COMBINATION OF TIME-FREQUENCY REPRESENTATIONS FOR BACKGROUND NOISE SUPPRESSION

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continuous wavelet transform, time-varying autoregressive process, short time Fourier transform, time-frequency representation, noise suppression

ABSTRACT
The aim of the paper is to propose approach for enhancement of time-frequency representation leading to the background noise suppression. The approach is based on combination of continuous wavelet analysis, time-varying autoregressive process and short time Fourier transform. By such combination we make the identification of important areas in the time-frequency representation easier. The proposed method is an alternative approach to significance tests which can be problematic in some cases. The performance of methods is presented on the gross domestic product of the United Kingdom and Group of 7. The results show that in the UK, oil crisis has a bigger impact compared to financial crisis, while from the perspectives of G7 countries, the impact of financial crisis was stronger. The obtained results can be also used for consequent econometric analysis which identify dependencies, relations, bilateral causalities or other economic aspects.

INTRODUCTION
The need to analyse the data can be found across most disciplines. Despite the diversity of disciplines, it is a common goal to obtain the maximum information from data analysis that will help to solve the tasks set. With respect to the scientific area such data are given as observations in the form of time series or input signals. The common analytical instruments are given in time or frequency domain. The linking of both approaches giving us a more compact view can be done via time-frequency techniques (TF).

The literature includes many interesting papers from application in engineering (Stankovic et al. 2012; Liu et al. 2011), biology and medicine (Faust et al. 2015), or economics (Maršálek et al. 2014; Fititi et al. 2014).

The estimation of TF representation of the data can be done via several approaches. The widely used is short time Fourier transformation (STFT) (Proakis et al. 2002), the time-frequency varying autoregressive process (TFAR) (Proakis et al. 2002), multiple window method (Cakrak and Loughlin 2001) and wavelet analysis (Rajmic 2014). There are also alternative approaches such as modified empirical mode decomposition (Sebesta et al. 2013), the usage of Wigner-Vile distribution (Orovic and Stankovic 2009) or methods for more complicated multicomponent signals (Stankovic et al. 2012).

Suitable methods for the analysis of non-stationary signals is STFT, continuous wavelet transform (CWT), multiple windows method or TFAR. The TFAR is a simplification of the general autoregressive moving average model. The comparison of these main methods can be found in Blumenstein et al. (2012) or Klejmová (2015). The results shows that while CWT has better time resolution, the TFAR has better frequency resolution.

In most economic application the wavelet analyses predominates. The reason can be found in its simple usage for non-stationary signal and better time resolution (Jiang and Mahadevan 2011). The use of CWT for estimation of co/cross-spectra, the co-movement analysis is very popular. Such an approach put in evidence the existence of both long run and short-run co-movement. Aguiar-Conraria and Soares (2014) generalize the concept of simple coherency to partial wavelet coherency and multiple wavelet coherency akin to partial and multiple correlations. Berdiev and Chang (2015) took TF framework to examine the strength of business cycle synchronization. Fidrmuc et al. (2014) apply wavelet spectrum analysis to study globalization and business cycles in China and G7 countries. And Maršálek et al. (2014) proposed an original method based on CWT for filtering out the global shocks from the time series.

In order to have better predictive power it is suitable to support and validate the obtained results via some testing. The basic work is given by Torrence and Compo (1998). The paper presents statistical significance tests for wavelet power spectra are developed by deriving theoretical wavelet spectra for white and red noise processes and using these to establish significance levels and confidence intervals. Similar approach to Torrence and Campo can be found in Schulte et al. (2015) or Ge (2007). Ge derived the sampling distributions of the wavelet power and power spectrum of a Gaussian White Noise (GWN) in a rigorous statistical work. He proved that the results given by Torrence and Compo (1998) are...
numerically accurate when adjusted by a factor of the sampling period. The different approach to model validation is proposed by Jiang and Mahadevan (2011). The author uses testing via Monte Carlo (MC) simulations to infer whether the model prediction and experimental observation represents two coherent processes.

