TWO-SIDED TRUNCATIONS OF INHOMOGENEOUS BIRTH-DEATH PROCESSES

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ABSTRACT
We consider a class of inhomogeneous birth-death queueing models and obtain uniform approximation bounds of two-sided truncations. Some examples are considered. Our approach to truncations of the state space can be used in modeling information flows related to high-performance computing.

INTRODUCTION
It is well known that explicit expressions for the probability characteristics of stochastic birth-death queueing models can be found only in a few special cases. Therefore, the study of the rate of convergence as time \( t \to \infty \) to the steady state of a process is one of two main problems for obtaining the limiting behavior of the process. If the model is Markovian and stationary in time, then, as a rule, the stationary limiting characteristics provide sufficient or almost sufficient information about the model. On the other hand, if one deals with inhomogeneous Markovian model then, in addition, the limiting probability characteristics of the process must be approximately calculated. The problem of existence and construction of limiting characteristics for time-inhomogeneous birth and death processes is important for queueing and some other applications, see for instance, [1], [3], [5], [8], [15], [16]. General approach and related bounds for the rate of convergence was considered in [13]. Calculation of the limiting characteristics for the process via truncations was firstly mentioned in [14] and was considered in details in [15], uniform in time bounds have been obtained in [17].

As a rule, the authors dealt with the so-called northwest truncations (see also [9]), namely they studied the truncated processes with the same first states \( 0, 1, \ldots, N \). In the present paper we consider a more general approach and deal with truncated processes on state space \( N_1, N_1 + 1, \ldots, N_2 \) for some natural \( N_1, N_2 \geq N_1 \).

Let \( X = X(t) \), \( t \geq 0 \) be a birth and death process (BDP) with birth and death rates \( \lambda_n(t), \mu_n(t) \) respectively.

Let \( p_{ij}(s, t) = Pr \{ X(t) = j | X(s) = i \} \) for \( i, j \geq 0, \ 0 \leq s \leq t \) be the transition probability functions of the process \( X = X(t) \) and \( p_i(t) = Pr \{ X(t) = i \} \) be the state probabilities.

Throughout the paper we assume that

\[
P ( X (t + h) = j | X(t) = i ) =
\[
= \begin{cases} 
q_{ij}(t) h + \alpha_{ij}(t, h) & \text{if } j \neq i, \\
1 - \sum_{k \neq i} q_{ik}(t) h + \alpha_{ii}(t, h) & \text{if } j = i,
\end{cases}
\]

(1)

where all \( \alpha_{ij}(t, h) \) are \( o(h) \) uniformly in \( i \), i.e. \( \sup_i \| \alpha_{ij}(t, h) \| = o(h) \). Here all \( q_{ii}(t) = \lambda_i(t), i \geq 0, q_{ii-1}(t) = \mu_i(t) i \geq 1 \), and all other \( q_{ij}(t) \equiv 0 \).

The probabilistic dynamics of the process is represented by the forward Kolmogorov system of differential equations:

\[
\begin{align*}
\frac{dp_0}{dt} &= -\lambda_0(t)p_0 + \mu_1(t)p_1, \\
\frac{dp_k}{dt} &= \lambda_{k-1}(t)p_{k-1} - (\lambda_k(t) + \mu_k(t))p_k + \mu_{k+1}(t)p_{k+1}, \quad k \geq 1.
\end{align*}
\]

(2)

By \( p(t) = (p_0(t), p_1(t), \ldots) \top \), \( t \geq 0 \), we denote the column vector of state probabilities and by \( A(t) = (a_{ij}(t)), \ t \geq 0 \) the matrix related to (2). One can see that \( A(t) = Q^\top(t) \), where \( Q(t) \) is the intensity (or infinitesimal) matrix for \( X(t) \).

