LQ DIGITAL CONTROL OF BALL & PLATE SYSTEM

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ABSTRACT

This paper proposes the design of linear quadratic (LQ) digital controller for Ball & Plate model and 2DOF structure of the controller. Unknown parameters of the controller are determined with the help of polynomial approach to controller design. Semi-optimal solution is obtained using minimization of linear quadratic criterion. Spectral factorization with the aid of the Polynomial Toolbox for MATLAB was used for minimization of this LQ criterion. Additional poles of characteristic polynomial are placed so that the process is subtle and without sudden changes in controller output. Results have shown that the controller is able to stabilize the ball in desired position on the plate, reject external disturbances and follow reference path without much effort. Controller was designed for step changing and harmonic reference signal to further examine its capabilities.

INTRODUCTION

The Ball & Plate model is system with two inputs and two outputs. It has integrating properties, hence it can be considered unstable. This paper deals with controller design for this system using polynomial approach, because it simplifies the design problem to operations on algebraic polynomial (Diophantine) equations (Kučera 1993). Minimization of linear quadratic (LQ) criterion is used to derive controller parameters, which leads to semi-optimal solution (half of poles of characteristic polynomial have to be user-defined (Bobál et al. 2005)). This is particulary useful because it is quite challenging to place multiple user-defined poles. This process is applied to 2 degrees of freedom (2DOF) controller structure, which provides separation of feed-back part (responsible for stabilization and disturbance rejection) and feed-forward part (responsible for reference tracking) (Matušů and Prokop 2013). The PID/PSD control in closed-loop feedback structure was applied in (Jadlovská et al. 2009), where Butterworth, Graham-Lathrop and Naslin's methods were used for calculating controller parameters. A double feedback loop structure based on fuzzy logic is tested in (Wang et al. 2007). Fuzzy supervision and sliding control are proposed in (Moarref et al. 2008) and a non-linear switching is described in (Tian et al. 2006).
The paper is organized as follows. A brief description of mathematical model of the Ball & Plate structure is in Section 2. The design of LQ controller is shown in Section 3. Section 4 contains results of simulation and Section 5 concludes the paper.

BALL & PLATE MATHEMATICAL MODEL

A rough scheme of Ball & Plate model is presented in Figure 1. The derivation of system equations makes use of general form of Euler-Lagrange equation of the second kind (Rumyantsev 1994):

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \tag{1}$$

where T is kinetic energy of the system, V is potential energy, Q_i is i-th generalized force and q_i is i-th generalized coordinate. It is assumed that servomotor used for tilting the plate is described by first-order transfer function G_m with MATLAB units sent to servomotors circuit as input and actual angle of the plate as output:

$$G_m(s) = \frac{K_m}{\tau_m s + 1} \tag{2}$$

where K_m = 0.1878 and τ_m = 0.187 are gain and time constants of the motor respectively. These constants were obtained from real model's manual pages (Humusoft 2006).

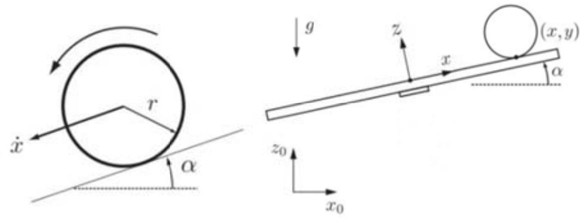

Figure 1: Ball & Plate scheme (Nokhbeh et al. 2011)

The system has only 2 generalized coordinates in total (ball position coordinates x and y), because plate angles are direct result of transfer function (2). Also the only external force acting on the system is gravitational force (friction is neglected for the sake of simplification). The

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derivation of specific equations from (1) is not the purpose of this paper, thus only final result will be presented. This result consists of a system of 2 ordinary second-order differential equations:

$$x: \left(m+\frac{I_b}{r^2}\right)\ddot{x}-m\left(\dot{\alpha}\dot{\beta}y+\dot{\alpha}^2x\right)+mg\sin\alpha=0 \quad (3)$$

$$y: \left(m+\frac{I_b}{r^2}\right)\ddot{y}-m\left(\dot{\alpha}\dot{\beta}x+\dot{\beta}^2y\right)+mg\sin\beta=0 \quad (4)$$

where m, r and I_b are mass, radius and moment of inertia of the ball respectively, g is gravitational acceleration, α and β are plate angles (α changes x coordinate and β changes y coordinate), $\dot{\alpha}$ and $\dot{\beta}$ are first time derivatives of plate angles, x and y are coordinates of the ball from center of the plate and $\ddot{x}$, $\ddot{y}$ are second time derivatives of ball coordinates.

