SIMULATION OF ROBUST ALGEBRAIC CONTROL OF A DELAYED HEAT EXCHANGER WITH CONTROLLER RATIONALIZATION

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ABSTRACT
Concise complete controller design with simulations for a laboratory circuit heating plant with feedback delays is the aim of this contribution. The particular steps of the design include a mathematical modelling summary of then plant, introduction of relevant algebraic tools, robust controller design and tuning, its discretization and rational simplification. The plant model contains internal delays, thus, it is infinite-dimensional. A special ring of stable and proper quasipolynomial functions represent the main algebraic mean for controller structure design. A possible range of adjustable controller parameters is then limited by using the robust stability and performance analysis. Their eventual values are consequently obtained by simulations of control responses. A discretization of the resulting controller is suggested and a method for its rational simplification is verified by simulations in MATLAB® and Simulink®. All the obtained results are promising for a practical implementation of the presented approach.

INTRODUCTION
Dynamical plants and processes, even linear ones, with internal (state) delays constitute a family of systems that are not trivial to be controlled by the use of many traditional control design approaches, or it is even impossible to implement them (Loiseau et al. 2009; Richard 2003; Vyhlidal et al. 2014). Some possible approaches, e.g. Lyapunov-Krasovskii methods, are too complex and troublesome to be practically implemented by engineers. Algebraic structures, in the contrary, proved to be effective tools for delayed system dynamics description and control design (Dostál et al. 2002; Loiseau 2000; Kučera 1991). Namely, the ring of proper and stable quasipolynomial meromorphic functions (R_{ms}) (Zítek and Kučera, 2003; Pekař 2012) as the main algebraic tool is used in this paper to determine the controller structure. The solution of the Bézout equation followed by the Youla-Kučera parameterization satisfy the controller properness and control feedback stability.

In this contribution, a circuit laboratory heat exchanger (Dostál et al. 2008) serves as a controlled plant since its mathematical model (Pekař et al. 2009) evinces internal feedback delays. The laboratory appliance can be considered as a small-scaled model of a real-word system, e.g. as the cooling system in cars. Recently, it has been utilized to perform the verification of solutions of various engineering and scientific tasks, see e.g. (Bobál et al. 2013; Prokop et al. 2012); however, usually plant models with only input-output delays have been used.

Once the controller structure by means of the R_{ms} ring is designed, its free and adjustable controller parameters ought to be set properly. To solve this task, possibilities are plentiful and many various ideas can be adopted. We decided to satisfy robust stability and robust performance under unstructured multiplicative uncertainty (Doyle et al. 1992) that is based on the well-known Nyquist criterion, the validity of which for delayed systems was proved e.g. in (Pekař et al. 2011). As the outcome of robust analysis, only possible controller parameters’ ranges are obtained; therefore, their values are precised via simulations in MATLAB® and Simulink® with some integral criteria evaluation.

Presented algebraic design usually yields controllers with delayed dynamics that is linear but far from the conventional proportional-integral-derivative (PID) control law. Since up to 95 % of control loops in industry are equipped by PID controllers (Desborough and Miller 2001), it is desirable to approximate the eventual controller structure by the PID one. The idea of the Padé expansion is adopted in this contribution where the whole controller transfer function is rationalized instead of the traditional approximation of separate delay elements.

A simple controller discretization for digital implementation based on delta models (Middleton and Goodwin 1990) in input-output space finalizes the controller design procedure. The method is easy to handle for practitioners - in contrast to some other more sophisticated methods operating in the state-space domain, see e.g. pseudospectral methods (Breda et al. 2005).
HEAT EXCHANGER MODEL

A photo and a rough scheme of the laboratory heat exchanger to be controlled is provided to the reader in Figures 1 and 2 (Dostálek et al. 2008). The model works as follows: Distilled water inside the piping is driven by a pump \( u_p(t) \) - continuously controllable via the voltage \( u_p(t) \) - through a flow heater \( \{1\} \) with maximum power \( P_h \) of 750 W. The heater output temperature, \( t_{HO} \), is measured by a platinum thermometer. Warmed liquid then goes through a 15 meters long insulated coiled pipeline \( \{2\} \) which causes the significant delay in the system. A heat-consuming appliance is represented by the air-water heat exchanger (cooler) \( \{3\} \) equipped with a continuously adjustable (by means of the voltage \( u_c(t) \)) and an on/off cooling fans \( \{4, 5\} \). Input and output temperatures of the cooler, \( t_{CI} \) and \( t_{CO} \) respectively, are measured by platinum thermometers as well. The expansion tank \( \{7\} \) compensates for the expansion effect of the heat fluid. Let the ambient temperature be \( t_A \).