We assume that all birth and death intensity functions \( \lambda_i(t) \) and \( \mu_i(t) \) are linear combinations of a finite number of functions which are locally integrable on \([0, \infty)\). Moreover, we suppose that

\[
\lambda_n(t) \leq \Lambda_n \leq L < \infty, \quad \mu_n(t) \leq \Delta_n \leq L < \infty,
\]

(3)
for almost all $t \geq 0$. Throughout the paper by $\| \cdot \|$ we denote the $l_1$-norm, i.e. $\|x\| = \sum |x_i|$, and $\|B\| = \sup_j \sum_{i,j} |b_{ij}|$ for $B = (b_{ij})_{i,j=0}^{\infty}$.

Let $\Omega$ be a set all stochastic vectors, i.e. $l_1$ vectors with nonnegative coordinates and unit norm. Then we have

$$\|A(t)\| \leq 2 \sup(\lambda_k(t) + \mu_k(t)) \leq 4L,$$

for almost all $t \geq 0$. Hence the operator function $A(t)$ from $l_1$ into itself is bounded for almost all $t \geq 0$ and locally integrable on $[0; \infty)$.

Therefore we can consider the system (2) as a differential equation

$$\frac{dp}{dt} = A(t)p, \quad p = p(t), \quad t \geq 0, \quad (4)$$

in the space $l_1$ with bounded operator function $A(t)$.

It is well known (see, for instance, [2]) that the Cauchy problem for differential equation (1) has unique solutions for arbitrary initial condition, and moreover $p(s) \in \Omega$ implies $p(t) \in \Omega$ for $t \geq s \geq 0$.

Therefore, we can apply the general approach to employ the logarithmic norm of a matrix for the study of the problem of stability of Kolmogorov system of differential equations associated with nonhomogeneous Markov chains. The method is based on the following two components: the logarithmic norm of a linear operator and a special similarity transformation of the matrix of intensities of the Markov chain considered, see the corresponding definitions, bounds, references and other details in [4], [5], [13], [15], [17].

**Definition.** A Markov chain $X(t)$ is called weakly ergodic, if $\|p^*(t) - p^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $p^*(0), p^{**}(0)$.

Here $p^*(t)$ and $p^{**}(t)$ are the corresponding solutions of (4).

Let $E_k(t) = E \{ X(t) | X(0) = k \}$ (then the corresponding initial condition of system (4) is the $k-th$ unit vector $e_k$).

**Definition.** Let $X(t)$ be a Markov chain. Then $\varphi(t)$ is called the *limiting mean* of $X(t)$ if

$$\lim_{t \to \infty} \varphi(t) = E_k(t) = 0,$$

for any $k$.

**AUXILIARY NOTIONS AND RESULTS**

The property $p(t) \in \Omega$ for any $t \geq 0$ allows to put $p_i(t) = 1 - \sum_{j \neq i} p_j(t)$, for arbitrary fixed $i$. Then we obtain the following system from (4)

$$\frac{dz(t)}{dt} = B(t)z(t) + f(t), \quad (5)$$

where $z(t)$ is $p(t)$ without coordinate $p_i$, namely, $z(t) = (p_0, p_1, \ldots, p_{i-1}, p_{i+1}, \ldots)$. Respectively we have $f(t) = (0, 0, \ldots, \mu_i, \lambda_i, 0, \ldots)$, and $B(t)$ is the following matrix:

$$B(t) = \begin{pmatrix}
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
-\lambda_{i-2} - \mu_{i-2} - \mu_{i-1} - \lambda_{i-1} - \mu_i & 0 & \cdots & \cdots & 0 \\
0 & -\lambda_{i-3} - \mu_{i-3} - \mu_{i-2} - \lambda_{i-2} - \mu_{i-1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}.$$

