OPTIMAL GAIN SCHEDULED CONTROLLER FOR A TWO FUNNEL LIQUID TANKS IN SERIES

Adam Krhovják, Petr Dostál, Stanislav Talaš and Lukáš Rušar
Department of Process Control
Faculty of Applied Informatics, Tomas Bata University in Zlín,
Nad Stráněmi 4511, 760 05 Zlín, Czech Republic
krhovjak@fai.utb.cz

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ABSTRACT
Motivated by the control complexity of nonlinear systems, we introduce an optimal gain scheduled controller for a nonlinear system of two funnel liquid tanks in series, based on a linearization of a nonlinear state equation of the system about selected operating points. Specifically, the proposed technique aims at extending the region of validity of linearization by introducing a parametrized linear model, which enables to construct a feedback controller at each point. Additionally, we present an integral control approach which ensures robust regulation under all parameter perturbations. The parameters of resulting family of feedback controllers are scheduled as functions of reference variables, resulting in a single optimal controller. Nonlocal performance of the gain scheduled controller for the nonlinear model has been checked by mathematical simulation.

INTRODUCTION
We live in a world of highly complex systems that exhibit nonlinear behavior. Engineers facing the control of these systems are required to design such mechanisms that would satisfy desired characteristics through the operating range. In many cases, situation is complicated by the fact that the tracking problem involves multiple variables interacting with each other. One consequence is that superposition principle, which is known from linear systems does not hold any longer and we are faced more challenging situations.

However, because of powerful tools we know for linear systems, the first step in designing a control for a nonlinear systems consists in linearization. There is no question that whenever it is possible, we should take advantage of design via linearization approach.

Nevertheless, we must bear in mind the basic limitation associated with this approach. The understanding that linearization represents only an approximation in the neighborhood of an operating point.

To put it differently, linearization cannot be viewed globally since it can only predict the local behavior of a nonlinear system. Therefore, conventional controllers with fixed parameters cannot guarantee performance beyond the vicinity of operating point. Interestingly enough, in many situation, it is possible to explicitly capture how the dynamic of the system changes in its equilibrium points by introducing a family of linear models which are parametrize by one or more scheduling variables. In such cases it is intuitively reasonable to linearize the nonlinear system about selected operating points, capturing key states of the system, design a linear feedback controller at each point, and interpolate the resulting family of linear controllers by monitoring scheduling variables. There has been a significant research in gain scheduling (GS). We refer the interest reader to (Rugh 1991; Shamma and Athans 1991; Shamma and Athans 1992; Lawrence and Rugh 1995) for deeper and more insightful understanding of the gain scheduling procedure.

Most efforts have been devoted to the analytical framework (Shamma and Athans 1990; Rugh 1991) and less attention has been paid to particular engineering applications except few remarkable applications in technological processes (Jiang 1994; Kaminer et al. 1995; Krhovják et al. 2015). Moreover, these intensive efforts have not paid attention to the question of optimal performance design which would make the concept much more attractive for a potential practitioner in industry. In order to address those needs we have stressed to illustrate gain scheduling strategy for a nonlinear system of two funnel liquid tanks in series (TFLT), extending region of validity of linearization approach by designing an optimal controller that is a prescription for moving from one design to another.

Thus, the problem of a designing an optimal control for a model of the nonlinear system of the TFLT has been reduced to a problem of designing a family of optimal feedback controllers that are interpreted as a single controller via scheduling variables.

Throughout the paper we gradually reveal the scheduling procedure satisfying a tracking problem as well as the design of an optimal control trajectory for the multivariable nonlinear system of two funnel tanks.

MODEL OF THE TFLT
A simplified model of the TFLT system taken from (Dostál et al. 2008) is schematically shown in Figure 1. The process consists of two liquid streams that are pumped into funnel tanks. Pump with a flow rate \( q_1 \) discharges liquid into the first tank (T1). The second tank
(T2) is fed by both liquid stream $q_{2f}$ and $q_{1}$, representing liquid that leaves the first tank through the opening in the base. There are no reactants or reaction kinetics and stoichiometry to consider. The model also includes hydraulic relationship for the tank outlet streams.

Both parameters of the tanks and initial liquid levels are captured in Table 1.

<table>
<thead>
<tr>
<th>Tank</th>
<th>$D$</th>
<th>$H$</th>
<th>$\bar{h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.5</td>
<td>2.5</td>
<td>1.8</td>
</tr>
<tr>
<td>2</td>
<td>1.5</td>
<td>2.5</td>
<td>14</td>
</tr>
</tbody>
</table>

In this case study we have used $k_{1} = 0.32 \text{m}^{1.5}/\text{min}$, $k_{2} = 0.3m^{1.5}/\text{min}$ and $q_{1f} = 0.2 \text{m}^{3}/\text{min}$, $q_{2f} = 0.1 \text{m}^{3}/\text{min}$.