The complete mathematical model was published in (Pekař et al. 2009). For the sake of this paper, however, it is sufficient to consider the relationship between \( \Delta u_p(t) \) and \( \Delta t_{CO} \) governed by the transfer function

\[
G(s) = \frac{\Delta t_{CO}(s)}{\Delta u_p(s)} = \frac{(b_{DD} \exp(-\tau_D s) + b_0) \exp(-\tau_h)}{s^2 + a_2 s^2 + a_1 s + a_0 + a_{DD} s \exp(-\tau_h)}
\]

where the prefix \( \Delta \) means the difference from an operating point. In particular, for the operating point

\[
\begin{bmatrix}
    u_p, u_c, P_h, \vartheta_{HO}, \vartheta_{CI}, \vartheta_{CO}, \vartheta_1 \\
    5 V, 3 V, 300 W, 44.1^\circ C, 43.8^\circ C, 36^\circ C, 24^\circ C
\end{bmatrix}
\]

the following particular values were determined

\[
\begin{align*}
    b_{DD} &= 2.334 \cdot 10^{-6}, \\
    b_0 &= -2.146 \cdot 10^{-7}, \\
    a_2 &= 0.1767, \\
    a_1 &= 0.009, \\
    a_0 &= 1.413 \cdot 10^{-4}, \\
    a_{DD} &= -7.624 \cdot 10^{-3}, \\
    \tau_D &= 1.5, \\
    \tau &= 131, \\
    \vartheta &= 143
\end{align*}
\]

ALGEBRAIC CONTROL DESIGN

A concise introduction of main algebraic tools and control design steps follows.

**Definition 1** (\( R_{MS} \) ring element (Pekař 2012)).

\[
T(s) = \frac{n(s)}{d(s)} \in R_{MS} \quad \text{if} \quad n(s), \quad d(s) \quad \text{are} \quad \text{quasipolynomials} \quad (\text{Loiseau 2000}) \quad \text{and} \quad n(s) = \tilde{n}(s) \exp(-\tau s), \quad \tau \geq 0. \text{ The term is formally stable and it must hold} \quad \sup_{Re[s] > 0} \left| T(s) \right| < \infty
\]

for some for \( R > 0 \) (i.e. it is proper, see (Partington and Bonnet 2004)).

Consider the simple negative feedback control loop as in Figure 3 where \( W(s) \) is the Laplace transform of the reference, \( D(s) \) stands for that of the load disturbance, \( E(s) \) is transformed control error, \( U_0(s) \) expresses the controller action, \( U(s) = \Delta P_{Pd}(s) \) represents the manipulated input affected by a load disturbance, and \( Y(s) = \Delta \vartheta_{CO}(s) \) is the plant output controlled signal. Note that all the presented signals are assumed to be ratios of elements from \( R_{MS} \), namely, \( W(s) = H_w(s) / P_w(s), \quad D(s) = H_d(s) / P_d(s) \) where \( H_w(s), \quad H_d(s), \quad F_p(s), \quad P_{Pd}(s) \in R_{MS} \). Moreover, \( G(s) = B(s) / A(s), \quad G_p(s) = Q(s) / P(s) \) are coprime fractions for the plant and the controller, respectively, with \( A(s), B(s), P(s), Q(s) \in R_{MS} \).
Theorem 1 (Stabilization (Kučera 1991; Zítek and Kučera, 2003)). Given a Bézout coprime pair \( A(s), B(s) \in R_{MS} \) the closed-loop system is stable (in \( R_{MS} \) sense) if and only if there exists a coprime pair \( P(s), Q(s) \in R_{MS} \) satisfying the Bézout identity

\[
A(s)P(s) + B(s)Q(s) = 1
\]  

(3)

Note that the Bézout coprimeness is satisfied if \( \inf_{\text{Re } s > \rho_0} \|A(s) + B(s)\| > 0 \) (Partington and Bonnet 2004).