Let $D^*$ be a matrix

$$D^* = \begin{pmatrix}
\alpha & 0 & 0 & \cdots & 0 \\
0 & \alpha & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \alpha \\
0 & 0 & \cdots & \cdots & 0 \\
\end{pmatrix}$$

then $D^*BD^{-1} =$

$$\begin{pmatrix}
-\lambda_0 - \lambda_1 & 0 & 0 & \cdots & 0 \\
0 & -\lambda_1 - \lambda_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -\lambda_{i-2} - \lambda_{i-1} - \lambda_i & 0 \\
0 & 0 & \cdots & 0 & -\lambda_i \\
\end{pmatrix}.$$
\[
z(t) = V(t, 0)z(0) + \int_0^t V(t, \tau)f(\tau)\,d\tau, \quad (8)
\]
where \(V(t, z)\) is the Cauchy operator of (5), see, for instance [13].

We have \(\|\mathbf{f}(t)\|_{L^D} = d_{i-1}\mu_i(t) + d_{i+1}\lambda_i(t) \leq d_{i-1}\Delta_i + d_{i+1}\Lambda_i\) for almost all \(t \geq 0\). On the other hand, if we put
\[
\beta_k(t) = \begin{cases}
\lambda_k(t) + \mu_{k+1}(t) + \frac{d_{k+1}}{\rho_k} \lambda_{k+1}(t) + \frac{d_{k+1}}{\rho_k} \mu_{k+1}(t), & k < i - 1 \\
\lambda_{i-1}(t) + \mu_{i}(t) + \frac{d_i}{\rho_{i-1}} \lambda_{i}(t) + \frac{d_i}{\rho_{i-1}} \mu_{i}(t), & k = i - 1 \\
\lambda_i(t) + \mu_{i+1}(t) + \frac{d_{i+1}}{\rho_{i+1}} \lambda_{i+1}(t) + \frac{d_{i+1}}{\rho_{i+1}} \mu_{i+1}(t), & k = i \\
\lambda_k(t) + \mu_{k+1}(t) + \frac{d_{k+1}}{\rho_k} \lambda_{k+1}(t) + \frac{d_{k+1}}{\rho_k} \mu_{k+1}(t), & k > i.
\end{cases}
\]
(9)

then one has
\[
\|B(t)\|_{L^D} = \sup_{k \geq 0} \beta_k(t) \leq 4L - \alpha(t),
\]
for almost all \(t \geq 0\).

Then \(f(t)\) and \(B(t)\) are bounded and locally integrable on \([0, \infty)\) as vector function and operator function in \(L^1_{\mathcal{D}}\) respectively.

Now we have the following bound for the logarithmic norm \(\gamma(B(t))\) in \(L^1_{\mathcal{D}}\):
\[
\gamma(B)_{L^1_{\mathcal{D}}} = \gamma \left( (DB(t)D^{-1})_t \right) = -\inf_{k \geq 0} (\alpha_k(t)) = -\alpha(t), \quad (10)
\]
in accordance with (7), see detailed discussion in our previous papers [4], [5], [15], [17].

Hence
\[
\|V(t, s)\|_{L^1_{\mathcal{D}}} \leq e^{-\int_s^t \alpha(\tau)\,d\tau}.
\]
(11)

Suppose now that there exist positive \(M\) and \(\alpha\) such that
\[
e^{-\int_s^t \alpha(\tau)\,d\tau} \leq Me^{-\alpha(t-s)}, \quad \text{for any } 0 \leq s \leq t.
\]
(12)

Then \(X(t)\) is exponentially weakly ergodic in \(L^1\) norm.

