STATE-SPACE CONSTRAINED MODEL PREDICTIVE CONTROL

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KEYWORDS

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ABSTRACT

Constrained State-space Model Predictive Control is presented in the paper. Predictive controller based on incremental linear state-space process model and quadratic criterion is derived. Typical types of constraints are considered – limits on manipulated, state and controlled variables. Control experiments with nonlinear model of multivariable laboratory process are simulated first and real experiment is realized afterwards.

INTRODUCTION

Model Predictive Control is very popular control method treated in academic area and used in industry as well (Clarke et al. 1987a; Clarke et al. 1987b; Clarke and Mohtadi 1989; Camacho and Bordons 2007). It is very general, easy to understand and intuitive concept how to solve optimal control problem. Future control actions are calculated to minimize quadratic cost function - at least future control movements and control errors are penalized. Dynamic model of the process is used for future plant behaviour – to calculate vector of future controlled variable. Only actual control action (first element from the vector of calculated control actions) is applied to the process and whole procedure is repeated – this is called receding horizon concept and it introduces feedback in some way. Many process models exist and also cost function formulations which gives arise to wide range of different methods. Some of them are more practical and some are treated by academicians more frequently.

In the paper Constrained State-space Model Predictive Control is presented as very powerful approach. Calculations are straightforward and easy to program. Using of state-space model leads naturally to matrix formulation of prediction equations even for multivariable systems, terminal state can be easily penalized in the cost function and constrains on the state variables can be respected too. The last mentioned property is a key feature for the paper – because our controlled process has state variables with limits that should not be exceeded. Furthermore not all the state variables are measured so they must be estimated which is another elegant task for state-space models. Predictive controller is derived fist and applied to a laboratory process afterwards.

STATE-SPACE PREDICTIVE CONTROLLER

Following finite horizon quadratic criterion is considered

$$J = \sum_{j=N_1}^{N_2} \sum_{m=1}^{n_y} r_m \left[w_m(k+j) - \hat{y}_m(k+j) \right]^2 + \\ + \sum_{j=1}^{N_3} \sum_{n=1}^{n_u} q_n \Delta u_n(k+j-1)^2 \qquad (1)$$

Where N_1 and N_2 are minimum and maximum prediction horizons, N_3 is control horizon - after first N_3 control moves the control signal is kept constant, n_y is number of the system outputs, n_u is number of the system inputs, $y_m(k+j)$ is an optimum j-step ahead prediction of the m-th system output, $w_m(k+j)$ is a future set-point or reference sequence for the m-th output. $\Delta u_n(k+j-1) = u_n(k+j-1) - u_n(k+j-2)$ is n-th control increment, r_m and q_n are positive weighting coefficients.

State-space model for multivariable process is expressed as

$$\begin{aligned} \bar{\mathbf{x}}(k+1) &= \overline{\mathbf{M}}\,\bar{\mathbf{x}}(k) + \overline{\mathbf{N}}\,\mathbf{u}(k) \\ \mathbf{y}(k) &= \overline{\mathbf{Q}}\,\bar{\mathbf{x}}(k) \end{aligned} \qquad (2)$$

$$\mathbf{u}(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_{n_u}(k) \end{bmatrix}, \quad \bar{\mathbf{x}}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n_x}(k) \end{bmatrix}, \quad \mathbf{y}(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_{n_y}(k) \end{bmatrix}$$

where $\bar{\mathbf{x}}(k)$ is the state vector composed from n_x state variables, $\mathbf{u}$(k) is vector of inputs, $\mathbf{y}$(k) is vector of outputs and $\overline{\mathbf{M}}$, $\overline{\mathbf{N}}$ and $\overline{\mathbf{Q}}$ are system matrices.

