EXACT FAST ALGORITHM FOR OPTIMAL LINEAR SEPARATION OF 2D DISTRIBUTION

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KEYWORDS

Fast Hough transform, image segmentation, linear separation, optimal cut, exhaustive search, Otsu criterion.

ABSTRACT

The paper presents a new fast computation scheme for linear separation in two-dimensional feature space. This scheme is based on a combination of several image processing techniques: fast Hough transform, cumulative sum computation and expression of optimized criterion as a function of additive statistics. It is shown that complexity of the scheme is $O(n^2 \log n)$ for chosen set of criteria. Two appropriate criteria are discussed, both being a 2D extension of well-known Otsu's criterion: standard one considering covariance trace and one considering covariance second eigenvalues. Applicability of the latter criterion for the color segmentation problem is discussed.

INTRODUCTION

Histogram analysis is a popular technique of the image processing. Particularly, spatial separation of histogram is a well-known method of image segmentation. In the case of 1D intensity histogram, segmentation becomes global binarization problem. Among global binarization algorithms Otsu's method (Otsu 1979) is undisputed leader. There are Otsu's method generalizations to the multidimensional space, e.g. two-dimensional case with intensity and gradient value axises (Nie et al. 2013). It is worth to notice that similar extensions could be useful for other segmentation tasks. For example, MRC document compression conduct segmentation in colour space, which leads to clusterization in 2D or 3D spaces (Vinogradova et al. 2015).

Otsu's method consists in minimization of functional (between class variance) on the set of possible histogram partitions. This functional is not convex in general case, so exhaustive search method is considered. Computation efficiency of such algorithms directly depends on complexity of functional calculation in the point. If every computation involves full range of the histogram, the complexity could be unacceptable. Luckily, usage of implementation of cumulative sum technique and precalculation of additive statistics leads to constant-time complexity of functional calculation in the point for Otsu's method. This trick could be used in two-dimensional space too. However, structure of the partition set in this case should be clarified, as optimal computational complexity heavily depends on this structure. Let us consider 2D histogram and its partitions into arbitrary orthotropic rectangle with vertex in origin and the rest of the histogram (as proposed in (Nie et al. 2013)). Obviously, number of such partitions will be the same as the number of histogram bins. While target functional is computable via additive statistics, 2D cumulative sum technique will help to keep constant-time complexity of functional evaluation. However, considered partition set looks suspicious. It seems more familiar to use arbitrary straight lines as a partition set. It is worth to notice that the number of partitions in this case will be same $O(n^2)$. Unfortunately, there are no known fast methods which allow to find optimal partition line in acceptable time. In this paper such method is proposed, based on the fast Hough transform (FHT).

Hough Transform (HT) was invented by Paul Hough for the analysis of photographs obtained with bumble chamber in 1959, and patented in 1961. Later HT was modified by R. O. Duda and P. E. Hart to eliminate cases with

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unbounded transformation space. More information about invention of HT can be found in (Hart 2009). HT is a very popular tool in image processing, a lot of applications of HT exist, for instance: edge detection, document orientation, vanishing point detection (Nikolaev et al. 2008), detection of circles and ellipses. Also HT is successfully implemented in robust regression analysis (Goldenshluger and Zeevi 2004; Ballester 1994; Bezmaternykh et al. 2012).

Mathematically HT is a special case of Radon $\mathcal{R}$ Transform (RT):

$$\mathcal{R}f(L) = \int\limits_{L} f(x)|dx|, \tag{1}$$

where f is a scalar function, Ω is a domain of f (for example coordinate space of image), $x \in \Omega \subset R^2$, L is a line with $L \subset \Omega$. According to equation (1) Radon Transform creates continuous set of lines, which usually is not required in practice problems. RT defined for discrete Ω is Hough transform.

It is easy to estimate complexity boundaries of HT for gray image $\mathcal{I}$, which is two dimensional scalar function. Suppose image $\mathcal{I}$ has square domain of size $n \cdot n$. Then the number of different lines on that image is proportional to n^2, length of any line is n, therefore complexity is $O(n^3)$. However better calculation scheme exists.

Fast Hough transform was invented in 1998 by Brady (Brady 1998), and several years later (Karpenko et al. 2004; Frederick et al. 2005) the version with in-place calculations was proposed. Complexity boundary for the FHT is $O(n^2 \log n)$ for square image with linear size n, similarly to the 2D fast Fourier transform. And yet, FHT doesn't involve complex arithmetic or even multiplications and could be computed in integer domain.