Figure 1: Two funnel liquid tanks in series

The only step needed to develop the model of SLT is to write conservation equation (Richardson 1989), representing material balance for a single material. Recall that the general form of a mass balance is given

\[ \text{INPUT} = \text{OUTPUT} + \text{ACCUMULATION} \]

It is easy to see that the simplified model of TFLT can be modeled by

\[ \pi \frac{D^2}{4H^2} h_1 \frac{dh_1}{dt} + q_1 = q_{1f} \]

(1)

\[ \pi \frac{D^2}{4H^2} h_2 \frac{dh_2}{dt} - q_1 + q_2 = q_{2f} \]

(2)

where $h_1$ and $h_2$ represent liquid levels from the bottom and $D$ is the diameter of the cross sectional area at the top of the tanks. As the liquid moves through the valves, we see dependence of $q$ on liquid level as

\[ q_1 = k_1 \sqrt{h_1 - h_2} \]

(3)

\[ q_2 = k_2 \sqrt{h_2} \]

(4)

where $k_1$ and $k_2$ are positive valve constants.

MODEL STRUCTURE FOR GS DESIGN

In the process of designing and implementing a gain scheduled controller for a nonlinear system, we have to find its approximations about the family of operating (equilibrium) points. Thus, the coupled nonlinear first-order ordinary differential equations (1)-(2) capturing the dynamics of the TFLT have to be transformed into its linearized form.

In view of our example, we shall deal with multi-input multi-output linearizable nonlinear system represented by

\[ \dot{x} = f(x, u) \]

(5)

\[ y = g(x) \]

(6)

where $\dot{x}$ denotes derivative of $x$ with respect to time variable and $u$ are specified input variables. We call the variable $x$ the state variable and $y$ the output variable. We shall refer to (5) and (6) together as the state-space model.

To obtain a state-space model of the TFLT, let us take $x_1 = h_1$, $x_2 = h_2$ as state variables and $u_1 = q_{1f}$, $u_2 = q_{2f}$ as control inputs. Then the state equations are

\[ \frac{dx_1}{dt} = \frac{4H^2}{\pi D^2 x_1^2} (u_1 - k_1 \sqrt{x_1 - x_2}) \]

(7)

\[ \frac{dx_2}{dt} = \frac{4H^2}{\pi D^2 x_2^2} (u_2 + k_1 \sqrt{x_1 - x_2} - k_2 \sqrt{x_2}) \]

(8)

and the output equations take the form

\[ y_1 = x_1 \]

(9)

\[ y_2 = x_2 \]

(10)

One can easily sketch the trajectory of steady-state characteristic by setting $\dot{x} = 0$ and solving for unknown vector $x$.

Therefore the equilibrium points correspond to the solution of

\[ 0 = \frac{4H^2}{\pi D^2 x_1^2} (u_1 - k_1 \sqrt{x_1 - x_2}) \]

(11)

\[ 0 = \frac{4H^2}{\pi D^2 x_2^2} (u_2 + k_1 \sqrt{x_1 - x_2} - k_2 \sqrt{x_2}) \]

(12)

Having calculated equilibrium points of state equation, our goal now is to approximate (5) about selected single operating point. Suppose $x \neq 0$ and $u \neq 0$, and consider the change of variables

\[ y_1 = x_1 \]

(13)

\[ y_2 = x_2 \]

(14)

\[ y_1 = x_1 \]

(15)

It should be noted that in the new variables system has equilibria in origin. Expanding the right hand side of (5) about point $(\bar{x}, \bar{u})$, we obtain

\[ f(x, u) \approx f(\bar{x}, \bar{u}) + \frac{\partial f(\bar{x}, \bar{u})}{\partial x} (x - \bar{x}) + \frac{\partial f(\bar{x}, \bar{u})}{\partial u} (u - \bar{u}) + \text{H.O.T.} \]

(16)

If we restrict our attention to a sufficiently small neighborhood of the equilibrium point such that the
higher-order terms are negligible, then we may drop these terms and approximate the nonlinear state equation by the linear state equation
\[
\dot{x}_a = Ax_a + Bu_a
\] 
where
\[
A = \frac{\partial f}{\partial x_{x-x,u-a}}
\]
\[
B = \frac{\partial f}{\partial u_{x-x,u-a}}
\]

**PARAMETRIZATION OF LINEAR MODELS**

Before we present a parametrization via scheduling variable, let us first examine configuration of the gain scheduled control system captured in Figure 2. From the figure, it can be easily seen that controller parameters are automatically changed in open loop fashion by monitoring operating conditions. From this point of view, presented gain scheduled control system can be understood as a feedback control system in which the feedback gains are adjusted using feedforward gain scheduler.