Significance tests proposed by Torrence and Compo (1998) or Ge (2007) require a priori knowledge of the noise character. As shown in Pomenkova (2017), MC results for CWT differs from Ge (2007) especially in case of heteroskerasticity in input data. Therefore, we propose combination of several TF methods as an alternative to these tests. In each method the background noise is depicted with different characteristics. However significant spectral components should be captured in most cases. Based on such an assumption we should be able to suppress the noise and highlight required components by using their combination. While in engineering background noise can be considered as the rest after removal periodic components (GWN, red noise etc.), in case of economic data the situation is not the same. Economic data can be viewed as a composition of several cyclical components which can occur in different time sub-period (not in whole time). The nature of an economic indicator plays an important role and can influence the character of nested cycles. In such way the background noise character is usually taken as a weakly stationary series and is obtained in dependence on analytical approaches.

The objective of the paper is propose approach for enhancement of TF representation leading to the background noise suppression. Thus, on the basis of the proposed method, we make the identification of important areas in the TF representation easier. The application of the proposed methodology on economic data allows easier interpretations from time and frequency perspectives. It can be also used for consequent macro/micro-econometric analysis of dependencies, relations with other economic aspects or analysis of bilateral causalities. The performance of methods is presented on the gross domestic product data of the United Kingdom and G7. These representatives were chosen because of available sample size and because they represent leading economies.

**METHODOICAL BACKGROUND**

**Continuous Wavelet transform (CWT)**

In order to describe the parameters of a signal not only in time but also in frequency domain, wavelet transform and its modifications can be used. One of these modifications, which is commonly used for assessing cyclical movements in different types of macroeconomics time series is continuous wavelet transform (CWT). It can be described as the integral of analysed signal with the base function (mother wavelet) (Walnut 2013):

\[
S_{CWT}(a, \tau) = \int_{-\infty}^{\infty} s(t) \frac{1}{\sqrt{a}} \psi \left( \frac{t - \tau}{a} \right) dt, \quad a > 0, \tau \in \mathbb{R},
\]

where \(s(t)\) is the time series, \(\psi \left( \frac{t}{a} - \tau \right)\) is a scaled version of the mother wavelet, \(\tau\) denotes the time shift, and \(a\) denotes the scale (or frequency) (Walnut 2013).

To be the invertible transform, basis (mother wavelets) functions must be mutually orthogonal, have zero mean value and limited to finite time interval. That is

\[
i) \quad \int_{-\infty}^{\infty} \psi \left( \frac{t}{a} - \tau \right) dt = 0, \\
ii) \quad \int_{-\infty}^{\infty} \psi^2 \left( \frac{t}{a} - \tau \right) dt = 1, \\
iii) \quad 0 < C_{\psi} = \int_{-\infty}^{\infty} |\Psi(\omega)|^2, \\
\Psi(\omega) = \int_{-\infty}^{\infty} \psi \left( \frac{t}{a} - \tau \right) e^{-i\omega t} dt,
\]

where \(\Psi(\omega)\) is the Fourier transform of \(\psi(\tau)\). To satisfy the assumptions for the time-frequency analysis, waves must be compact in time as well as in the frequency representation. There are several types of mother wavelets which can be used (e.g. Gaussian, Haar, Daubechies, Morlet etc.). In this paper, we use the complex Morlet wavelet (Walnut 2013):

\[
\psi(t) = \exp \left( -\frac{t^2}{2\sigma^2} \right) \exp(i\omega_0 t),
\]

where \(\sigma\) is a Gaussian window width in time and \(\omega_0\) is the central frequency of the wavelet. The complex Morlet wavelet is a substantially complex exponential modulated by a Gaussian envelope. In order to recalculate the local frequency, corresponding to the scale \(a\), the following equation can be used (Walnut 2013).

\[
\omega(\tau) = \frac{\omega_0}{a(\tau)},
\]

where \(\omega = 2\pi f\) and \(\tau\) is the time shift.