Put now \(z(0) = 0\) (i.e., \(p(0) = e_i\)). Then we have
\[
\|z(t)\|_{L^1_{\mathcal{D}}} \leq \|V(t, 0)\|_{L^1_{\mathcal{D}}} \|z(0)\|_{L^1_{\mathcal{D}}} +
\int_0^t \|V(t, s)\|_{L^1_{\mathcal{D}}} \|f(s)\|_{L^1_{\mathcal{D}}} ds \leq
\leq \frac{1}{2} M (d_{i-1}\Delta_i + d_{i+1}\Lambda_i) .
\]
(13)

On the other hand
\[
\|z\|_{L^1_{\mathcal{D}}} = (d_0 + \cdots + d_{i-1}) p_0 + (d_1 + \cdots + d_{i-1}) p_1 +
\cdots + d_{i-1}p_{i-1} + d_{i+1}p_i + (d_i + d_{i+2}) p_{i+2} + \cdots.
\]

Denote
\[
g_k = \sum_{j=k}^{i-1} d_j, \quad G_k = \sum_{j=i+1}^{k} d_j.
\]

Then we have
\[
\|z(t)\|_{L^1_{\mathcal{D}}} = \|Dz(t)\|_{L^1_{\mathcal{D}}} = \sum_{k < i} p_k(t) g_k +
\sum_{k \geq i} p_k(t) G_k \geq \begin{cases}
p_k(t) g_k, & k < i \\
p_k(t) G_k, & k \geq i
\end{cases}, \quad (14)
\]
Hence
\[
p_k(t) \leq \begin{cases}
\frac{M(d_{i-1}\Delta_i + d_{i+1}\Lambda_i)}{\alpha_k}, & k < i \\
\frac{M(d_{i-1}\Delta_i + d_{i+1}\Lambda_i)}{\alpha G_k}, & k \geq i
\end{cases}, \quad (15)
\]
for any \(k\).

## TWO-SIDED TRUNCATIONS

Consider truncated BDP on state space \(N_1, N_1 + 1, \ldots, N_2\) with intensities \(\lambda^*_{k}(t) = \lambda_k(t)\) if \(N_1 \leq k < N_2\), and \(\mu^*_{k}(t) = \mu_k(t)\) if \(N_1 < k \leq N_2\) and suppose other birth and death rates equal to zero. We will denote by \(A^*(t), p^*(t)\) and so on the correspondent characteristics of truncated BDP.

We have for the truncated process the following equation
\[
\frac{dp^*(t)}{dt} = A^*(t) p^*(t), \quad (16)
\]
instead of (4). Now, the property \(p^*(t) \in \Omega\) for any \(t \geq 0\) allows to put \(p^*_i(t) = 1 - \sum_{j \neq i} p_j^*(t)\), for arbitrary \(i\). Then we obtain from (16)
\[
\frac{dz^*(t)}{dt} = B^*(t) z^*(t) + f^*(t), \quad (17)
\]
instead of (5).

Rewrite (17) in the form:
\[
\frac{dz^*}{dt} = B(t) z^*(t) + (B^*(t) - B(t)) z^*(t) + f^*(t) . \quad (18)
\]
Then we have the following equality for the solutions of (5) and (18):
\[
\Phi(t) - \Phi^*(t) = \int_0^t V(t, s) (\Phi(s) - \Phi^*(s)) \, ds +
\int_0^t V(t, s) (f(s) - f^*(s)) \, ds . \quad (19)
\]

We will suppose that \(z(0) = z^*(0) = 0\) (i.e., \(p(0) = p^*(0) = e_i\) or \(X(0) = X^*(0) = i\)), where \(N_1 < i < N_2\). Then \(f(s) = f^*(s)\), for any \(s\).

Hence we have
\[
z(t) - z^*(t) = \int_0^t V(t, s) (B(s) - B^*(s)) z^*(s) \, ds , \quad (20)
\]
and

\[
((B(s) - B^*(s)) z^*(s)) = \\
(0, \cdots, 0, \mu_i N_i p_{N_i}, -\mu_i N_i p_{N_i}, 0, \cdots, \\
0, -\lambda N_i p_{N_i}, \lambda N_i p_{N_i}, 0, \cdots)^T.
\]