Let us follow now Theorem 1 for the plant (1). Introduce a Bézout coprime factorization e.g. as

\[
G(s) = \frac{(b_{0D} \exp(-\tau_0 s) + b_0) \exp(-\tau s)}{s^3 + a_2s^2 + a_1s + a_0 + a_{10D} \exp(-\tau b)} = \frac{B(s)}{A(s)},
\]

\[m_0 > 0\]

A particular solution of (3) can be

\[
Q_0 = 1, P_0(s) = \frac{(s + m_0)^3 - (b_{0D} \exp(-\tau_0 s) + b_0) \exp(-\tau s)}{s^3 + a_2s^2 + a_1s + a_0 + a_{10D} \exp(-\tau b)}
\]

Theorem 2 (Youla-Kučera parameterization (Kučera 1991; Zítek and Kučera, 2003)). A particular solution \( P_0(s), Q_0(s) \in R_{MS} \) of (3) can be parameterized as

\[
P(s) = P_0(s) + B(s)Z(s) \quad \text{and} \quad Q(s) = Q_0(s) + A(s)Z(s)
\]

(4)

where \( Z(s) \in R_{MS} \) is arbitrary.

Theorem 3. If (3) holds, then

i) the reference signal \( w(t) = L^{-1}\{W(s)\} \) is asymptotically tracked if and only if \( F_p(s) \) divides the product \( A(s)P(s) \) in \( R_{MS} \).

ii) the load disturbance \( d(t) = L^{-1}\{D(s)\} \) is asymptotically rejected if and only if \( F_p(s) \) divides \( B(s)P(s) \) in \( R_{MS} \).

Note that the conditions in Theorem 3 are achieved by means of (4).

For instance, the setting

\[
Z(s) = \left( \frac{m_0^3}{b_{0D} + b_0} - 1 \right) \frac{(s + m_0)^3}{s^3 + a_2s^2 + a_1s + a_0 + a_{10D} \exp(-\tau b)}
\]

ensures the validity of Theorem 3 for the stepwise reference and load disturbance, and the controller transfer function is of a relatively simple form

\[
G_c(s) = \frac{m_0^3}{b_{0D} + b_0} \frac{s^3 + a_2s^2 + a_1s + a_0 + a_{10D} \exp(-\tau b)}{s^3 + a_2s^2 + a_1s + a_0 + a_{10D} \exp(-\tau b)}
\]

(5)

Such a controller can easily be implemented by amplifiers, integrators, delay and summing elements.

**ROBUST ANALYSIS**

There is a double motivation for robust analysis of the heat exchanger here. As first, some physical quantities of the laboratory model can vary and, moreover, measurement and identification uncertainties can naturally appear when modelling. As second, we intend to determine a possible range for the free parameter \( m_0 \) such that the control system is robust against these perturbations. The analysis is based on the following two known theorems for robust stability and robust performance, respectively (Doyle et al. 1992), for the control system as in Figure 3.

Theorem 4 (Robust stability). The feedback control system is robustly stable if and only if

\[
\|W_M(j\omega)G_c(j\omega)\|_{\infty} < 1
\]

(6)

where \( T_0(s) = G_{0s} \) is the nominal complementary sensitivity function with the nominal plant transfer function \( G_0(s) \), \( W_M(s) \) expresses a fixed stable weight function, and for the family of perturbed plant it holds that

\[
G(s) = (1 + \Delta(s)W_M(s))G_0(s), \quad \|\Delta(s)\|_{\infty} \leq 1
\]

(7)

**Definition 2** (Nominal performance). Nominal performance is expressed by the condition

\[
\|W_p(j\omega)S_0(j\omega)\|_{\infty} < 1
\]

