ON CONVERGENCE OF THE DISTRIBUTIONS OF RANDOM SUMS AND STATISTICS CONSTRUCTED FROM SAMPLES WITH RANDOM SIZES TO EXPONENTIAL POWER LAWS

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ABSTRACT

Limit theorems are proved establishing criteria of convergence of the distributions of random sums and statistics constructed from samples with random sizes to exponential power laws.

INTRODUCTION. EXPONENTIAL POWER DISTRIBUTIONS

Let $0 < \alpha \leqslant 2$. *Exponential power distribution* is the absolutely continuous distribution defined by its Lebesgue probability density

$$\ell_\alpha(x) = \frac{\alpha}{2\Gamma(\frac{1}{\alpha})} \cdot e^{-|x|^\alpha}, \quad -\infty < x < \infty. \tag{1}$$

To simplify the notation and calculation, here and in what follows we will use a single parameter α in representation (1) since this parameter is in some sense characteristic and determines the shape of distribution (1). With $\alpha = 1$ relation (1) defines the classical Laplace distribution with zero mean and variance 2. With $\alpha = 2$ relation (1) defines the normal (Gaussian) distribution with zero mean and variance $\frac{1}{2}$.

The class of distributions (1) was introduced and studied in (Subbotin 1923). Along with the term *generalized Laplace distribution* going back to the original paper (Subbotin 1923) at least four other different terms are used for distribution (1). For example, in (Box and Tiao 1973) this distribution is called *exponential power distribution*, in (Evans et al. 2000) and (Leemis and McQueston 2008) it is called *generalized error distribution*, in (Morgan 1996) the term *generalized exponential distribution* is used whereas in (Nadaraja 2005) and (Varanasi and Aazhang 1989) this distribution is called *generalized normal* and, *generalized Gaussian* respectively. Distributions of type (1) are widely used in Bayesian analysis and various applications from astronomy to signal and image processing.

In applied probability there is a convention, apparently historically going back to the book (Gnedenko and Kolmogorov 1954), according to which a model distribution can be regarded as reasonable and/or justified enough only if it is an *asymptotic approximation*, that is, there exist a more or less simple setting and the corresponding limit theorem in which the model under consideration is a limit distribution. An interrelation of this convention with the principle of the non-decrease of uncertainty in closed systems was traced in the book (Gnedenko and Korolev 1996). A well-known reasonable numerical characteristic of uncertainty is the entropy. As we have already seen, with $0 < \alpha \leqslant 2$ the exponential power distribution is a scale mixture of normal laws. At the same time, the normal distribution has the maximum (differential) entropy among all laws with the finite second moment whose support is the whole real axis. According to the principle of the non-decrease of entropy which often manifests itself in probability theory in the form of limit theorems for sums of independent random variables (see (Gnedenko and Korolev 1996)), if the modeled system were information-isolated from the outer medium, then the observed statistical distributions of its characteristics would have been very close to the normal law. But since any mathematical model by its definition cannot make account of all the factors which influence the current state or the evolution of the modeled system, then the param-

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eters of this normal law vary depending on the evolution of the medium exogenous with respect to the system under consideration. In other words, these parameters should be regarded as random depending on the information flows between the system and exogenous medium. Thus, in many situations reasonable mathematical models of statistical regularities of the behavior of the observed characteristics of complex systems should have the form of mixtures of normal laws, the particular case of which is the exponential power distribution (1).

Probably, by now the simplicity of representation (1) has been the main (at least, important) reason for using the exponential power distributions in many applied problems as a heavy-tailed (for $0 < \alpha < 2$) alternative to the normal law. The "asymptotic" reasons of possible adequacy of this model have not been provided yet. In this paper we will demonstrate that the exponential power distribution can be limiting in rather simple limit theorems for regular statistics constructed from samples with random sizes, in particular, in the scheme of random summation. Hence, along with the normal law, this distribution can be regarded as an asymptotic approximation for the distributions of some processes, say, similar to (non-homogeneous) random walks.

The main part of the paper is organized as follows. Normal mixture representation is studied in Section 2. We obtain a criterion of convergence of the distributions of random sums to exponential power distributions in Section 3. In Section 4 we consider a criterion of convergence of the distributions of regular statistics constructed from samples with random sizes to exponential power distributions. In Section 5 we discuss the obtained results. Finally, in Section 6 we obtain estimates on the rate of convergence of the distributions of random sums to exponential power laws.