Let us show how FHT could be applied to the segmentation problem solution.

METHOD FRAMEWORK

Proposed algorithm computational scheme consist of several separate computer vision techniques: fast Hough transform, calculation of cumulative sum, additive statistics forming and exhaustive search. Firstly, we describe computation scheme principle foundation. Secondly, its implementation to two-dimensional distribution linear separation.

Computational scheme

In this part we use classical algorithm for calculating cumulative sum over positive discrete function, defined on some discrete rectangle set $\Omega \subset \mathcal{R}^2$. For simplicity let's define Ω as square field with size n, then $F[i, j]$ is two-dimension array with $i, j \in [1, .., n]$.

Cumulative sum computation scheme is presented as function in algorithm 1. This function returns two-dimensional array F_I, where $F_I(i^*, j^*)$ is a sum over elements with coordinates satisfy $j = j^*$ and $i \leq i^*$, i.e. all underlying points.

Consider summation of functions F and F_I along straight horizontal line. Note that sum of F_I along line L is equal to the sum of all elements of F lying under the line L including line elements, that is sum over semiplane. Moreover, summation of F_I over a horizontal line with $i = n$ is equal to summation of F over the whole definition area. Let's introduce a line set with this property.

Algorithm 1 Cumulative sum computation

function CUMULATIVESUM(F)
    $F_I = F$
    for $i \leftarrow 2$ **to** n **do**
        for $j \leftarrow 1$ **to** n **do**
            $F_I(i, j) = F_I(i-1, j) + F(i, j)$
        end for
    end for
    return F_I
end function

Line L is called primary-horizontal (PH) if angle α is between normal to a line and direction of y-axis satisfy $\alpha \in [-\frac{\pi}{4}; \frac{\pi}{4}]$. Positive direction of angle is counter-clockwise from y-axis which is down-directed (see Figure 1). At the same time equivalence between summation over a semiplane over F and PH-line over F_I isn't preserved for set of primary-vertical lines. Line L is primary-vertical when $\alpha \in [\frac{\pi}{4}; \frac{3\pi}{4}]$. Nevertheless, to establish the same property for PV-lines it is necessary to transpose F, perform cumulative summation of F^T and transpose the result: $F_I = (CumulativeSum(F^T))^T$.

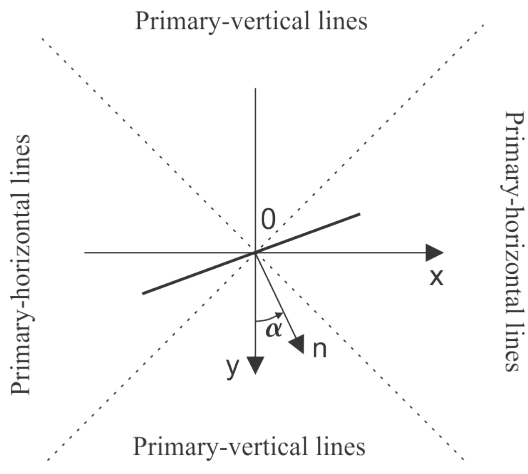

Figure 1: Coordinate system. Primary-horizontal and primary-vertical lines definition.

In such way, we are able to perform summation over semiplanes by summing over corresponding lines. To find sum over all possible lines, Hough transform is usually applied. Its naive computational complexity is very high $O(n^3)$. Luckily, faster algorithm exists. It is fast Hough transform, which complexity boundaries are $O(n^2 \log n)$. FHT produces two dimensional array with each cell containing sum over semiplane.

Consequently, new computational scheme allows us to perform fast summation of a function over semiplanes.

Coupled with exhaustive search in Hough space, new computation scheme can be applied for solution tasks involving linear separation, such as colour segmentation problem, two-dimensional global binarization by Otsu's method.

Additive statistics technique

Note that our scheme implementation can perform fast computation of additive operations. Therefore it is necessary to construct computational scheme using only additive statistic and additive-preserved operations. Concept of additive statistic can be illustrated with help of two sets: let two disjointed sets have statistics α_1 and α_2, then conjoint statistics is equal to $\alpha_1 + \alpha_2$. For example, weight is an additive statistic.