(8)

where \( W_p(j\omega) \) stands for the sensitivity weighting function and \( S_0(s) = G_{0s}(s)/W(s) \) means the sensitivity function for the nominal plant.
Theorem 5 (Robust performance). Robust performance comprising both robust stability (see Theorem 4) and performance (8) for the family of perturbed plants (7) is satisfied if and only if

$$\left\| P_s(j\omega)P_o(j\omega) + W_{sp}(j\omega)S_o(j\omega) \right\| < 1$$

By taking year-long variations in $\varphi(t)$ inside the laboratory room and identification uncertainties of transmission coefficients of the exchanger model into consideration, the following estimation of $W_{sp}(s)$ can be written

$$W_{sp}(s) = 0.36 \frac{(200s + 1)(10s + 1)}{(340s + 1)(15s + 1)} = \frac{720s^2 + 75.6s + 0.36}{5100s^2 + 355s + 1}$$

see Figure 4.

Let us select possible values $m_0 = [0.005, 0.01, 0.015, 0.02]$ and test the robust stability condition (6) as displayed in Figure 5. Apparently, all the four values comply with the condition.

The weighting function $W_p(s)$ is selected in order to keep the condition (8) for all four values of $m_0$, see Figure 6. Its possible eventual Laplace form might be

$$W_p(s) = \frac{(350s + 1)(90s + 1)}{900s(40s + 1)} = \frac{31500s^2 + 440s + 1}{36000s^2 + 900s}$$

where the level of conservativeness is a rather low except for middle frequencies. Once functions $W_{sp}(s)$ and $W_p(s)$ are fixed, possible values of the free controller parameter can be benchmarked by simulations of robust performance (9). Corresponding results are provided to the reader in Figure 7. As can be seen from the figure, the option $m_0 = 0.005$ fails on low frequencies; whereas $m_0 = 0.02$ is unacceptable due to (9) does not hold on middle frequencies. As a conclusion, the range $m_0 \in [0.008, 0.016]$ has been chosen as a candidate for further control response simulations etc.
CONTROL RESPONSES SIMULATIONS

Control responses for particular settings \( m_0 = \{0.008, 0.01, 0.012, 0.016\} \), where the step load disturbance \( d(t) = -10 \text{ W} \) enters at \( t = 3000 \), are displayed in Figures 8 and 9. As evident from these figures, higher values of \( m_0 \) lead to faster yet higher control actions. The option \( m_0 = 0.016 \) gives \( u_{0,\text{max}} > 450 \text{ W} \) which is not physically acceptable since in the operating point \( P_{H,0} = 300 \text{ W} \), the maximum possible action reads 450 W. Real time steering, for these cases, would claim the use of anti-windup control action calculations.

A quantitative benchmark comparison is given in Table 1 where the Integrated Squared Error (ISE) and the Integrated Squared Time Error (ISTE) criteria are evaluated. Note that these criteria are defined as

\[
J_{\text{ISE}} = \int_0^\infty [e^2(t) + \varphi u^2(t)] dt, \quad J_{\text{ISTE}} = \int_0^\infty [e^2(t) + \varphi u^2(t)] dt
\]

respectively. The weighting coefficient is chosen as \( \varphi = 10 \).

Table 1: ISE and ISTE Criteria Values for Controller (5)

<table>
<thead>
<tr>
<th>( m_0 )</th>
<th>ISE</th>
<th>ISTE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.008</td>
<td>4.3583 \cdot 10^4</td>
<td>4.3583 \cdot 10^7</td>
</tr>
<tr>
<td>0.01</td>
<td>4.0582 \cdot 10^4</td>
<td>9.1137 \cdot 10^6</td>
</tr>
<tr>
<td>0.012</td>
<td>3.9875 \cdot 10^4</td>
<td>8.2889 \cdot 10^6</td>
</tr>
<tr>
<td>0.016</td>
<td>4.2634 \cdot 10^4</td>
<td>1.0483 \cdot 10^7</td>
</tr>
</tbody>
</table>

Apparently, \( m_0 = 0.012 \) yields the best (minimum) values; hence, this option is selected as the final controller parameter value.

CONTROLLER RATIONALIZATION

Linear delayed (anisochronic) controllers belong to a wide family of infinite-dimensional systems and they are characterized by an infinite spectrum. Due to practical reasons, it is mostly desirable to approximate them by a finite-dimensional control law, namely, by the PID rule.