Normal mixture representation

In (West 1987) it was noticed that for $0 < \alpha \leqslant 2$ the distributions of type (1) are representable as scale mixtures of normal laws (also see (Choy and Smith 1997)). For the sake of convenience of further references here we retell the proof of this result from (West 1987) in other terms.

By $G_{\alpha,\theta}(x)$ and $g_{\alpha,\theta}(x)$ we will respectively denote the distribution function and probability density of the strictly stable law with characteristic exponent α and parameter θ defined by the characteristic function

$$\mathfrak{g}_{\alpha,\theta}(t) = \exp\Big\{-|t|^{\alpha}\exp\Big\{-\frac{i\pi\theta\alpha}{2}\mathrm{sign}t\Big\}\Big\},\quad t\in\mathbb{R}, \qquad (2)$$

with $0 < \alpha \leqslant 2$, $|\theta| \leqslant \theta_{\alpha} = \min\{1, \frac{2}{\alpha} - 1\}$ (see, e. g., (Zolotarev 1986)). Let

$$h_{\alpha/2}(z) = \frac{\alpha}{\Gamma(\frac{1}{\alpha})}\sqrt{\frac{\pi}{2}}\cdot\frac{g_{\alpha/2,1}(z)}{\sqrt{z}},\quad z\geqslant 0,$$

$$w_{\alpha/2}(z) = \frac{h_{\alpha/2}(z^{-1})}{z^2} = \frac{\alpha}{\Gamma(\frac{1}{\alpha})}\sqrt{\frac{\pi}{2}}\cdot\frac{g_{\alpha/2,1}(z^{-1})}{z^{3/2}},\quad z\geqslant 0.$$

Below, in the proof of lemma 1, we will show that both $h_{\alpha/2}(z)$ and $w_{\alpha/2}(z)$ are probability densities. Assume that all the random variables mentioned in this paper are defined on the same probability space $(\Omega, \mathfrak{A}, \mathsf{P})$ which is rich enough. The symbol $\stackrel{d}{=}$ denotes the coincidence of distributions. If $V_{\alpha/2}$ and $U_{\alpha/2}$ are non-negative absolutely continuous random variables with densities $h_{\alpha/2}(z)$ and $w_{\alpha/2}(z)$, respectively, then, as is easily seen,

$$U_{\alpha/2} \stackrel{d}{=} V_{\alpha/2}^{-1}. \qquad (3)$$

It is well known that if $\zeta_{\alpha,\theta}$ is a random variable with the stable distribution corresponding to characteristic function (2), then $\mathsf{E}|\zeta_{\alpha,\theta}|^p < \infty$ for every $p < \alpha$. Therefore, from the definition of the density $h_{\alpha/2}(z)$ it follows that $\mathsf{E}V_{\alpha/2}^p < \infty$ for any $p < (\alpha+1)/2$ and hence, (3) implies that $\mathsf{E}U_{\alpha/2}^q < \infty$ for any $q > 0$.

The distribution functions corresponding to the densities $\ell_{\alpha}(x)$, $h_{\alpha/2}(z)$ and $w_{\alpha/2}(z)$ will be denoted by the capital letters $L_{\alpha}(x)$, $H_{\alpha/2}(z)$ and $W_{\alpha/2}(z)$, respectively. The standard normal distribution function ($\alpha = 2$) and its density will be respectively denoted $\Phi(x)$ and $\varphi(x)$,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2},\quad \Phi(x) = \int_{-\infty}^{x}\varphi(z)dz.$$

LEMMA 1. *If $0 < \alpha \leqslant 2$, then the exponential power distribution (1) is a scale mixture of normal laws*:

$$L_{\alpha}(x) = \int_0^{\infty}\Phi\big(x\sqrt{z}\big)dH_{\alpha/2}(z),\quad x\in\mathbb{R}, \qquad (4)$$

$$L_{\alpha}(x) = \int_0^{\infty}\Phi\Big(\frac{x}{\sqrt{z}}\Big)dW_{\alpha/2}(z),\quad x\in\mathbb{R}. \qquad (5)$$

PROOF. From (2) it follows that the characteristic function of the symmetric ($\theta = 0$) strictly stable distribution has the form

$$\mathfrak{g}_{\alpha,0}(t) = e^{-|t|^{\alpha}},\quad t\in\mathbb{R}. \qquad (6)$$