Let's consider spatial separation of two-dimensional distribution into two classes C_1 and C_2. Denote P as normalized F, meaning

$$P[i,j] = \frac{F[i,j]}{\sum\limits_{i,j\in\Omega} F[i,j]}, [i,j] \in [1,..,n]$$

Note that P can be interpreted as two-dimensional distribution, so we can derive all necessary statistics for it, particularly expected values and covariance matrices for each class.

Lets denote weight of each class:

$$\omega_{C1} = \sum_{(i,j)\in C_1} P[i,j],\ \omega_{C2} = \sum_{(i,j)\in C_2} P[i,j],$$

therefore $\omega_{C_1} + \omega_{C_2} = 1$. Then expected values for each class are

$$E_X(i) = \frac{1}{\omega_X} \sum_{i,j\in X} iP[i,j] = \frac{1}{\omega_X} D_X(i),$$

$$E_X(j) = \frac{1}{\omega_X} \sum_{i,j\in X} jP[i,j] = \frac{1}{\omega_X} D_X(j),$$

$$E_X(i^2) = \frac{1}{\omega_X} \sum_{i,j\in X} i^2P[i,j] = \frac{1}{\omega_X} D_X(i^2),$$

$$E_X(j^2) = \frac{1}{\omega_X} \sum_{i,j\in X} j^2P[i,j] = \frac{1}{\omega_X} D_X(j^2),$$

$$E_X(ij) = \frac{1}{\omega_X} \sum_{i,j\in X} ijP[i,j] = \frac{1}{\omega_X} D_X(ij),$$

where $X = \{C_1, C_2\}$. Then expected value for P is $E_P(i) = D_{C_1}(i) + D_{C_2}(i)$ and same for other variables. Note that $D_X(i)$, $D_X(j)$, $D_X(ij)$, $D_X(i^2)$, $D_X(j^2)$ are additive statistics, like weight. Therefore we can perform fast calculation of weight and $D_X(\cdot)$ for expected value computation. Let's derive equation for covariance matrix of each class, denote for that:

$$\sigma_X^2(i) = \frac{1}{\omega_X} D_X(i^2) - (\frac{1}{\omega_X} D_X(i))^2,$$

$$\sigma_X^2(j) = \frac{1}{\omega_X} D_X(j^2) - (\frac{1}{\omega_X} D_X(j))^2,$$

$$Cov_X(i,j) = \frac{1}{\omega_X} D_X(ij) - \frac{1}{\omega_X^2} D_X(i) D_X(j)$$

Then equation for covariance matrix is:

$$\sigma_X = \begin{pmatrix} \sigma_X^2(i) & Cov_X(i,j) \\ Cov_X(i,j) & \sigma_X^2(j) \end{pmatrix} \quad (2)$$

In this manner, to define covariance matrices for each class we need to calculate additive statistics: ω, D and combine them into (2). As shown before, with algorithm 1 we can obtain additive statistics by summation of $CumulativeSum(F)$ over straight primary-horizontal and $(CumulativeSum(F^T))^T$ over straight primary-vertical lines and, as shown in (2), covariance matrix for each class can be computed by pre-calculated additive statistics combination.

Criteria

It is clear, that for search separation line we need to establish some criterion. In this paper we use two different criteria. First is global Otsu's binarization in two-dimensional feature space (Otsu 1979) criterion: between-class variance trace minimization:

$$\sigma(C_1, C_2) = \omega_{C_1}\sigma_{C_1} + \omega_{C_2}\sigma_{C_2}, \quad (3)$$

and Otsu's problem for two-dimensional feature space becomes

$$L^* = \arg\min_{L\in\Omega} trace(\sigma(C_1(L), C_2(L))), \quad (4)$$

therefore we need to solve optimization problem for finding optimal separation line L^*.

Nevertheless (4) has application boundaries. For instance, when variance of distribution along one axis is greater than variance along other one, separation result may be wrong. For example, colour histogram of scene which consist of two monochrome objects illuminated by achromatic light source, (Usilin et al. 2010). Indeed, in case of single object illumination, color-histogram is represented by straight line as different parts of an object have different lightning intensity. In real tasks this line become cloud of points spread along the line. In case when scene consist of two monochrome objects illuminated by achromatic light, colour histogram consist of two point clouds spread along two different lines (Nikolaev and Nikolayev 2004).