If we extend the state vector with the last control action $\mathbf{u}(k-1)$, we get the state-space system with control increments as inputs (incremental form)

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$$\underbrace{\begin{bmatrix}\overline{\mathbf{x}}(k+1)\\ \mathbf{u}(k)\end{bmatrix}}_{\mathbf{x}(k+1)}=\underbrace{\begin{bmatrix}\overline{\mathbf{M}} & \overline{\mathbf{N}}\\ \mathbf{0}_{n_u\times n_x} & \mathbf{I}_{n_u\times n_u}\end{bmatrix}}_{\mathbf{M}}\underbrace{\begin{bmatrix}\overline{\mathbf{x}}(k)\\ \mathbf{u}(k-1)\end{bmatrix}}_{\mathbf{x}(k)}+$$

$$+\underbrace{\begin{bmatrix}\overline{\mathbf{N}}\\ \mathbf{I}_{n_u\times n_u}\end{bmatrix}}_{\mathbf{N}}\Delta\mathbf{u}(k) \tag{3}$$

$$\mathbf{y}(k)=\underbrace{\begin{bmatrix}\overline{\mathbf{Q}} & \mathbf{0}_{n_y\times n_u}\end{bmatrix}}_{\mathbf{Q}}\underbrace{\begin{bmatrix}\overline{\mathbf{x}}(k)\\ \mathbf{u}(k-1)\end{bmatrix}}_{\mathbf{x}(k)}$$

The optimum one-step ahead prediction of the model output is

$$\begin{aligned}\hat{\mathbf{x}}(k+1)&=\mathbf{M}\,\mathbf{x}(k)+\mathbf{N}\,\Delta\mathbf{u}(k)\\ \hat{\mathbf{y}}(k+1)&=\mathbf{Q}\,\hat{\mathbf{x}}(k+1)=\mathbf{QM}\,\mathbf{x}(k)+\mathbf{QN}\,\Delta\mathbf{u}(k)\end{aligned} \tag{4}$$

The optimum two-step ahead prediction is

$$\begin{aligned}\hat{\mathbf{x}}(k+2)&=\mathbf{M}\,\hat{\mathbf{x}}(k+1)+\mathbf{N}\,\Delta\mathbf{u}(k+1)=\mathbf{M}^2\,\mathbf{x}(k)+\\ &\quad+\mathbf{MN}\,\Delta\mathbf{u}(k)+\mathbf{N}\,\Delta\mathbf{u}(k+1)\\ \hat{\mathbf{y}}(k+2)&=\mathbf{Q}\,\hat{\mathbf{x}}(k+2)=\mathbf{QM}^2\,\mathbf{x}(k)+\mathbf{QMN}\,\Delta\mathbf{u}(k)+\\ &\quad+\mathbf{QN}\,\Delta\mathbf{u}(k+1)\end{aligned} \tag{5}$$

Generally the optimum j-step ahead prediction is

$$\hat{\mathbf{y}}(k+j)=\mathbf{QM}^j\,\mathbf{x}(k)+\sum_{i=0}^{j-1}\mathbf{QM}^{j-i-1}\mathbf{N}\,\Delta\mathbf{u}(k+i) \tag{6}$$

Set of N_2 j-step ahead predictions starting from N_1 and with N_3 future control moves in matrix form can be expressed as

$$\mathbf{Y}=\mathbf{H}\Delta\mathbf{U}+\mathbf{F}\mathbf{x}(k) \tag{7}$$

where $\mathbf{Y}$, $\Delta\mathbf{U}$ and $\mathbf{F}$ are

$$\mathbf{Y}=\begin{bmatrix}\hat{\mathbf{y}}(k+N_1)\\ \hat{\mathbf{y}}(k+N_1+1)\\ \vdots\\ \hat{\mathbf{y}}(k+N_2)\end{bmatrix},\ \Delta\mathbf{U}=\begin{bmatrix}\Delta\mathbf{u}(k)\\ \Delta\mathbf{u}(k+1)\\ \vdots\\ \Delta\mathbf{u}(k+N_3-1)\end{bmatrix},$$

$$\mathbf{F}=\begin{bmatrix}\mathbf{QM}^{N_1}\\ \mathbf{QM}^{N_1+1}\\ \vdots\\ \mathbf{QM}^{N_2}\end{bmatrix}.$$

and $\mathbf{H}$ is block lower triangular matrix with its non-null i-row and j-column elements defined by $(\mathbf{H})_{ij}=\mathbf{QM}^{i-j}\mathbf{N}$. First term of (7) is called as forced response and second part as free response – response from actual state without changing the manipulated variables.