On the other hand, it is well known that the symmetric strictly stable distribution with parameter α is a scale mixture of normal laws in which the one-sided ($\theta = 1$) stable law with parameter $\alpha/2$ is the mixing distribution:

$$G_{\alpha,0}(x) = \int_0^{\infty}\Phi\Big(\frac{x}{\sqrt{z}}\Big)dG_{\alpha/2,1}(z),\quad x\in\mathbb{R} \qquad (7)$$

(see, e. g., (Zolotarev 1986), theorem 3.3.1). Write relation (7) in terms of characteristic functions with the account of (6):

$$e^{-|t|^{\alpha}} = \int_0^{\infty}\exp\Big\{-\frac{t^2z}{2}\Big\}g_{\alpha/2,1}(z)dz. \qquad (8)$$

Then, re-denoting the argument $t \longmapsto x$ and making some formal transformations of equality (8), we obtain

$$\ell_\alpha(x) = \frac{\alpha}{2\Gamma(\frac{1}{\alpha})}e^{-|x|^\alpha} = \frac{\alpha}{\Gamma(\frac{1}{\alpha})}\sqrt{\frac{\pi}{2}}\int_0^\infty \frac{\sqrt{z}}{\sqrt{2\pi}}\exp\Big\{-\frac{x^2 z}{2}\Big\} =$$

$$\frac{g_{\alpha/2,1}(z)}{\sqrt{z}}dz = \int_0^\infty \sqrt{z}\varphi(x\sqrt{z})h_{\alpha/2}(z)dz. \qquad (9)$$

It can be easily verified that $h_{\alpha/2}(z)$ is the probability density of a nonnegative random variable. Indeed, for any $z > 0$ we have

$$\int_{-\infty}^{\infty} \sqrt{z}\varphi(x\sqrt{z})dx = 1.$$

Therefore it follows from (9) that

$$1 = \int_{-\infty}^{\infty} \ell_\alpha(x)dx = \int_{-\infty}^{\infty}\int_0^\infty \sqrt{z}\varphi(x\sqrt{z})h_{\alpha/2}(z)\,dz\,dx =$$

$$= \int_0^\infty h_{\alpha/2}(z)\Big(\int_{-\infty}^{\infty}\sqrt{z}\varphi(x\sqrt{z})\,dx\Big)dz = \int_0^\infty h_{\alpha/2}(z)dz.$$

Relation (6) can be written in a somewhat different form in terms of the density $w_{\alpha/2}$ with the account of (3):

$$\ell_\alpha(x) = \frac{\alpha}{\Gamma(\frac{1}{\alpha})}\sqrt{\frac{\pi}{2}}\int_0^\infty \frac{1}{\sqrt{2\pi z}}\exp\Big\{-\frac{x^2}{2z}\Big\}\frac{g_{\alpha/2,1}(z^{-1})}{z^{3/2}}dz =$$

$$\int_0^\infty \frac{1}{\sqrt{z}}\varphi\Big(\frac{x}{\sqrt{z}}\Big)w_{\alpha/2}(z)dz. \qquad (10)$$

Thus, representations (4) and (5) follow from (9) and (10), correspondingly. The lemma is proved.

If Z_α is a random variable having the exponential power distribution with parameter α, then relation (6) with the account of (3) mean that $Z_\alpha \stackrel{d}{=} X\cdot\sqrt{U_{\alpha/2}}$, where X and $U_{\alpha/2}$ are independent random variables such that X has the standard normal distribution.

Since the function $h_{\alpha/2}(x)$ is a probability density, then its definition implies the following interesting statement providing the possibility to calculate $\mathsf{E}U_{\alpha/2,1}^{-1/2}$ explicitly although, in general, the density $g_{\alpha/2,1}(z)$ cannot be written out in an explicit form in terms of elementary or simple special functions. However, this statement has only a tangent relation to the main topic of this paper.

COROLLARY 1. *Let $Y_{\alpha,1}$ be a random variable having the one-sided stable distribution with characteristic exponent $\alpha\in(0,1)$. Then $\mathsf{E}Y_{\alpha,1}^{-1/2} = \frac{1}{\alpha}\Gamma(\frac{1}{2\alpha})/\sqrt{2\pi}$.*

EXAMPLE 1. Consider the case $\alpha = 1$. Then, as is known, $G_{1/2,1}(x)$ is the Lévy distrbution (a particular case of the inverse Gaussian distribution, the distribution of the time until the standard Wiener process hits the unit level). The corresponding density is

$$g_{1/2,1}(z) = \frac{1}{z^{3/2}\sqrt{2\pi}}\exp\Big\{-\frac{1}{2z}\Big\}, \quad z > 0.$$