New criterion for spatial separation of two-dimensional colour histogram is proposed:

$$L^* = \arg\min_{L\in\Omega} \lambda_2(\sigma(C_1(L), C_2(L))), \quad (5)$$

where λ_2 equals second eigenvalue of covariance matrix σ. Indeed, distribution can be interpreted as ellipse of inertia. For two-dimensional case it is necessary to pose ellipse so, that its second axis is minimized, which equals to second eigenvalue minimization.

Spatial separation algorithm

In this section it is convenient to consider FHT as a black box algorithm $HS = FHT(F)$, where F is two-dimensional array input and H is two-dimensional array output, with indices corresponding to straight line parameters and value is a sum along this line. To simplify the notation, we propose algorithm for PH-lines, nevertheless, as mentioned before, transposing is the only difference between PH- and PV-lines algorithm. Denote $L = fhtline(i, j)$ as array of line $L \in \Omega$ points parametrized by (i, j) in Hough space.

Combining all mentioned computational methods it is possible to construct a new spatial separation scheme. Following (2) to find covariance matrix it is necessary to calculate a range of additive statistics. Let's simplify notation by introducing vector of matrices $data = [D_X(i), D_X(j), D_X(i^2), D_X(j^2), D_X(ij), \omega_1]$, therefore $data[6]$ is ω_1 and so on. Also it is necessary to mention that each element of $data_H$ has size equal to Hough space, i.e. $[n \cdot 4n]$ (for particular implementation). Therefore σ is four-dimentional matrix, with size $[n \cdot 4n \cdot 2 \cdot 2]$. Algorithm results in an array of line L points. Algorithmic scheme in mnemonic language is presented in algorithm 2.

Algorithm 2 spatial separation algorithm (PH-lines)

Input: $data$;
Initialize: $data_{CS}$;
for $k \leftarrow 1$ **to** 6 **do** ▷ Counting cumulative sum
    $data_{CS}[k] \leftarrow CumSum(data[k])$;
end for
Initialize: $data_H$
for $k \leftarrow 1$ **to** 6 **do** ▷ Perform Fast Hough Transform
    $data_H[k] \leftarrow FHT(data_{CS}[k])$;
end for
for $X \in \{C_1, C_2\}$ **do** ▷ First moments computation
    for $t \leftarrow 1$ **to** 5 **do**
        $E_X(t) = (1/data_H[6]) \cdot data_H[t]$; ▷ Matrix element-wise division
    end for
    Computing: $\sigma^2_X(i), \sigma^2_X(j), Cov_X(i.j)$ with $E_X(t)$;
    Computing: σ_X; according to equation (3)
end for
$\sigma = \omega_{C_1}\sigma_{C_1} + \omega_{C_2}\sigma_{C_2}$; ▷ Between-class covariance matrix
$T = trace(\sigma)$ ▷ Trace element-wise computation
$[i^*, j^*] = argmax(T)$;▷ Find indices of max element
$L^* = fhtline(i^*, j^*)$; ▷ Array of points
Return: L^*;

DISCUSSION

Complexity

In this section we show that algorithmic scheme complexity boundaries are estimated as $O(n^2 \log n)$. As mentioned above, algorithmic scheme consist of several stages: fast Hough transform, cumulative summation, collecting of additive statistics and full search over Hough space.

To perform collection of additive statistics for squared Ω elementwise multiplication of six $n \cdot n$ matrices is required, which means complexity of such operation is $O(n^2)$. Complexity of cumulative summation is n^2, as we perform n summation of lines(vectors) which consist of n elements. Scheme requires to repeatedly perform the cumulative summation, which means that cost of this operations is proportional to $O(n^2)$.

Next steps of the scheme are fast Hough transform and linear operations over Hough space, which cost $O(n^2 \log n)$ and $O(n^2)$. Final step of the scheme is exhaustive search over Hough space with cost $O(n^2)$. Total results are in Table 1.

Table 1: Analysis of Algorithm Complexity.

Method	**Complexity**
Cumulative summation	$O(n^2)$
Additive statistics containing	$O(n^2)$
FHT	$O(n^2 \log n)$
Exhaustive search	$O(n^2)$
Total	$O(n^2 \log n)$

Results

First, let's investigate cases when corner separator (Jian et al. 1996) and line separator usage is optimal. Trace minimization criterion is used in both cases. See results in Figures 2,3. Top left corner corresponds to origin of coordinate system, vertical axe is down-directed and horizontal axe directs to the right, as shown in Figure 1.