Criterion (1) can be rewritten in matrix form as

$$J=(\mathbf{W}-\mathbf{Y})^T\,\overline{\mathbf{R}}(\mathbf{W}-\mathbf{Y})+\Delta\mathbf{U}^T\,\overline{\mathbf{Q}}\Delta\mathbf{U} \tag{8}$$

where $\mathbf{W}$ is

$\mathbf{W}=\begin{bmatrix}\mathbf{w}(k+N_1)\\ \mathbf{w}(k+N_1+1)\\ \vdots\\ \mathbf{w}(k+N_2)\end{bmatrix}$ and $\overline{\mathbf{R}}$ and $\overline{\mathbf{Q}}$ are diagonal matrices filled with diagonal sub matrices of weighting coefficients.

If we substitute predictions from (7) into equation (8) we get

$$\begin{aligned}J&=(\mathbf{W}-\mathbf{H}\Delta\mathbf{U}-\mathbf{F}\mathbf{x}(k))^T\,\overline{\mathbf{R}}(\mathbf{W}-\mathbf{H}\Delta\mathbf{U}-\mathbf{F}\mathbf{x}(k))+\\ &\quad+\Delta\mathbf{U}^T\,\overline{\mathbf{Q}}\Delta\mathbf{U}=\mathbf{U}^T\mathbf{S}\Delta\mathbf{U}+2\mathbf{s}^T\Delta\mathbf{U}+k\end{aligned} \tag{9}$$

where $\mathbf{S}$, $\mathbf{s}$ and k are

$$\mathbf{S}=\mathbf{H}^T\,\overline{\mathbf{R}}\mathbf{H}+\overline{\mathbf{Q}},\quad \mathbf{s}=-\mathbf{H}^T\,\overline{\mathbf{R}}(\mathbf{W}-\mathbf{F}\mathbf{x}(k)),$$
$$k=(\mathbf{W}-\mathbf{F}\mathbf{x}(k))^T\,\overline{\mathbf{R}}(\mathbf{W}-\mathbf{F}\mathbf{x}(k)).$$

Without constraints explicit solution - minimization of (9) can be expressed as

$$\begin{aligned}\Delta\mathbf{U}_{N3}&=-\mathbf{S}^{-1}\cdot\mathbf{s}=\\ &=\underbrace{\left(\mathbf{H}^T\,\overline{\mathbf{R}}\mathbf{H}+\overline{\mathbf{Q}}\right)^{-1}\mathbf{H}^T\,\overline{\mathbf{R}}}_{\mathbf{L}}(\mathbf{W}-\mathbf{F}\mathbf{x}(k))\end{aligned} \tag{10}$$

Control law is a linear gain matrix $\mathbf{L}$ that multiplies predicted control error – difference between future reference and free response of the plant. Receding strategy means that only first element of the sequence $\Delta\mathbf{U}_{N3}$ is applied to process and next control action is calculated according to a new state of the process which must be measured or observed.

Numerical optimization method must be used if manipulated, state or controlled variables are constrained or other types of constrains occurs. Minimization of (9) with respect to linear inequalities

$$\mathbf{A}\cdot\Delta\mathbf{U}\le\mathbf{b} \tag{11}$$

is called Quadratic Programming (QP).