In this case

$$w_{1/2}(z) = \sqrt{\frac{\pi}{2}}\cdot\frac{g_{1/2,1}(z^{-1})}{z^{3/2}} = \frac{\sqrt{\pi}z^{3/2}e^{-z/2}}{\sqrt{2}\sqrt{2\pi}z^{3/2}} = \tfrac{1}{2}e^{-z/2},$$

that is, $w_{1/2}(z) = \frac{1}{2}e^{-z/2}$ is the density of the exponential distribution with parameter $\frac{1}{2}$. As this is so, according to (7) we have

$$\ell_1(x) = \tfrac{1}{2}e^{-|x|} = \int_0^\infty \frac{1}{\sqrt{z}}\varphi\Big(\frac{x}{\sqrt{z}}\Big)\frac{e^{-z/2}}{2}dz,$$

which is a well-known property of the Laplace distribution, see, e. g., (Korolev et al. 2011a), lemma 12.7.1.

A criterion of convergence of the distributions of random sums to exponential power distributions

Everywhere in what follows the symbol $\Longrightarrow$ denotes convergence in distribution.

Consider a sequence of independent identically distributed random variables $X_1, X_2, \ldots$, defined on a probability space $(\Omega,\,\mathfrak{A},\,\mathsf{P})$. Assume that

$$\mathsf{E}X_1 = 0, \quad 0 < \sigma^2 = \mathsf{D}X_1 < \infty. \qquad (11)$$

For a natural $n \geqslant 1$ let $S_n = X_1 + \ldots + X_n$. Let $N_1, N_2, \ldots$ be a sequence of nonnegative integer random variables defined on the same probability space so that for each $n \geqslant 1$ the random variable N_n is independent of the sequence $X_1, X_2, \ldots$ For definiteness, hereinafter we will assume that $\sum_{j=1}^{0} = 0$. In what follows convergence will meant as $n\to\infty$ unless otherwise specified.

A random sequence $N_1, N_2, \ldots$ is said to be infinitely increasing ($N_n \longrightarrow \infty$) in probability, if $\mathsf{P}(N_n \leqslant m) \longrightarrow 0$ for any $m\in(0,\infty)$.

The following important statement was firstly proved in (Korolev 1994).

LEMMA 2. *Assume that the random variables $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above and $N_n\longrightarrow\infty$ in probability. A distribution function $F(x)$ such that $\mathsf{P}\left(S_{N_n} < x\sigma\sqrt{n}\right) \Longrightarrow F(x)$ exists if and only if there exists a distribution function $Q(x)$ satisfying the conditions $Q(0) = 0$,*

$$F(x) = \int_0^\infty \Phi\left(\frac{x}{\sqrt{y}}\right)dQ(y),\ x\in\mathbb{R}, \quad \mathsf{P}(N_n < nx) \Longrightarrow Q(x).$$

THEOREM 1. *Assume that the random variables $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above and $N_n \longrightarrow \infty$ in probability. Then $\mathsf{P}\left(S_{N_n} < x\sigma\sqrt{n}\right) \Longrightarrow L_\alpha(x)$, if and only if $\mathsf{P}(N_n < nx) \Longrightarrow W_{\alpha/2}(x)$.*

PROOF. This statement is a direct consequence of lemma 2 with $Q(x) = W_{\alpha/2}(x)$ and representation (5).

A criterion of convergence of the distributions of regular statistics constructed from samples with random sizes to exponential power distributions

For $n \geqslant 1$ let $T_n = T_n(X_1, \ldots, X_n)$ be a statistic, that is, a measurable function of the random variables $X_1, \ldots, X_n$. For each $n \geqslant 1$ define the random variable T_{N_n} by letting $T_{N_n}(\omega) = T_{N_n(\omega)}\big(X_1(\omega), \ldots, X_{N_n(\omega)}(\omega)\big)$ for every elementary outcome $\omega \in \Omega$.