Let \( \{d^*_k\} \) be a sequence of positive numbers such that there exist positive \( M^* \) and \( \alpha^* \) such that

\[
-\int_s^t \alpha^*(\tau) \, d\tau \leq M^* e^{-\alpha^*(t-s)},
\]

for any \( 0 \leq s \leq t \), instead of (12), where

\[
\alpha^*(t) = \min \alpha^*_k(t),
\]

and

\[
\alpha^*_k(t) = \\
\left\{ \begin{array}{ll}
\lambda (t) + \nu k (t) - \frac{d^*_{k+1}}{\alpha^*} \lambda (t) + \nu k (t) - \frac{d^*_{k+1}}{\alpha^*} \lambda (t) + \nu k (t), & k < i - 1 \\
\lambda (t) + \nu k (t) - \frac{d^*_{k+1}}{\alpha^*} \lambda (t) + \nu k (t) - \frac{d^*_{k+1}}{\alpha^*} \lambda (t) + \nu k (t), & k = i - 1 \\
\lambda (t) + \nu k (t) - \frac{d^*_{k+1}}{\alpha^*} \lambda (t) + \nu k (t) - \frac{d^*_{k+1}}{\alpha^*} \lambda (t) + \nu k (t), & k = i \\
\lambda (t) + \nu k (t) - \frac{d^*_{k+1}}{\alpha^*} \lambda (t) + \nu k (t) - \frac{d^*_{k+1}}{\alpha^*} \lambda (t) + \nu k (t), & k > i.
\end{array} \right.
\]

Hence we obtain

\[
\| (B(s) - B^*(s)) z^*(s) \|_{1D} \leq \\
\| g_{N_i-1} + g_{N_i} | \mu N_i(s) p_{N_i}(s) + \\
| G_{N_i+1} + G_{N_i} | \lambda N_i(s) p_{N_i}(s) \| \\
\leq 2 g_{N_i-1} \Delta N_i p_{N_i}(s) + 2 G_{N_i+1} \Delta N_i p_{N_i}(s).
\]

Put \( g^*_k = \sum_{j=k}^{i-1} d^*_j \) and \( G^*_k = \sum_{j=k+1}^{i} d^*_j \).

Instead of (15) we have now

\[
p^*_k(t) \leq \left\{ \begin{array}{ll}
\frac{M^* (\Delta, d^*_i+1 + \Lambda i d^*_i)}{\alpha^* G^*_k}, & k < i \\
\frac{M^* (\Delta, d^*_i+1 + \Lambda i d^*_i)}{\alpha^* G^*_k}, & k > i.
\end{array} \right.
\]

Therefore (20), (25) and (26) imply the bound

\[
\| z(t) - z^*(t) \|_{1D} \leq \\
2 M^* (\Delta, d^*_i+1 + \Lambda i d^*_i) \\
\cdot \left( \frac{g_{N_i-1} \Delta N_i}{g_{N_i}^*} + \frac{G_{N_i+1} \Delta N_i}{G_{N_i}^*} \right).
\]

Let

\[
d = \min (d_{i-1}, d_i), \quad W = \inf_k \left( \frac{g_k}{k}, \frac{d}{i}, \frac{G_k}{i} \right).
\]

We have the inequalities:

\[
|p_i - p^*_i| \leq |p_0 - p^*_0| + |p_{i-1} - p^*_{i-1}| + \\
|p_{i+1} - p^*_{i+1}| + \cdots \leq \frac{1}{d} \| z(t) - z^*(t) \|_{1D},
\]

\[
\| p(t) - p^*(t) \| \leq \frac{2}{d} \| z(t) - z^*(t) \|_{1D},
\]

\[
2 \| z \|_{1D} \geq 1 \frac{d_1 + \cdots + d_{i-1}}{i} p_1 + \cdots + \\
(i - 1) \frac{d_{i-1}}{i - 1} p_i + \frac{d}{i} \ p_i + (i + 1) \frac{d_i}{i + 1} p_{i+1} + \\
(i + 2) \frac{d_{i+2}}{i + 2} p_{i+2} + \cdots \geq W \| p \|_{1E}.
\]