In the following text transformation to the form of equation (11) is shown for typical and often used constraints types.

a) Limits on the manipulated variables

$$\mathbf{u}_{\min} \leq \mathbf{u}(i) \leq \mathbf{u}_{\max}, \quad i \in \{k, k+N_3-1\} \tag{12}$$

$$\begin{bmatrix} \mathbf{I}_{n_u \times n_u} & \cdots & \mathbf{0}_{n_u \times n_u} \\ \vdots & \ddots & \vdots \\ \mathbf{I}_{n_u \times n_u} & \cdots & \mathbf{I}_{n_u \times n_u} \end{bmatrix} \Delta\mathbf{U} \leq \begin{bmatrix} \mathbf{I}_{n_u \times n_u} \\ \vdots \\ \mathbf{I}_{n_u \times n_u} \end{bmatrix} \left(\mathbf{u}_{\max} - \mathbf{u}(k-1)\right) \tag{13}$$

$$\begin{bmatrix} -\mathbf{I}_{n_u \times n_u} & \cdots & \mathbf{0}_{n_u \times n_u} \\ \vdots & \ddots & \vdots \\ -\mathbf{I}_{n_u \times n_u} & \cdots & -\mathbf{I}_{n_u \times n_u} \end{bmatrix} \Delta\mathbf{U} \leq \begin{bmatrix} -\mathbf{I}_{n_u \times n_u} \\ \vdots \\ -\mathbf{I}_{n_u \times n_u} \end{bmatrix} \left(\mathbf{u}_{\min} - \mathbf{u}(k-1)\right) \tag{14}$$

b) Limits on the state variables

$$\mathbf{x}_{\min} \leq \mathbf{x}(i) \leq \mathbf{x}_{\max}, \quad i \in \{k+1, k+N_2\} \tag{15}$$

$$\begin{bmatrix} \mathbf{N} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{MN} & \mathbf{N} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}^{N_2-1}\mathbf{N} & \mathbf{M}^{N_2-2}\mathbf{N} & \cdots & \mathbf{N} \end{bmatrix} \Delta\mathbf{U} \leq \begin{bmatrix} \mathbf{I}_{n_x \times n_x} \\ \mathbf{I}_{n_x \times n_x} \\ \vdots \\ \mathbf{I}_{n_x \times n_x} \end{bmatrix} \mathbf{x}_{\max} - \begin{bmatrix} \mathbf{M} \\ \mathbf{M}^2 \\ \vdots \\ \mathbf{M}^{N_2} \end{bmatrix} \mathbf{x}(k) \tag{16}$$

$$-\begin{bmatrix} \mathbf{N} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{MN} & \mathbf{N} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}^{N_2-1}\mathbf{N} & \mathbf{M}^{N_2-2}\mathbf{N} & \cdots & \mathbf{N} \end{bmatrix} \Delta\mathbf{U} \leq \begin{bmatrix} -\mathbf{I}_{n_x \times n_x} \\ -\mathbf{I}_{n_x \times n_x} \\ \vdots \\ -\mathbf{I}_{n_x \times n_x} \end{bmatrix} \mathbf{x}_{\min} + \begin{bmatrix} \mathbf{M} \\ \mathbf{M}^2 \\ \vdots \\ \mathbf{M}^{N_2} \end{bmatrix} \mathbf{x}(k) \tag{17}$$

b) Limits on the controlled variables

$$\mathbf{y}_{\min} \leq \mathbf{y}(i) \leq \mathbf{y}_{\max}, \quad i \in \{k+N_1, k+N_2\} \tag{18}$$

$$\mathbf{H}\Delta\mathbf{U} \leq \begin{bmatrix} \mathbf{I}_{n_y \times n_y} \\ \vdots \\ \mathbf{I}_{n_y \times n_y} \end{bmatrix} \mathbf{y}_{\max} - \mathbf{Fx}(x) \tag{19}$$

$$-\mathbf{H} \cdot \Delta\mathbf{U} \leq \begin{bmatrix} -\mathbf{I}_{n_y \times n_y} \\ \vdots \\ -\mathbf{I}_{n_y \times n_y} \end{bmatrix} \mathbf{y}_{\min} + \mathbf{Fx}(x) \tag{20}$$

Practically it means that we have to calculate Hess matrix **S** (as function of process model and penalizations only) and gradient **s** (except process model and penalization also function of future set-point and actual state – predicted control error), fill linear inequalities matrix **A** and vector **b** and use QP to find optimal future control actions. Again only first element is applied as control action to the process and whole procedure is repeated in next sample time – only gradient **s** is updated and QP is solved.