We will say that the statistic T_n is asymptotically normal, if there exist $\delta > 0$ and $\theta \in \mathbb{R}$ such that

$$\mathsf{P}\left(\delta\sqrt{n}\big(T_n - \theta\big) < x\right) \Longrightarrow \Phi(x). \qquad (12)$$

LEMMA 3. *Assume that $N_n \longrightarrow \infty$ in probability as $n \to \infty$. Let the statistic T_n be asymptotically normal in the sense of* (12). *Then a distribution function $F(x)$ such that $\mathsf{P}\left(\delta\sqrt{n}\big(T_{N_n} - \theta\big) < x\right) \Longrightarrow F(x)$ exists if and only if there exists a distribution function $Q(x)$ satisfying the conditions $Q(0) = 0$, $F(x) = \int_0^\infty \Phi\big(x\sqrt{y}\big)dQ(y)$, $x \in \mathbb{R}$, $\mathsf{P}(N_n < nx) \Longrightarrow Q(x)$.*

PROOF. Actually, this lemma is a particular case of theorem 3 in (Korolev 1995), the proof of which is, in turn, based on general theorems on convergence of superpositions of independent random sequences (Korolev 1994, Korolev 1996). Also see (Gnedenko and Korolev 1996), theorem 3.3.2.

THEOREM 2. *Assume that $N_n \longrightarrow \infty$ in probability as $n \to \infty$. Let the statistic T_n be asymptotically normal in the sense of* (12). *Then $\mathsf{P}\left(\delta\sqrt{n}\big(T_{N_n} - \theta\big) < x\right) \Longrightarrow L_\alpha(x)$ if and only if $\mathsf{P}(N_n < nx) \Longrightarrow H_{\alpha/2}(x)$.*

PROOF. This statement is a direct consequence of lemma 3 with $Q(x) = H_{\alpha/2}(x)$ and representation (4).

Discussion

The convergence of the distributions of the normalized indices N_n to the distributions $W_{\alpha/2}$ and $H_{\alpha/2}$ is the main condition in theorems 1 and 2, respectively. Now we will give a rather general example of the situation where these conditions can hold. For this purpose we introduce a useful construction of nonnegative integer-valued random variables which, under an appropriate normalization, converge to a given nonnegative (not necessarily discrete) random variable, whatever the latter is.

In the book (Gnedenko and Korolev 1996) it was proposed to model the evolution of non-homogeneous chaotic stochastic processes, in particular, the dynamics of financial assets by compound doubly stochastic Poisson processes (compound Cox processes). This approach got further grounds and development in the books (Bening and Korolev 2002, Korolev and Sokolov 2008, Korolev et al. 2011a, Korolev 2011). In (Korolev and Skvortsova 2006, Korolev 2011) this approach was successfully applied to modeling the processes of plasma turbulence. Similar methods were considered in (Granovsky and Zeifman 2000, Zeifman 1991). According to this approach the flow of informative events, each of which generates the next observation, is described by the stochastic point process $M(\Lambda(t))$ where $M(t)$, $t \geq 0$, is a homogeneous Poisson process with unit intensity and $\Lambda(t)$, $t \geq 0$, is a random process independent of $M(t)$ possessing the properties: $\Lambda(0) = 0$, $\mathsf{P}(\Lambda(t) < \infty) = 1$ for any $t > 0$, the trajectories $\Lambda(t)$ are non-decreasing and right-continuous. The process $M(\Lambda(t))$, $t \geq 0$, is called the doubly stochastic Poisson process (Cox process).

Within this model for each t the distribution of the random variable $M(\Lambda(t))$ is mixed Poisson. Consider the case where in this model the time t remains fixed (say, $t = 1$) and $\Lambda(t) = nU_{\alpha/2}$, where n is an auxiliary natural-valued parameter, $U_{\alpha/2}$ is a random variable with the distribution function $W_{\alpha/2}(x)$ independent of the standard Poisson process $M(t)$, $t \geqslant 0$. Here the asymptotic $n \to \infty$ can be interpreted as that the (stochastic) intensity of the flow of informative events is assumed very large. For each natural n let

$$N_n = M(nU_{\alpha/2}) \qquad (13).$$

It is obvious that the random variable N_n so defined has the mixed Poisson distribution

$$\mathsf{P}(N_n = k) = \mathsf{P}\big(M(nU_{\alpha/2}) = k\big) = \int_0^\infty e^{-nz}\frac{(nz)^k}{k!}w_{\alpha/2}(z)dz \quad k = 0, 1, \ldots$$

This random variable N_n can be also interpreted as the number of events registered up to time n in the Poisson process with the stochastic intensity having the density $w_{\alpha/2}(z)$. Assume that the random variable $U_{\alpha/2}$ and the Poisson process $M(t)$ are independent of the sequence $X_1, X_2, \ldots$. Then, obviously, for each n the random variable N_n is also independent of this sequence.

