law ($\alpha=2$, $\theta=0$) will be denoted $\Phi(x)$. It is well known that

$$G_{\alpha,0}(x)=\int_0^\infty\Phi\Big(\frac{x}{\sqrt{u}}\Big)dG_{\alpha/2,1}(u),\quad x\in\mathbb{R}\qquad(1)$$

(see, e.g., (Zolotarev 1986) or (Feller 1971)). To representation (1) there corresponds the representation in terms of characteristic functions:

$$\mathfrak{g}_{\alpha,0}(s)=\int_0^\infty\exp\Big\{-\frac{s^2u}{2}\Big\}dG_{\alpha/2,1}(u),\quad s\in\mathbb{R}.\quad(2)$$

A Lévy process $X(t)$, $t\in[0,1]0$, is called *α-stable*, if $\mathsf{P}\big(X(1)<x\big)=G_{\alpha,\theta}(x)$, $x\in\mathbb{R}$. It can be shown that if $X(t)$, $t\in[0,1]$, is a Lévy process, then $X(t)$ is α-stable if and only if $X(t)\stackrel{d}{=}t^{1/\alpha}X(1)$, $t\in[0,1]$ (see, e.g., (Embrechts and Maejima 2002)).

Consider a sequence of compound Cox processes

$$Z_n(t)=\sum_{i=1}^{N_1^{(n)}(\Lambda_n(t))}X_{n,i},\quad t\ge0,\qquad(3)$$

where $\{N_1^{(n)}(t),\ t\in[0,1]\}_{n\ge1}$ are Poisson processes with unit intensity; for each $n=1,2,...$ the random variables $X_{n,1},X_{n,2},...$ are identically distributed, moreover, for each $n\ge1$ the random variables $X_{n,1},X_{n,2},...$ and the process $N_1^{(n)}(t)$, $t\in[0,1]$, are independent; for each $n=1,2,...$ the random measure $\Lambda_n(t)$, $t\in[0,1]$, is a Lévy process independent of the process

$$X_n(t)=\sum_{i=1}^{N_1^{(n)}(t)}X_{n,i},\quad t\ge0,$$

such that $\Lambda_n(0)=0$, $\Lambda_n(1)\stackrel{d}{=}k_nU_{\alpha,1}^{(n)}$, where $\{k_n\}_{n\ge1}$ is an infinitely increasing sequence of natural numbers and $U_{\alpha,1}^{(1)},U_{\alpha,1}^{(2)},...$ is a sequence of identically distributed a.s. positive random variables having one-sided strictly stable distribution with parameters $\alpha\in(0,1]$ and $\theta=1$. For definiteness, we assume that $\sum_{i=1}^0=0$. From the abovesaid it follows that $\mathsf{E}\Lambda_n^\beta(1)<\infty$ for any $\beta<\alpha$ and

$$\Lambda_n(t)\stackrel{d}{=}t^{1/\alpha}\Lambda_n(1)\stackrel{d}{=}t^{1/\alpha}k_nU_{\alpha,1}^{(n)}\stackrel{d}{=}t^{1/\alpha}k_nU_{\alpha,1}^{(1)},\quad t\ge0.\qquad(4)$$

Assume that

$$\mathsf{E}X_{n,1}=0\ \text{ and }\ 0<\sigma_n^2\equiv\mathsf{E}X_{n,1}^2<\infty.\qquad(5)$$

Let $t=1$. Denote $N_n=N_1^{(n)}(\Lambda_n(1))$. Assume that, as $n\to\infty$,

$$\mathsf{P}(X_{n,1}+...+X_{n,k_n}<x)\longrightarrow\Phi(x),\qquad(6)$$

with the same $\{k_n\}_{n\ge1}$ as in the definition of the random measures $\Lambda_n(t)$. From the classical theory of limit theorems it is known that (6) holds, if, as $n\to\infty$,

$$k_n\sigma_n^2\longrightarrow1\qquad(7)$$

and

$$k_n\mathsf{E}X_{n,1}^2\mathbb{I}(|X_{n,1}|\ge\epsilon)\longrightarrow0\qquad(8)$$

for any $\epsilon>0$.

Moreover, by virtue of (4) it is obvious that

$$\frac{\Lambda_n(1)}{k_n}\stackrel{d}{=}\frac{k_nU_{\alpha,1}^{(1)}}{k_n}=U_{\alpha,1}^{(1)}.$$

therefore, formally,

$$\frac{\Lambda_n(1)}{k_n}\Longrightarrow U_{\alpha,1}^{(1)}.\qquad(9)$$

But, as it was shown in (Gnedenko and Korolev 1996) (also see, e.g., (Bening and Korolev 2002) or (Korolev et al. 2011)), (9) is equivalent to

$$\frac{N_n}{k_n}\Longrightarrow U_{\alpha,1}^{(1)}.\qquad(10)$$

By the Gnedenko–Fahim transfer theorem (Gnedenko and Fahim 1969) (also see theorem 2.9.1 in (Korolev et al. 2011)) conditions (6) and (10) imply that, as $n\to\infty$,

$$Z_n(1)=X_{n,1}+...+X_{n,N_n}\Longrightarrow Z,\qquad(11)$$

where Z is the random variable with the characteristic function

$$\mathfrak{f}(s)=\int_0^\infty\exp\Big\{-\frac{s^2u}{2}\Big\}d\mathsf{P}(U_{\alpha,1}^{(1)}<u),\quad s\in\mathbb{R}.$$

But by virtue of (2)

$$\mathfrak{f}(s) = \int\limits_0^\infty \exp\Big\{-\frac{s^2u}{2}\Big\}dG_{\alpha,1}(u) = \mathfrak{g}_{2\alpha,0}(s), \quad s \in \mathbb{R},$$

that is, the limit random variable Z in (11) has the symmetric strictly stable distribution with the characteristic exponent $\alpha_0 = 2\alpha$.

Consider an α_0-stable Lévy process $Z(t)$, $t \in [0,1]$, such that $Z(1) \stackrel{d}{=} Z$. Since $Z_n(t)$ and $Z(t)$ are Lévy processes, almost all their sample paths belong to the Skorokhod space $\mathcal{D}$.

Using theorem 15.6 from (Billingsley 1968) we obtain the following result.

THEOREM. *Let $\alpha \in (0,1]$ and a compound Cox process $Z_n(t)$ (see (3)) be controlled by the Lévy process $\Lambda_n(t)$ such that $\Lambda_n(1) \stackrel{d}{=} k_n U_{\alpha,1}^{(n)}$, where $\{k_n\}_{n\geq 1}$ is an infinitely increasing sequence of natural numbers and $U_{\alpha,1}^{(1)}, U_{\alpha,1}^{(2)}, \ldots$ is a sequence of identically distributed a.s. positive random variables having one-sided strictly stable distribution with parameters α and $\theta = 1$. Assume that the random jumps $\{X_{n,j}\}_{j\geq 1}$, $n = 1, 2, \ldots$, of the compound Cox process $Z_n(t)$ satisfy conditions (5), (7) and (8) with the same numbers k_n. Then the random walks generated by these compound Cox processes weakly converge in the Skorokhod space $\mathcal{D} = (D[0,1], d_0)$ to a 2α-stable Lévy process $Z(t)$ with $\mathsf{P}\big(Z(1) < x\big) = G_{2\alpha,0}(x)$.*

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