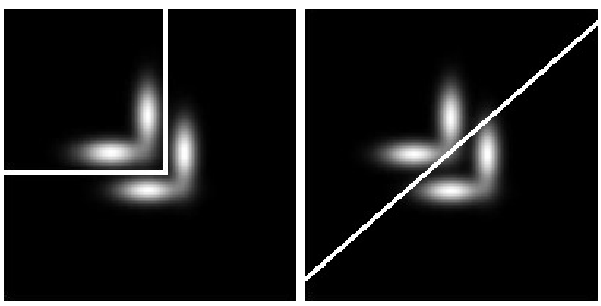

Figure 2: Form of two-dimensional distribution that implies corner separator application. Corner separator (left), line separator (right)

Nevertheless there are cases when line separation is preferential, see Figure 3.

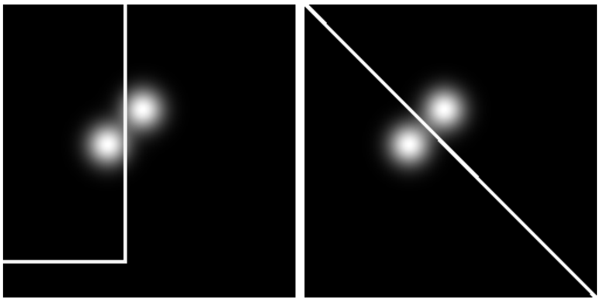

Figure 3: Form of two-dimensional distribution that implies line separator application. Corner separator (left), line separator (right)

The paper contains description of two different optimization spatial separation criteria based on between-class covariance matrix computation: trace minimization and second eigenvalue minimization of this matrix.

In case when classes variance is nearly equal in both directions, trace minimization criteria perform correct separation (see Figure 4). Note that such distribution frequently appears in 2D Otsu's problem, which explains applying of trace minimization criterion.

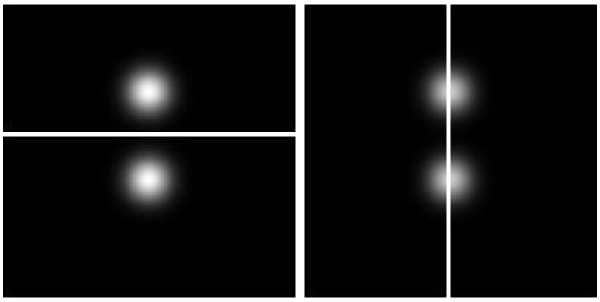

Figure 4: Difference between two criteria: trace minimization (left) and second eigenvalue minimization (right).

As mentioned before, one of potential implementation of this computational scheme is spatial separation in colour histogram space. Simulated historgram of a scene with two monochrome objects illuminated by single achromatic source of light is shown in Figure 5. It is a pair of two-dimensional distributions spreaded along some direction.

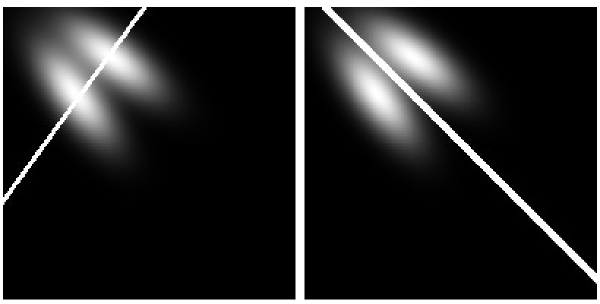

Figure 5: Difference between two criteria: trace minimization (left) and second eigenvalue minimization (right).

It appears that in case when we simulate color segmentation problem, trace minimization barely give as desired result. At the same time new second value minimization criterion allows to achieve appropriate separation.

CONCLUSION

In this paper a new computationally effective and numerically precise scheme for spatial separation of two-dimensional distribution is proposed, which is based on combination of common image processing methods, such as fast Hough transform, cumulative summation, collecting of additive statistics and exhaustive search. One of the most important tool in the proposed scheme is FHT, which allows to improve complexity boundaries from $O(n^3)$ to $O(n^2 \log n)$ without loss of precision. Also a new line separation criteria was proposed, namely between-class covariance matrix second value minimization, which is perspective candidate for segmentation problems solution.

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