Denote $A_n(z) = \mathsf{P}(N_n < nz)$, $z \geqslant 0$ ($A_n(z) = 0$ for $z < 0$). It is easy to see that $A_n(z) \Longrightarrow W_{\alpha/2}(z)$. Indeed, as is known, if $\Pi(x;\ell)$ is the Poisson distribution function with the parameter $\ell > 0$ and $E(x;c)$ is the distribution function with a single unit jump at the point $c \in \mathbb{R}$, then $\Pi(\ell x;\ell) \Longrightarrow E(x;1)$ as $\ell \to \infty$. Since for $x \in \mathbb{R}$ $A_n(x) = \int_0^\infty \Pi(nx;nz)dW_{\alpha/2}(z)$, then by the Lebesgue dominated convergence theorem, as $n \to \infty$, we have

$$A_n(x) \Longrightarrow \int_0^\infty E(x/z;1)dW_{\alpha/2}(z) = \int_0^x dW_{\alpha/2}(z) = W_{\alpha/2}(x),$$

that is, the random variables N_n defined above satisfy the condition of lemma 2 with $Q(x) = W_{\alpha/2}(x)$. Moreover, $N_n \longrightarrow \infty$ in probability since $\mathsf{P}(U_{\alpha/2} = 0) = 0$.

Similarly, let $V_{\alpha/2}$ be a random variable with the distribution function $H_{\alpha/2}(x)$ independent of the standard

Poisson process $M(t)$, $t \geqslant 0$. For each natural n let $N_n = M\big(nV_{\alpha/2}\big)$. The distribution of the random variable N_n is mixed Poisson,

$$\mathsf{P}(N_n = k) = \frac{1}{k!}\int_0^\infty e^{-nz}(nz)^k h_{\alpha/2}(z)dz,\ k = 0,1,2,\ldots$$

This random variable N_n can be also interpreted as the number of events registered up to time n in the Poisson process with the stochastic intensity having the density $h_{\alpha/2}(z)$. Assume that the random variable $V_{\alpha/2}$ and the Poisson process $M(t)$ are independent of the sequence $X_1, X_2, \ldots$. Then, obviously, for each n the random variable N_n is also independent of this sequence.

As above, it is easy to make sure that $\mathsf{P}(N_n < nz) \Longrightarrow H_{\alpha/2}(z)$, that is, these random variables N_n satisfy the condition of lemma 3 with $Q(x) = H_{\alpha/2}(x)$. Moreover, $N_n \longrightarrow \infty$ in probability since $\mathsf{P}(V_{\alpha/2} = 0) = 0$.

REMARK 1. Using this simple construction of random indices N_n through the random change of time in the Poisson process one can easily obtain examples of random variables N_n participating in lemmas 2 and 3 whatever a distribution function $Q(x)$ with $Q(0) = 0$ is. Indeed, if U is a random variable with the distribution function $Q(x)$ and $M(t)$, $t \geqslant 0$, is the standard Poisson process independent of U, then the random variables $N_n = M(nU)$ satisfy the condition $\mathsf{P}(N_n < nx) \Longrightarrow Q(x)$.

Estimates on the rate of convergence of the distributions of random sums to exponential power laws

Here we will consider the rate of convergence in theorem 1. In addition to the conditions on the random variables $X_1, X_2, \ldots$ imposed in Sect. 2, assume that

$$\beta^3 = \mathsf{E}|X_1|^3 < \infty. \tag{14}$$

Let the random variable N_n be defined by (13). Denote $D_{n,\alpha} = \sup_x \big|\mathsf{P}\big(S_{N_n} < x\sigma\sqrt{n}\big) - L_\alpha(x)\big|$.

THEOREM 3. *Let conditions* (11) *and* (14) *hold and let the random variable N_n be defined by* (13). *For any $n \geqslant 1$ we have*

$$D_{n,\alpha} \leqslant 0.3812 \cdot \frac{\alpha}{\Gamma\big(\frac{1}{\alpha}\big)}\frac{\beta^3}{\sigma^3\sqrt{n}}.$$

PROOF. The distribution of the random variable N_n is mixed Poisson. Hence, by the Fubini theorem

$$\mathsf{P}\big(S_{N_n} < x\sigma\sqrt{n}\big) = \mathsf{P}\big(S_{M(nU_{\alpha/2,1})} < x\sigma\sqrt{n}\big) =$$

$$= \int_0^\infty \mathsf{P}\big(S_{M(nz)} < x\sigma\sqrt{n}\big) w_{\alpha/2}(z)dz. \tag{15}$$