**Time-frequency varying AR process (TFAR)**

This method uses a parametric approach and creates a model generating an input signal. The analysed signal \(s(n)\) is then regarded as the output of a linear filter influenced by white noise \(w\) with variance \(\sigma_w^2\). The autoregressive process can be described by the AR\((p)\) model given by the equation

\[
s(n) = c + \sum_{i=1}^{p} a_i s_{n-i} + w_n,
\]

where \(a_i, i = 1, \ldots, p\) are the parameters of the autoregressive model of the order \(p\), \(c\) is a constant and \(w_n\) is white noise. The output spectrum can be described

\[
S(f) = |H \left( e^{j2\pi fT} \right)|^2 \sigma_w^2,
\]

where \(H \left( e^{j2\pi fT} \right)\) is a linear time variant filter. Thus the spectrum estimation, when we use the AR\((p)\) process, is done according to the formula (Proakis 2002)

\[
S_{AR}(f) = \frac{\hat{\sigma}_w^2}{\left| 1 + \sum_{i=1}^{p} \hat{a}_i e^{-j2\pi fT} \right|^2},
\]

where \(\hat{a}_i\) are estimates of the AR\((p)\) parameters and \(p\) is the lag order. Several methods for estimating AR\((p)\)
We selected the complex Morlet with center frequency range of 1 year to 10 years, with 257 individual scales. Our analysis consists of several steps. First, we analyse the data before this even which can be considered a structural break affecting the data. Additionally, because of the Brexit we were interested in setting of TF methods that would support the validation of proposed method. The data sets were chosen to be overlapping, and thus supporting the validation of proposed method. We use seasonally adjusted quarterly data of the gross domestic product (GDP), volume index in OECD reference year 2010 (OECD 2017) of the United Kingdom (UK) in 1956/01-2016/03 and Group of 7 (G7) in 1961/02-2016/03. All variables are in first differences of logarithms (Fig. 1), further they will be denoted as GDP. The motivation for the data selection was appropriate data sample size for testing the proposed method. We needed sufficient data range to have detailed time resolution. Both the UK and G7 data meet these requirements. Also, the data sets were chosen to be overlapping, and thus supporting the validation of proposed method. Additionally, because of the Brexit we were interested in analysing the data before this even which can be considered a structural break affecting the data.

Setting of TF methods
Our analysis consists of several steps. First, we analyse the data using CWT. We set scales to correspond to the range of 1 year to 10 years, with 257 individual scales. We selected the complex Morlet with center frequency $f_b = 1.5$ as mother wavelet (Poměnková and Klejmová 2015). The complex Morlet wavelet is based on the standard Morlet with the advantage of providing complex results making it possible to obtain a phase part (quadrature) of spectrum. In case of TF estimation via TFAR process we used Burg approach for coefficient estimates on 30 samples with 29 samples overlapping and with the Hann window. The optimal value of lag order was based on the AIC criteria (Klejmová 2015). The parameters of STFT were set to correspond to the TFAR settings (30 samples, 29 samples overlapping, Hann window) to simplify the process of combining the methods.

Combination of TF methods
Significance tests based on Ge (2007) require the knowledge of the noise character. This assumption can be broken when the data are heteroscedastic. Thereafter, Monte Carlo simulation for CWT can differs from Ge (2007) approach. To avoid this complication we suggest combination of several TF approaches as an alternative to significance tests. To obtain the best possible TF representation we combined results from the CWT, TFAR and STFT approach. Since the main focus was on the amplitude part of the spectra, we omitted the phase part of complex spectra $S_{CWT}$ and $S_{STFT}$. When focusing on the amplitude and phase components whole signal can be used for subsequent processing.