Theorem 2. Let birth-death processes \( X(t) \) and \( X^*(t) \) be such that (12) and (22) hold. Let \( p(0) = p^*(0) = \epsilon_i \) (i.e., \( X(0) = X^*(0) = i \)). Then the following bounds hold:

\[
\| p(t) - p^*(t) \| \leq \\
\frac{4 M^* (\Delta, d^*_i+1 + \Lambda i d^*_i)}{\alpha^* g_{N_i}^*},
\]

and

\[
\| p(t) - p^*(t) \|_{1E} \leq \\
\frac{4 M^* (\Delta, d^*_i+1 + \Lambda i d^*_i)}{\alpha^* g_{N_i}^*}.
\]

Corollary 1. Let under assumptions of Theorem 2 \( N_2 = \infty \). Then the following bounds hold

\[
\| p(t) - p^*(t) \| \leq \\
\frac{4 M^* (\Delta, d^*_i+1 + \Lambda i d^*_i) g_{N_i-1} \Delta N_i}{\alpha^* g_{N_i}^*},
\]

for any \( i > N_1 \).

Corollary 2. Let under assumptions of Theorem 2 \( N_1 = 0 \). Then the following bounds hold

\[
\| p(t) - p^*(t) \| \leq \\
\frac{4 M^* (\Delta, d^*_i+1 + \Lambda i d^*_i) g_{N_i+1} \Delta N_i}{\alpha^* G_{N_2}},
\]

for any \( i < N_2 \).
**EXAMPLES**

**Example 1.** Consider a birth-death process with periodic birth and death rates:

\[
\lambda_k (t) = \lambda (t) = 10 + \cos t
\]

\[
\mu_k (t) = 1 + \sin t, \ 0 < k < 1000
\]

\[
\mu_k (t) = 24 + \sin t, \ k \geq 1000.
\]

Let \( i = 1000. \)

Firstly we put

\[
\ldots, d_{997} = 1.5^2, d_{998} = 1.5, d_{999} = 1,
\]

\[
d_{1001} = 1.5, d_{1002} = 1.5^2, d_{1003} = 1.5^3, \ldots ;
\]

and

\[
\ldots, d^*_{997} = 4, d^*_{998} = 2, d^*_{999} = 1,
\]

\[
d^*_{1001} = 2, d^*_{1002} = 4, d^*_{1003} = 8, \ldots .
\]

Then we obtain

\[
M = 1, \alpha = \frac{1}{2}, M^* = 1, \alpha^* = \frac{1}{2};
\]

\[
d = 1.5, W = \frac{1.5}{1001};
\]

\[
\Lambda_i = 11, \Delta_i = 25.
\]

Hence Theorem 2 implies the following bounds:

\[
\| \mathbf{p} (t) - \mathbf{p}^* (t) \| \leq 10^{-12},
\]

\[
\| \mathbf{p} (t) - \mathbf{p}^* (t) \|_{1E} \leq 10^{-9},
\]

for \( N_1 = 350, \ N_2 = 1650. \)

Let now \( \{d^*_k\} \) be the same sequence, and let \( \{d_k\} \) be the such that

\[
\ldots, d_{997} = 1.1^2, d_{998} = 1.1, d_{999} = 1,
\]

\[
d_{1001} = 1.1, d_{1002} = 1.1^2, d_{1003} = 1.1^3, \ldots .
\]

Then one has

\[
M = 1, \alpha = \frac{1}{2}, d = 1.1,
\]

and the following bounds hold

\[
\| \mathbf{p} (t) - \mathbf{p}^* (t) \| \leq 10^{-14},
\]

\[
\| \mathbf{p} (t) - \mathbf{p}^* (t) \|_{1E} \leq 10^{-11},
\]

for \( N_1 = 800, \ N_2 = 1200. \)