Further, according to (5), the exponential power distribution with parameter α is a scale mixture of normal laws in which the mixing distribution is $W_{\alpha/2}$. From (15) and (5) it follows that

$$D_{n,\alpha} \leqslant \int_0^\infty \sup_x \bigg|\mathsf{P}\Big(\frac{S_{M(nz)}}{\sigma\sqrt{n}} < x\Big) - \Phi\Big(\frac{x}{\sqrt{z}}\Big)\bigg| dW_{\alpha/2}(z) =$$

$$= \int_0^\infty \sup_x \bigg|\mathsf{P}\Big(\frac{S_{M(nz)}}{\sigma\sqrt{nz}} < x\Big) - \Phi(x)\bigg| dW_{\alpha/2}(z). \tag{16}$$

For the estimation of the integrand in (16) we will use the following analog of the Berry–Esseen inequality for Poisson random sums in terms of non-central Lyapunov fractions.

LEMMA 4. *Let random variables $X_1, X_2, \ldots$ be identically distributed with $\mathsf{E}X_1 = 0$ and $\mathsf{E}|X_1|^3 < \infty$. Let M_λ be a Poisson random variable with parameter $\lambda > 0$ such that the random variables $M_\lambda, X_1, X_2, \ldots$ are jointly independent. Denote $Z_\lambda = X_1 + \ldots + X_{M_\lambda}$. Then*

$$\sup_x \big|\, \mathsf{P}\big(Z_\lambda < x\sqrt{\mathsf{D}Z_\lambda} < x\big) - \Phi(x)\big| \leqslant \frac{0.3041}{\sqrt{\lambda}} \cdot \frac{\mathsf{E}|X_1|^3}{(\mathsf{E}X_1^2)^{3/2}}.$$

The PROOF of this statement was given in (Korolev and Shevtsova 2012), also see (Korolev et al. 2011a), theorem 2.4.3.

We will also use the following statement which makes it possible to calculate $\mathsf{E}U_{\alpha/2}^{-1/2}$ although, in general, the density $w_{\alpha/2}(z)$ cannot be written out in an explicit form in terms of elementary or simple special functions.

LEMMA 5. *For any $\alpha \in (0,2)$ we have* $\mathsf{E}U_{\alpha/2}^{-1/2} = \frac{\alpha}{2}\sqrt{\pi}/\Gamma\big(\frac{1}{\alpha}\big)$.

PROOF. Obviously, $\mathsf{E}U_{\alpha/2}^{-1/2} = \mathsf{E}V_{\alpha/2}^{1/2} = \int_0^\infty \sqrt{z}h_{\alpha/2}(z)dz$. Further, from the definition of the density $h_{\alpha/2}$ it follows that

$$\int_0^\infty \sqrt{z}h_{\alpha/2}(z)dz = \frac{\alpha}{\Gamma\big(\frac{1}{\alpha}\big)}\sqrt{\frac{\pi}{2}}\int_0^\infty \frac{\sqrt{z}}{\sqrt{z}}g_{\alpha/2,1}(z)dz =$$

$$\frac{\alpha}{\Gamma\big(\frac{1}{\alpha}\big)}\sqrt{\frac{\pi}{2}}\int_0^\infty g_{\alpha/2,1}(z)dz = \frac{\alpha}{\Gamma\big(\frac{1}{\alpha}\big)}\sqrt{\frac{\pi}{2}},$$

since $g_{\alpha/2,1}(z)$ is a probability density. The lemma is proved.

Continuing (16) with the account of lemmas 4 and 5 we obtain

$$D_{n,\alpha} \leqslant 0.3041 \cdot \frac{\beta^3}{\sigma^3\sqrt{n}} \cdot \mathsf{E}U_{\alpha/2}^{-1/2} = 0.3041 \cdot \frac{\alpha}{\Gamma\big(\frac{1}{\alpha}\big)}\sqrt{\frac{\pi}{2}} \cdot \frac{\beta^3}{\sigma^3\sqrt{n}}.$$

The theorem is proved.

Asymmetric generalization of exponential power distributions by variance-mean mixing

All the distributions of type (1) are symmetric. There were some attempts to propose asymmetric (skew) generalization of distributions (1), see, e. g. (Fernandez et al. 1995), (Theodossiou 2000), (Komunjer 2007) where the so-called *skew exponential power distributions* were considered. In (Ayebo and Kozubowski 2004) basic properties of the skew exponential power distributions were considered. In (Zhu and Zinde-Walsh 2009) the so-called *asymmetric exponential power distributions* were proposed. However, all these generalizations are rather formal and do not assume the property of a "generalized" distribution to be a limit law in some simple asymptotic setting.