Linearized Model

For small angles of the plate, one can write $\sin\alpha\approx\alpha$ and $\sin\beta\approx\beta$. It is also assumed that the rate of change in plate inclination is small around the linearization point, thus $\dot{\alpha}\dot{\beta}\approx 0$, $\dot{\alpha}^2\approx 0$ and $\dot{\beta}^2\approx 0$. The moment of inertia of a hollow sphere (spherical shell) can be ideally expressed as $I_b=\frac{2}{3}mr^2$. These simplifications applied to (3) and (4) result in

$$x: \quad \ddot{x}=K_b\alpha \quad (5)$$

$$y: \quad \ddot{y}=K_b\beta \quad (6)$$

where K_b is constant dependent only on the gravitational acceleration g and the type of ball. The two dimensional problem is considered to be symmetric (see (3) and (4)), thus it is possible to express the mathematical model (by merging (2) with (5) or (6)) in one continuous transfer function $G(s)$ with generalized coordinate as output $Y(s)$ and generalized angle as input $U(s)$:

$$G(s)=\frac{Y(s)}{U(s)}=\frac{K}{s^2\left(\tau_m s+1\right)}=\frac{K}{\tau_m s^3+s^2} \quad (7)$$

where $K = K_bK_mC_x$ is velocity gain of the integrating system ($C_x = 5\ m^{-1}$ is conversion coefficient from meters to normalized coordinates).
Equation (7) can be generaly discretized into:

$$G(z^{-1})=\frac{B(z^{-1})}{A(z^{-1})}=\frac{b_1z^{-1}+b_2z^{-2}+b_3z^{-3}}{1+a_1z^{-1}+a_2z^{-2}+a_3z^{-3}} \quad (8)$$

where $B(z^{-1})$ and $A(z^{-1})$ are polynomials with unknown coefficients. Because the Ball & Plate model has double integrator, discrete transfer function (8) can be simplified as follows:

$$G(z^{-1})=\frac{b_1z^{-1}+b_2z^{-2}+b_3z^{-3}}{(1-z^{-1})^2(1-c_1z^{-1})} \quad (9)$$

2DOF LQ CONTROLLER DESIGN

Control Law

The controller is designed for two degree of freedom (2DOF) closed-loop control system shown in Figure 2, where G is controlled plant, C_f and C_b are feed-forward and feed-back parts of the controller respectively, $1/K(z^{-1})=1/(1-z^{-1})$ is the summation part of the controller (it is extracted from denominators of C_f and C_b for practical reasons), $w(k)$ is reference signal, $y(k)$ is output of the system, $u(k)$ is output of the controller, $n(k)$ is load disturbance and $v(k)$ is disturbance signal. It is assumed that no disturbances act on the system. This is obviously not true for real system, but it simplifies the design and structure of the controller.

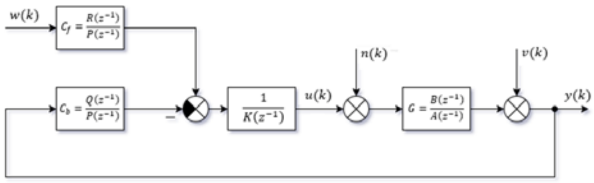

Figure 2: Structure of 2DOF controller

As mentioned, the controller is designed using polynomial approach. By taking signals from Figure 2 in their discrete forms (and omitting z^{-1} in polynomials' notation), one can write a relation between reference signal and output of the system:

$$Y(z^{-1})=\frac{BR}{AKP+BQ}W(z^{-1}) \quad (10)$$

The characteristic polynomial $D(z^{-1})$ can be extracted from (10) creating a Diophantine equation:

$$D=AKP+BQ \quad (11)$$

All polynomials in transfer functions will be called by their respective letter from now on, because omitting the term "(z^{-1})" will simplify the notation. Degree of polynomials Q, R and P can be obtained by determining the degree of the characteristic polynomial D, as described in (Bobál et al. 2005), from where it should be 6 for this specific case:

$$D=\sum_{i=0}^{6}d_iz^{-i} \quad (12)$$

Thus controllers C_b and C_f are

$$C_b(z^{-1})=\frac{Q}{P}=\frac{q_0+q_1z^{-1}+q_2z^{-2}+q_3z^{-3}}{1+p_1z^{-1}+p_2z^{-2}} \quad (13)$$

$$C_f(z^{-1})=\frac{R}{P}=\frac{r_0}{1+p_1z^{-1}+p_2z^{-2}} \quad (14)$$

where Q and P are polynomials with unknown coefficients, computed from (11) by method of undetermined coefficients. Polynomial R has one unknown coefficient r_0, which can be calculated for step-changing signal (see (Bobál et al. 2005)) as:

$$r_0 = \frac{d_0 + d_1 + d_2 + d_3 + d_4 + d_5 + d_6}{b_1 + b_2 + b_3} = \sum_{i=0}^{3} q_i \qquad (15)$$