Firstly we align the time axis (time resolution) of spectral representations $S_{CWT}$, $S_{AR}$ and $S_{STFT}$ so that each spectrum corresponds to one another. All three time vectors have linearly increasing trend, therefore for the time axis alignment the only requirement was to adjust the starting and ending point for each method. We omitted the first and last 15 columns of $S_{CWT}$, we denoted this remaining matrix as $S'_{CWT}$. By doing this, we ensured corresponding the time axis for all three methods.

Secondly we needed to align the frequency/scale axis of $S'_{CWT}$, $S_{AR}$ and $S_{STFT}$. The frequency range of $S_{AR}$ and $S_{STFT}$ was cropped to correspond to the range of $S'_{CWT}$ which was 1 year to 10 years cycles. He resulting frequency/business cycles vectors $f_{AR}$ and $f_{STFT}$ had a linearly increasing trend, however, the trend of $f_{CWT}$ was non-linear. To obtain the corresponding vectors we matched each point of $f_{CWT}$ with one value of $f_{AR}/f_{STFT}$ with 1.4% tolerance:

$$|f_{CWT} - f_{STFT}| \leq 0.014 \max(f_{CWT}; f_{STFT})$$

$$|f_{CWT} - f_{AR}| \leq 0.014 \max(f_{CWT}; f_{AR})$$

With this step, we have gained the adjusted TF matrices $S'_{AR}$ and $S'_{STFT}$ making all three methods aligned. For the combination of methods we selected a simple multiplication (Klejmová and Poměnková 2017). We used combination of CWT and AR ($S'_{CWT,AR}$) and the combination of CWT, AR and STFT ($S'_{CWT,AR,STFT}$):

$$S_{CWT,AR} = S'_{CWT}S'_{AR}$$

$$S_{CWT,AR,STFT} = S'_{CWT}S'_{AR}S'_{STFT}.$$
Figure 1: GDP of UK and G7 in time domain

(a) UK

(b) G7

Figure 2: Spectrum of GDP of UK and G7 in time domain

(a) CWT UK

(b) CWT G7

(c) AR UK

(d) AR G7

(e) STFT UK

(f) STFT G7
There are four types of figures. Namely the time representation of GDP for the UK and G7 (Fig. 1a-b), TF transform via CWT (Fig. 2a-b), TF transform via AR (Fig. 2c-d), transformation via STFT (Fig. 2e-f) and the adjustment of CWT picture with the help of AR (Fig. 3a-b) and with the help of AR+STFT (Fig. 3c-d).

Focusing on the time representation given in Fig. 1a-b, we can conclude the following. In the time representation of the United Kingdom data, we can see two sub-periods with different volatility, between 1956-1988 and 1989-2015. There are also visible several moments with higher/lower levels of the data, i.e. structural breaks (1958, 1964, 1968, 1973, 1979 and 2008) given by events in the UK economy such as oil crises, financial crisis. In the case of G7, there is a similar problem with volatility, but it is not visible as in the UK. In contrast with the UK, the G7 data has a slowly decreasing trend with a higher volatility between 1961-1988. Afterwards (1989-2015) the data character looks similar to the UK. In G7 we can see similar structural breaks (1973, 1979 and 2008).

After a pilot analysis of time representation of the data, we applied TF approaches. Firstly we modelled CWT (Fig. 2a-b), consequently TFAR (Fig. 2c-d) and STFT (Fig. 2e-f). As expected, CWT provides results with a very good time resolution. We can see several important areas across time and frequency. Focusing on TFAR representation the results are not so clear from time perspective as CWT, but they give us better information from frequency perspective in a similar way to STFT. Therefore, we decided to adjust the CWT picture with the help of TFAR and TFAR+STFT according to the calculation (eq. 9,10) described in Combination of TF methods.