Instead, here we consider a more natural asymmetric extension of the class of distributions (1). Fot this purpose we will use an approach similar to the one used by O. Barndorff-Nielsen in 1977 to introduce the class of generalized hyperbolic distributions as special variance-mean mixtures of normal laws (Barndorff-Nielsen 1977). The base of the corresponding reasoning is representation (5).

Let $\alpha \in (0,2]$, $\mu \in \mathbb{R}$. The probability distribution whose distribution function has the form

$$L_{\alpha,\mu}(x) = \int_0^\infty \Phi\Big(\frac{x-\mu z}{\sqrt{z}}\Big)\,dW_{\alpha/2}(z), \quad x \in \mathbb{R}, \qquad (17)$$

will be called a *skew exponential power distribution* or *skew generalized Laplace distribution*) with shape parameter α and asymmetry parameter μ. Formally, in mixture (17) the mixing is carried out with respect to both parameters of the normal law. However, by virtue of the fact that in (17) these parameters are tightly linked and the expectations (means) of the mixed normal laws turn out to be proportional to their variances, actually, (17) is a one-parameter mixture. That is why O. Barndorff-Nielsen and his colleagues called such mixtures *variance-mean mixtures* (Barndorff-Nielsen et al 1982).

If X is a random variable with the standard normal law independent of the random variable $U_{\alpha/2}$ introduced above, then it is easy to see that the distribution function $L_{\alpha,\mu}(x)$ (see (17)) corresponds to the random variable $Z_{\alpha,\mu} = X\sqrt{U_{\alpha/2}} + \mu U_{\alpha/2}$.

The moments of $Z_{\alpha,\mu}$ were found in (Grigoryeva and Korolev 2013).

A criterion of convergence of the distributions of random sums to skew exponential power distributions

Let $\{X_{n,j}\}_{j\geqslant 1}$, $n = 1,2,\ldots$ be a double array of row-wise identically distributed random variables. Let $\{N_n\}_{n\geqslant 1}$ be a sequence of integer-valued nonnegative random variables such that for each $n \geqslant 1$ the random variables $N_n, X_{n,1}, X_{n,2},\ldots$ are independent. Let $S_{n,k} = X_{n,1} + \ldots + X_{n,k}$. As above, to avoid misunderstanding we assume $\sum_{j=1}^{0} = 0$.

As it was demonstrated in (Korolev 2013, Zaks and Korolev 2013), variance-mean mixtures of normal laws (3) turn out to be identifiable, since for each fixed $\mu \in \mathbb{R}$ the one-parameter family of distributions $\big\{\Phi\big((x-\mu z)/\sqrt{z}\big) : z \geqslant 0\big\}$ is additively closed. In (Korolev 2013) the following general statement was proved (also see (Zaks and Korolev 2013)).

LEMMA 6. *Assume that there exist a sequence $\{k_n\}_{n\geqslant 1}$ of natural numbers and a number $\mu \in \mathbb{R}$ such that*

$$\mathsf{P}\big(S_{n,k_n} < x\big) \Longrightarrow \Phi(x-\mu). \qquad (18)$$

Assume that $N_n \to \infty$ in probability. Then the distributions of random sums weakly converge to some distribution function $F(x)$: $\mathsf{P}\big(S_{n,N_n} < x\big) \Longrightarrow F(x)$, if and only if there exists a distribution function $Q(x)$ such that $Q(0) = 0$,

$$F(x) = \int_0^\infty \Phi\Big(\frac{x-\mu z}{\sqrt{z}}\Big)\,dQ(z), \qquad (19)$$

and

$$\mathsf{P}(N_n < xk_n) \Longrightarrow Q(x). \qquad (20)$$

The following theorem is actually a particular case of lemma 6.

THEOREM 4. *Assume that there exist a sequence $\{k_n\}_{n\geqslant 1}$ of natural numbers and a number $\mu \in \mathbb{R}$ such that convergence* (6) *takes place. Assume that $N_n \to \infty$ in probability. Then convergence*

$$\mathsf{P}\big(S_{n,N_n} < x\big) \Longrightarrow L_{\alpha,\mu}(x) \qquad (21)$$

takes place if and only if

$$\mathsf{P}(N_n < xk_n) \Longrightarrow W_{\alpha/2}(x). \qquad (22)$$

Some estimates of the rate of convergence in theorem 4 were presented in (Grigoryeva and Korolev 2013).

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