**Example 2.** Consider a birth-death process with periodic birth and death rates:

\[
\lambda_k (t) = \lambda (t) = 10 + \cos t
\]

\[
\mu_k (t) = \mu (t) = 1 + \sin t, \ 0 < k < 10^6.
\]

\[
\mu_k (t) = 24 + \sin t, \ k \geq 10^6.
\]

Put \( i = 10^6. \)

We consider the similar sequences \( \{d_k\} \) and \( \{d^*_k\}. \)

Namely, let

\[
\ldots, d_{999997} = 1.5^2, d_{999998} = 1.5, d_{999999} = 1,
\]

\[
d_{1000001} = 1.5, d_{1000002} = 1.5^2, d_{1000003} = 1.5^3, \ldots ;
\]

and

\[
\ldots, d^*_{999997} = 4, d^*_{999998} = 2, d^*_{999999} = 1,
\]

\[
d^*_{1000001} = 2, d^*_{1000002} = 4, d^*_{1000003} = 8, \ldots .
\]

For the first sequence \( \{d_k\} \) we have \( d = 1.5 \) and the following bounds hold:

\[
\| \mathbf{p} (t) - \mathbf{p}^* (t) \| \leq 10^{-12},
\]

\[
\| \mathbf{p} (t) - \mathbf{p}^* (t) \|_{1E} \leq 10^{-6},
\]

if \( N_1 = 99350, \ N_2 = 100650. \)

Let \( \{d_k\} \) be the such that

\[
\ldots, d_{999997} = 1.1^2, d_{999998} = 1.1, d_{999999} = 1,
\]

\[
d_{1000001} = 1.1, d_{1000002} = 1.1^2, d_{1000003} = 1.1^3, \ldots .
\]

Then we have \( d = 1.1 \) and the following bound holds

\[
\| \mathbf{p} (t) - \mathbf{p}^* (t) \| \leq 10^{-14},
\]

\[
\| \mathbf{p} (t) - \mathbf{p}^* (t) \|_{1E} \leq 10^{-8},
\]

for \( N_1 = 90800, \ N_2 = 102000. \)

**Example 3.** Consider a birth-death process with large periodical birth and death rates:

\[
\lambda_k (t) = \lambda (t) = 1000 (10 + \cos t)
\]

\[
\mu_k (t) = 1000 (1 + \sin t), \ 0 < k < 10^3
\]

\[
\mu_k (t) = 1000 (24 + \sin t), \ k \geq 10^3.
\]

Let \( i = 10^3. \)

Consider the same sequences \( \{d_k\} \) and \( \{d^*_k\}. \) For the first sequence \( \{d_k\} \) we have \( d = 1.5 \) and

\[
\Lambda_i = 11000, \Delta_i = 25000, \Delta_N = 2000, \Lambda_N = 11000.
\]

Hence the following bounds hold:

\[
\| \mathbf{p} (t) - \mathbf{p}^* (t) \| \leq 10^{-6},
\]

\[
\| \mathbf{p} (t) - \mathbf{p}^* (t) \|_{1E} \leq 10^{-3},
\]

if \( N_1 = 350, \ N_2 = 1650. \)

For the second sequence \( \{d_k\} \) we have \( d = 1.1 \) and the bounds:

\[
\| \mathbf{p} (t) - \mathbf{p}^* (t) \| \leq 10^{-8},
\]

\[
\| \mathbf{p} (t) - \mathbf{p}^* (t) \|_{1E} \leq 10^{-5},
\]

for \( N_1 = 800, \ N_2 = 1200. \)
CONCLUSIONS

In this paper we considered a class of inhomogeneous birth-death queueing models and obtained uniform approximation bounds of two-sided truncations. Such approximations can be used in studying the information flows related to high-performance computing. The development of methodology for other classes of inhomogeneous Markovian queueing models seems to be a promising direction of research.

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