LABORATORY PROCESS

Laboratory Hydraulic-Pneumatic System (HPS) was designed and realized at Department of Process Control University of Pardubice (Klán et al. 2005; Honc and Dušek 2012) - see Fig. 1. It includes a combination of hydraulic and pneumatic components. The pneumatic circuits H and L create cross coupling between both classical double tank sections and form a multivariable system with non-typical feature – multivariable effect exists in transient state only. Water is pumped by two pumps into upper tanks LH and RH, flows into lower tanks LL and RL and from here back into the reservoir. Water flow rates are controlled by input signal of the pumps u_L, u_R. The levels in lower tanks are measured indirectly by difference pressure sensors with output signals y_L, y_R.

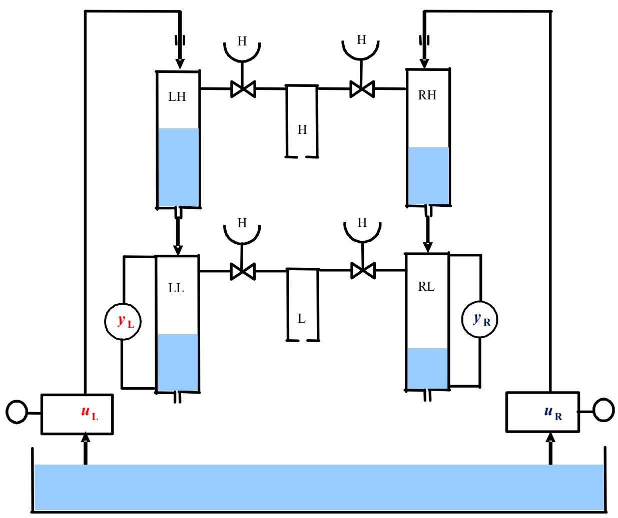

Figure 1: Scheme of HPS

CONTROL OBJECTIVE

Our task is to control water levels in lower tanks – to control multivariable system with two inputs and two outputs. Problem from the control point of view is easy overflow or underflow of higher tanks. Higher water levels are not measured so no direct safety actions can be applied. For conventional controllers only ad-hoc solution for specific set-point shape, change size or working point is possible by tuning the controller, setting slew rate limits on the control signal or by set-point modification. If we have state-space model of the process and use state observer we have full information about the state variables (water levels in all tanks and pressure in pneumatic loop) and we can solve problem synergistically as "state variables constrained model predictive control". We will calculate optimal control actions without violation higher water tanks maximum or minimum levels and also with respect to the input and output process variables limits.