The results of the adjustments can be seen in Fig. 3a-d. Focusing on the UK situation and comparing the enhanced figure (Fig. 3a) with the CWT figure (Fig. 2a), we can see a sharper picture with suppressed noise. Therefore, we can easily identify the most important events in the UK data from the time and frequency perspectives. We can find three most important events in the UK. The first is between 1960-1988 and can be divided into two sub-periods; 1960-1972 and 1973-1988. These results correspond to the time domain description. In addition to the time domain, we find that such an event results in a reaction in approximately 4 years (in the first sub-period 1960-1972) and around 5 years (in the second sub-period 1972-1988) cycles. The second important can be identified between years 1973-1976 and results in an economic reaction in short cycles of approximately 1.5 year. The last important area is between 2007-2010. It cover business cycle frequencies (from 4 to 2 year frequencies) and if compared to the previous one, it does not seem to have such impact in the UK as the previous events. To confirm this conclusion and to validate the results, we add an additional adjustment of three TF approaches leading to Fig. 3c. The result confirms the conclusion results from the adjustment of CWT and TFAR and the fact that the events between 1970-1976 have a stronger impact on the UK economy.

If we focus on the results for G7, we can find some similarities as well as dissimilarities. Again, we can see (Fig. 3h) that the most important area is between 1970-1974 in 4 years cycles and 1974-1981 in 5 years cycles consisting one period 1970-1982. The second important area is between 2007-2010 (financial crisis) which covers the range of business cycles, i.e. 1.5-5 years cycles. After adding the second adjustment for the cross validation of the obtained results presented in Fig. 3d, we see conformity in identification of important area. Also in this case, to confirm and validate the results, we added adjustment of TF approaches leading to Fig. 3d. The result confirms conclusion from the adjustment of CWT and TFAR and the fact that the events between 1970-1976 have a stronger impact on G7 economy than the financial crisis in 2007-2009.

Comparing the results from the economy point of view, we can see, that in the UK, the oil crisis has a bigger impact than the financial crisis, while from the perspectives of G7 countries, the impact of financial crisis was stronger. The obtained results can be used for consequence macro or micro-econometric analysis searching for dependencies or relations with other economic aspects. Moreover, they can motivate researcher to investigate further steps, e.g. decomposition analysis on specific component of the corresponding frequency which can be used for analysing bilateral causalties.

If we review the results from a methodological point of view, we successfully adjusted CWT with others TF approaches and suppressed the background noise. Consequently, certain events occurred to be much more visible and, further, the time as well as frequency identification was easier. Other possible research is significance testing according to Torenc and Compo (1998) or Ge (2007) with the investigation of Monte Carlo simulation or with the investigation of background noise description.

**COMPARISON WITH SIGNIFICANCE TEST**

To assess the performance of the proposed approach (combinations of TF methods), we carried out a significance test described in Ge (2007). To do this we assumed the background noise character to be GWN. In the case of different noise character or heteroskedasticity data character, the Monte Carlo simulation may be carried out (for details see Pomenkova (2017)). Significant parts of CWT spectrograms based on Ge (2007) can be seen in Fig. 4a-b. When applying the proposed method for the UK case, we identify the most important area of approximately 4-5 years cycles during 1960-1988 and the second important area between 2007-2010, covering business cycles. In the case of G7, the Ge approach was able to identify the most important area between 1970-1974 in 4 years cycles, 1974-1981 in 5 years cycles and also area between 2007-2010 covering the range of business cycles. In both (UK and G7) cases, the Ge approach tends to omit spectral peaks with shorter fluctuations.
CONCLUSION

The presented paper deals with enhancement of time-frequency representation leading to the background noise suppression. The new approach is based on the combination of CWT, TFAR and STFT making it easier to identify important areas in the TF representation.

The methods were performed on the GDP of the United Kingdom and Group of 7. The results show that in the UK, the oil crisis has a bigger impact compare to the financial crisis, while from the perspectives of G7 countries, the impact of the financial crisis was stronger. From a methodological point of view, we successfully adjusted CWT with others TF approaches and enhanced the time-frequency picture of the UK and G7 GDP with the suppressed background noise.

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