![Figure 2: Gain scheduled control](image)

Then it comes as no surprise that first and the most important step in designing a controller is to find an appropriate scheduling strategy. Once the strategy is found, it can be directly embedded into the controller design.

In order to understand the idea behind the gain scheduling let us first consider the nonlinear system
\[
\dot{x} = f(x, u, \alpha)
\] 
\[
y = g(x, \alpha)
\] 
We can see that the nonlinear system is basically same as the system that we have introduced in the previous section by equations (5) and (6). The only difference here is that both state and output equations are parameterized by a new scheduling variable \( \alpha \) representing the operating conditions.

To illustrate this motivating discussion let us consider this crucial point in the context of our example.

Suppose the system is operating at steady state and we want to design controller such that \( x \) tracks a reference signal \( w \). In order to maintain the output of the plant at the value \( \bar{x}_1 \) and \( \bar{x}_2 \), we have to generate the corresponding input signal to the system at \( \bar{u}_1 = k_1 \sqrt{x_1 - \bar{x}_1} \) and \( \bar{u}_2 = k_2 \sqrt{x_2 - \bar{x}_2} \), respectively. This implies that for every value of \( w \) in the operating range, we can define the desired operating point by \( x = y = w \) and \( u = \Pi(w) \).

Thus it means, that we can directly schedule on a reference trajectory.

Having identified a scheduling variable, the common scheduling scenario takes this form
\[
\dot{x}_a = A(\alpha)x_a + B(\alpha)u_a
\] 
Intuitively speaking, the parameters of (21) are scheduled as functions of the scheduling variable \( \alpha \). Since our model is simple nonlinear TITO system, we need to calculate elements of \( A, B \), corresponding to the structure of (18). In other words, the key how to move from one operating point to another is given by
\[
a_{01}(\alpha) = \frac{4H^2 k_1 \sqrt{\alpha_1 - \alpha_2}}{2D^2 \pi \alpha_1^2 (\alpha_1 - \alpha_2)}
\]
\[
a_{02}(\alpha) = \frac{4H^2 k_1 \sqrt{\alpha_1 - \alpha_2}}{2D^2 \pi \alpha_1^2 (\alpha_1 - \alpha_2)}
\]
\[
a_{03}(\alpha) = \frac{4H^2 k_1 \sqrt{\alpha_1 - \alpha_2}}{2D^2 \pi \alpha_1^2 (\alpha_1 - \alpha_2)}
\] 
\[
a_{04}(\alpha) = \frac{4H^2 k_1 \sqrt{\alpha_1 - \alpha_2}}{2D^2 \pi \alpha_1^2 (\alpha_1 - \alpha_2)} + \frac{k_2 \sqrt{\alpha_2}}{\alpha_2}
\]
\[
b_{01}(\alpha) = \frac{4H^2}{D^2 \pi \alpha_1^2}, \quad b_{04}(\alpha) = \frac{4H^2}{D^2 \pi \alpha_2^2}
\]

An important feature of our analysis is that even if \( \alpha \) represents reference vector, the equations (22) still capture the behavior of the system around equilibria.

**REGULATION VIA INTEGRAL CONTROL**

Since the previous sections resulted in a family of parametrized linear models we want to design state feedback control such that
\[
y \rightarrow y, \quad \text{as} \quad t \rightarrow \infty
\]

Further, we assume that we can physically measure the controlled output \( y \). In order to ensure zero steady-state tracking error in the presence of uncertainties, we want to use integral control. The regulation task will be achieved by stabilizing system at an equilibrium point where \( y = y_e \).

To maintain the system at that point it must be true, that there exists a pair of \( (\bar{x}, \bar{u}) \) such that
\[
\theta = f(\bar{x}, \bar{u}, \alpha)
\]
\[
\theta = g(\bar{x}, \alpha) - y_e
\]

Note, that for equations (24)-(25) we assume a unique solution \( (\bar{x}, \bar{u}) \).
In searching for an optimal control $u = -Kx$ we have to design gain matrix $K$ that minimizes quadratic cost function (omit dependence on $\alpha$)

$$J = \int_0^\infty [x^T Q x + u^T R u + 2x^T Nu] dt$$

(35)

where $Q$, $R$ a symmetric, positive (semi-) definite weighting matrices and $N$ represents

The traditional problem is solved using algebraic Riccati equation

$$A^T P + PA - PB + N R - I (B^T P + N^T ) + Q = 0$$

(36)

Then the gain matrix $K$ is derived from $P$ by

$$K = R^{-1}(B^T P + N^T )$$

(37)

In view of the procedure that we have just described, one can notice that three main issues are involved in the development of gain scheduled controller; namely linearization of TFLT about the family of operating regions, design of a parametrized family of linear matrix feedback controllers for the parametrized family of linear systems and construction of gain scheduled controller.