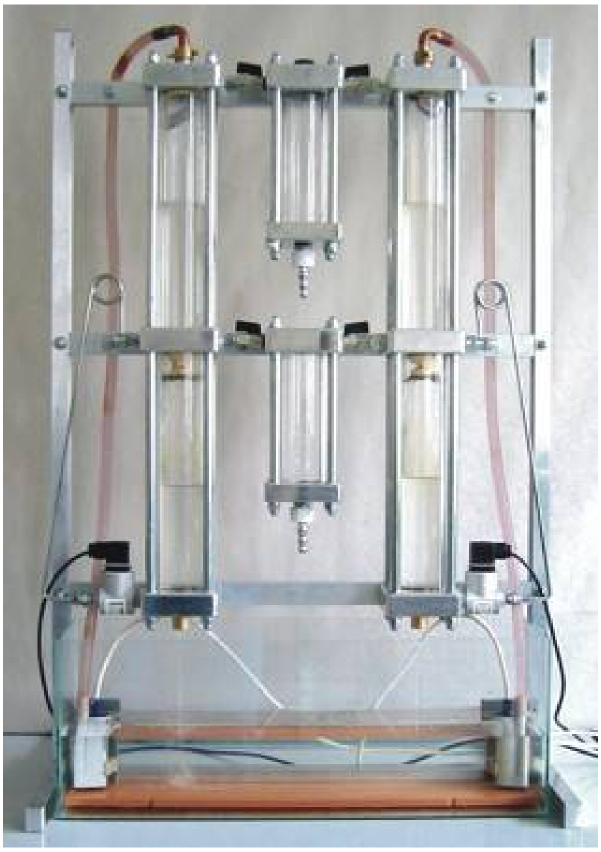

Figure 2: Hydraulic-Pneumatic System

CONTROL EXPERIMENTS

Control experiments are simulated with nonlinear process model first and real experiment is carried out afterwards. Nonlinear model together with its linearized form was presented in (Honc and Dušek 2012). Predictive controller is using state-space 5th order linear model with two inputs and two outputs. Parameters of predictive controller are listed in Table 1. Set-points are changed in time 10 and 20 minutes as step changes. Sample time of the controller is 10 s. Response of the simulated control experiment are plotted in Fig. 3 and Fig. 4. Real control experiment is in Fig. 5 and Fig. 6. Meaning of the used variables is in Table 2.

Table 1: MPC Parameters

Parameter	Value
N_1	1
N_2	18
N_3	18
r_1, r_2	1
q_1, q_2	5

Table 2: Variables Notation

Variable	Meaning
u_L, u_R	pump control voltages
y_L, y_R	pressure sensors signals
h_{LL}	water level in left lower tank
h_{RL}	water level in right lower tank
h_{LH}	water level in left higher tank
h_{RL}	water level in right higher tank