For controller (5), we decided to use the Padé approximation which is usually performed in such a way that the approximation is applied to separate exponential terms. This technique leads to high-order approximation models. Contrariwise, the whole transfer function approximation based on the Maclaurin series expansion is utilized here, which enables to control the matching model order.

Let the finite-dimensional approximating controller be

\[
\overline{G}_n(s) = \frac{G(s)}{P(s)} = \frac{q_2 s^2 + q_1 s + q_0}{s(s + p_0)}
\]

The matching rule can be expressed as

\[
\left[ \frac{d^i}{ds^i} G_0^n(s) \right]_{s=0} = \left[ \frac{d^i}{ds^i} \overline{G}_n(s) \right]_{s=0}, \quad i = 0,1,2,3,4
\]

Conditions (11) are worth noting and discussing. Since \( G_0(n) \to \infty \), the inversions are used in the equations. Moreover, the identity \( [1/G_0(s)]_{n=0} = [1/\overline{G}_n(s)]_{n=0} \) applied on (5), gives a dull solution \( 0 = 0 \), therefore four rather than three conditional equations are to be set. Finally, it is possible to calculate identities (11) at a different point from \( s = 0 \), e.g. in the neighborhood of a frequency where a good approximation is desired.

Hence, the solution of (11) for (5) results in

\[
\overline{G}_n(s) = \frac{-36.873 s^2 + 0.87184 s + 2.91172 \cdot 10^3}{s(s + 3.59065 \cdot 10^{-2})}
\]
The simulation comparison of original control responses by means of controller (5) with those using the simplified rational PID controller (12) are displayed in Figures 10 and 11. These results prove a very good agreement of the original and approximating responses with the exception of the abrupt change in \( u_c(t) \) at \( t = 0 \) and “undershoot” of \( y(t) \) near the zero.

**CONTROLLER DISCRETIZATION**

In this section, the reader is briefly provided with the idea of a possible controller discretization based on the transfer function description, delta models (Middleton and Goodwin 1990) and linear delay interpolation (Vyhlidal and Zítek 2005).

Delay exponential terms are subjected to the transformation \( \exp(-\eta s)X(s) \rightarrow x(t - \eta) \) and interpolated as

\[
x(t - \eta_i) = (1 - \alpha_i)x(t - \tau_{d,i}) + \alpha_i x(t - \tau_{d,i-1})
\]

where \( \eta_i = [\eta_i / T_s] \), \( \tau_{d,i} = d_i T_s \), \( \tau_{d,i+1} = (d_i + 1)T_s \), \( \tau_{d,i} \leq \eta_i \leq \tau_{d,i+1} \), \( \alpha_i = (\eta_i - \tau_{d,i})/T_s \) and \( T_s \) means the sampling period.

Then \( x(t - kT_s) \rightarrow z^{-k}X(z) \) where \( z \) is the z-transform variable associated with the shifting operator \( q \).

The approximation of derivatives resides in the introduction of variable \( \gamma \) associated with the delta operator \( \delta \) defined as

\[
\gamma = \frac{z^{-1}}{\beta T_s z + (1 - \beta)W_s}
\]

where \( \beta \in [0,1] \) represents a weighting parameter. In this paper, the Tustin approximation governed by the setting \( \beta = 0.5 \) is used.

The eventual particular discretized control law with \( T_s = 1 \text{s} \) is omitted because of the limited space. Similarly, a graphical comparison of continuous-time and discrete-time simulation control responses is useless since both the courses are almost identical and indistinguishable. Nevertheless, the eventual comparison of simulated and real-measured control responses with the digital implementation of controller (5) are finally given in Figures 12 and 13.
CONCLUSIONS
We have presented a complex control design for a laboratory circuit heating plant (exchanger). Its mathematical model evinces internal delays and thus the process ought to be considered as an infinite-dimensional one. The introduced utilization of the ring of stable and proper meromorphic functions has resulted in a delayed controller with an unknown (tunable) parameter which has been determined by using robust control tools and simulation experiments. For practitioners, a rationalization procedure and a possible digital implementation of the control law have been suggested; both the results have given a very good agreement with the original control responses. A multi-input, multi-output control design might be a suitable task for the future research on the exchanger.

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