So far, we have formed the basic idea of the control problem. All that remains now is to simulate the performance of the gain scheduling procedure with the help of the integral control.

**SIMULATIONS AND RESULTS**

In this section, we simulate the gain scheduled control of TFLT. We have developed a custom MATLAB function based on the simulator introduced by (Krinhovják et al. 2014) that simulates adequately the behavior of TFLT. Idealistic model has been implemented according to equations (1) and (2). The popular ODE solver based on Runge-Kutta methods (Hairer et al. 1993) was considered to calculate numerical solution.

The simulation results of gain scheduled control are presented in Figures 4-8. Figure 4 shows the optimal responses of the control system to sequences of step changes in reference signals. As can be seen a step change in reference signals causes a new calculation of the equilibrium point of the system.

![Figure 4: The responses of the closed-loop system to a sequence of step changes](image)
Figure 5 shows the response of the closed-loop system to a slow ramp that takes the set points over a period of 500 minutes. These observations are consistent with a common gain scheduling rule-of-thumb about the behavior of gain scheduled controller under slowly varying scheduling variable.

In contrast, the figure 6 shows the response to a faster ramp signal. As the slope of the ramp increases, tracking performance deteriorates. If we keep increasing the slope of the ramp, the system will eventually go unstable.

To appreciate what we gain by gain scheduling, Figure 7 and Figure 8 illustrates responses of the closed-loop system to the same sequence of changes. In the first case, a gain scheduled controller is applied, while in the second case a fixed-gain controller evaluated at $\alpha = [-1 -1]$ is used.

From this illustration it is evident why we have to modify the gain scheduled controller. While stability and zero steady-state tracking error are achieved, as predicted by our analysis, the responses deteriorates significantly as the reference is far from operating point. In some situations it may be possible to reach a large value of the reference signal by a sequence of step changes, as in the Figure 4 where we allow enough time for the system to settle done after each step change. This can be viewed as another possible way how to change the reference set point.

CONCLUSION

This paper addressed the problem of the gain scheduling procedure as well as the design of optimal control for a nonlinear multi-input multi-output system of two funnel liquid tanks in series. First, we have detailed studied the simplified model of the technological process. Based on the model, we have followed a general analytical framework for gain scheduling. We have also pointed out that selection of scheduling variable depends on particular characteristics of the system. This observation has critical importance and leads us to the conclusion that rule of scheduling on reference variable can be applied for other technological processes. The main advantage of this approach is that linear design methods can be applied to the linearized system at each operating point. Thanks to this feature, the presented procedure leaves room for many linear control methods. As our results show, presented integral control approach ensures robust regulation under all parameter perturbations. However the strength of the presented feedback lies in the optimal design of the gain matrix. In addition, we have demonstrated that a gain scheduled control system has the potential to respond rapidly changing operating conditions.
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AUTHOR BIOGRAPHIES

ADAM KRHOVJÁK studied at the Tomas Bata University in Zlin, Czech Republic, where he obtained his master degree in Automatic Control and Informatics in 2013. He now attends PhD. study in the Department of Process Control, Faculty of Applied Informatics of the Tomas Bata University in Zlin. His research interests focus on modeling and simulation of continuous time technological processes, adaptive and nonlinear control.

PETR DOSTÁL studied at the Technical University of Pardubice, where he obtained his master degree in 1968 and PhD. degree in Technical Cybernetics in 1979. In the year 2000 he became professor in Process Control. He is now Professor in the Department of Process Control, Faculty of Applied Informatics of the Tomas Bata University in Zlin. His research interest are modeling and simulation of continuous-time chemical processes, polynomial methods, optimal and adaptive control.

STANISLAV TALAŠ studied at the Tomas Bata University in Zlin, Czech Republic, where he obtained his master degree in Automatic Control and Informatics in 2013. He now attends PhD. study in the Department of Process Control, Faculty of Applied Informatics of the Tomas Bata University in Zlin. His e-mail address is talas@fai.utb.cz.

LUKÁŠ RUŠAR studied at the Tomas Bata University in Zlin, Czech Republic, where he obtained his master degree in Automatic Control and Informatics in 2014. He now attends PhD. study in the Department of Process Control, Faculty of Applied Informatics of the Tomas Bata University in Zlin. His research interests focus on model predictive control. His e-mail address is rusar@fai.utb.cz.