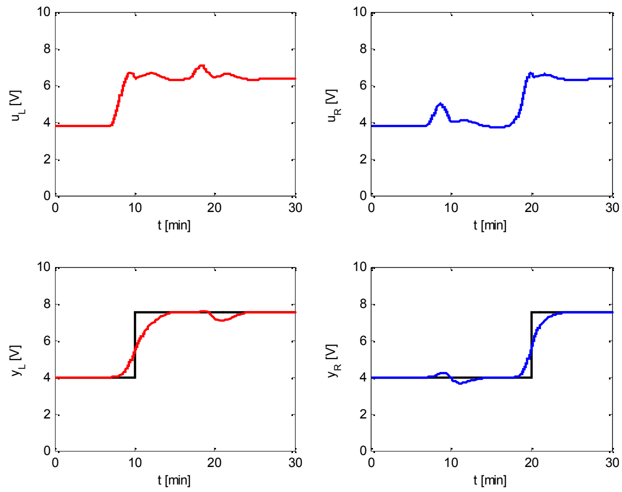

Figure 3: Simulated Control Experiment

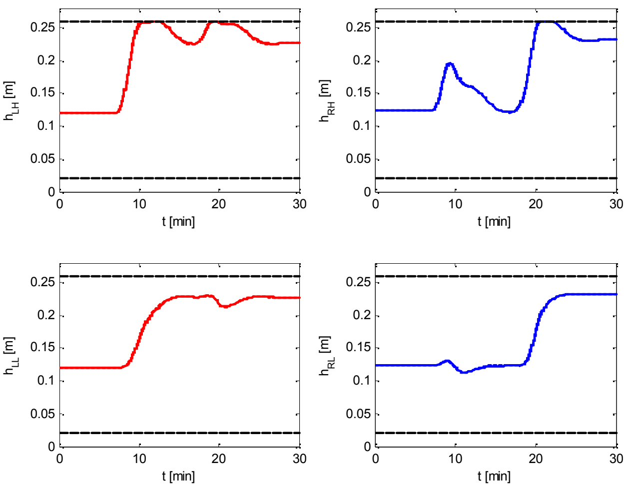

Figure 4: Water Levels by Simulated Experiment

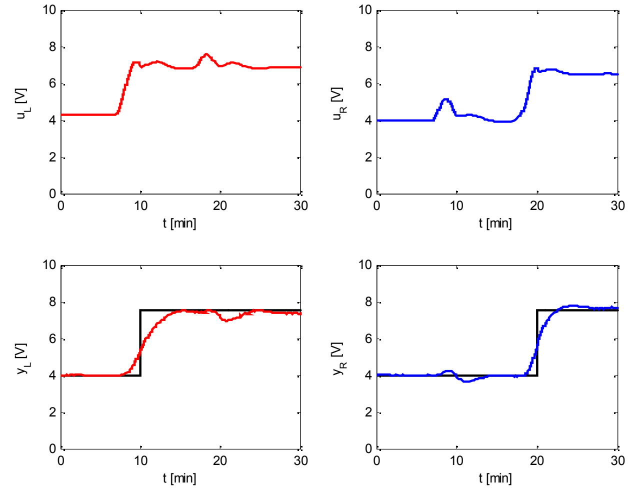

Figure 5: Real Control Experiment

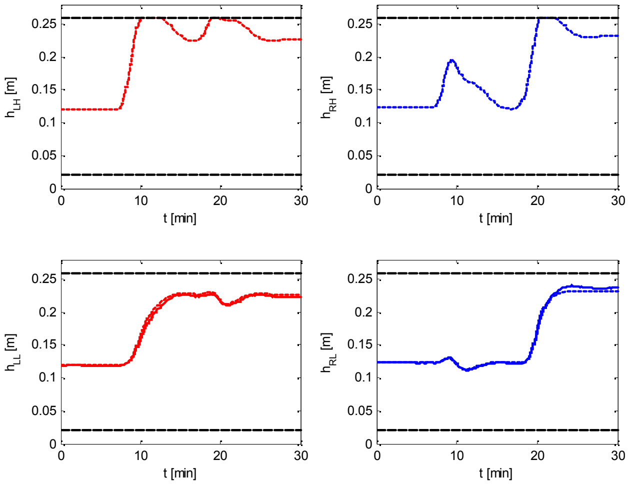

Figure 6: Water Levels by Real Experiment

CONCLUSIONS

State-space Constrained Model Predictive Control was presented and applied in the paper. The key feature for our laboratory application can be seen in Fig. 4 and Fig. 6 – water levels in higher tanks LH and RH do not exceed limits marked by horizontal dashed line – controller respects state variables constraints. There are two trends – solid line for simulated or measured variable and dotted line for its estimation. By real experiment we do not have information about water level in higher tanks – we can only estimate those variables. The other interesting feature is that the control actions starts even before the set-point step change. This is because the predictive controller knows future set-points on whole control horizon and can react in "predictive" way. This information together with multidimensionality of the controller significantly reduces cross effects – changes in other manipulated variables act as disturbances. Similarity of responses in Fig. 3 and Fig. 5 can be seen as a proof of model quality. Model Predictive Control with state-space process model is very elegant and straightforward example of modern control technique. Its application is enabled with increase of computational power and development of automation means. Existence of state-space model of the controlled system is the prerequisite. Derivation and programming of the prediction equations is easy compared to solving of Diophantine equations in case of multivariable input-output process model. Also criterion extension of terminal state or other requirements is possible as well as dealing with constraints on state variables. State estimation allows us to take into account also unmeasured or immeasurable information and it is from some point of view optimal way of filtering. Some disadvantage of using the state for the controller arises if measurement or estimation is not unbiased or model mismatches. Then the controller leaves steady-state control error – the controlled variable estimation follows the set-point but the real output has an offset.

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