In the case where reference signal is not step-changing, but harmonic, the polynomial R will be of higher degree and another Diophantine equation has to be solved:

$$SD_W + BR = D \qquad (16)$$

where S is an auxiliary polynomial not needed in controller parameters and D_w is denominator of harmonic reference signal $D_w = 1 - 2z^{-1}\cos(\omega T_0) + z^{-2}$, where ω is its angular frequency and T_0 is sampling period.

Minimization of LQ Criterion

A semi-optimal solution can be obtained by minimizing linear quadratic (LQ) criterion, which is closely described in (Bobál et al. 2005):

$$J = \sum_{k=0}^{\infty} \left\{ \left[e(k)\right]^2 + q_u \left[u(k)\right]^2 \right\} \qquad (17)$$

where $e(k) = w(k) - y(k)$ is error, $u(k)$ is controller output and q_u is penalization constant, which influences the controller output during minimization process.

According to (Bobál et al. 2005), this criterion can be minimized for input-output description of the model by applying spectral factorization on the following equation:

$$A(z^{-1}) q_u A(z) + B(z^{-1}) B(z) = D(z^{-1}) \delta D(z) \qquad (18)$$

where δ is chosen so that coefficient $d_0 = 0$ for the sake of simplification and $A(z)$, $B(z)$, $D(z)$ are conjugate polynomials of their respective counterparts. There is no analytical solution of spectral factorization for polynomials with degree 3 or higher, thus it has to be solved numerically by iterative methods (A is 3rd degree polynomial). The Polynomial Toolbox for MATLAB (Šebek 2014) contains tools for solving spectral factorization. The result of spectral factorization in this problem offers 3 roots of characteristic polynomial (12) that are optimal. Remaining 3 roots (poles) have to be user-defined. For a fully optimal solution, these poles can be placed to zero, but they are placed closer to a unit circle to make the controller more robust and its output properly bounded. Polynomial (12) can be now obtained and unknown coefficients of polynomial Q, P and R computed from (11) and (15) or (16).

RESULTS

It is important to note that controlled model in simulation was non-linear model described in (3) and (4). Its linearized form was used only for the design of the controller. Transfer function of the system is obtained after parameters K and τ_m are introduced into (7):

$$G(s) = \frac{K_b}{s^2} \frac{K_m}{\tau_m s + 1} C_x = \frac{-5.0706}{s^2 \left(0.187s + 1\right)} \qquad (19)$$

Transfer function (19) can be discretized for the sampling period $T_s = 0.1$s:

$$G(z^{-1}) = \frac{0.00396z^{-1} + 0.01394z^{-2} + 0.00304z^{-3}}{1 - 2.5871z^{-1} + 2.1743z^{-2} - 0.5871z^{-3}} \qquad (20)$$

The result of spectral factorization of (18) for $q_u = 1$ are 3 optimal poles $0.8477 \pm 0.1409i$ and 0.5821. User-defined poles were chosen to be 0.8, 0.8 and 0.8. Controller parameters for step-changing reference signal were calculated from (11) and (15) and substituted into (13) and (14):

$$C_b(z^{-1}) = \frac{-3.0311 + 7.4940z^{-1} - 6.0558z^{-2} + 1.5860z^{-3}}{1 - 1.1023z^{-1} + 0.3830z^{-2}} \qquad (21)$$

$$C_f(z^{-1}) = \frac{-0.0069}{1 - 1.1023z^{-1} + 0.3830z^{-2}} \qquad (22)$$

For harmonic reference signal (with period 5s), the polynomial R is 1st degree polynomial and feedforward part of the controller (22) has the following form:

$$C_f(z^{-1}) = \frac{0.0438 - 0.0529z^{-1}}{1 - 1.1023z^{-1} + 0.3830z^{-2}} \qquad (23)$$

Figure 3 and Figure 4 show step reference tracking capabilities of designed controller. Because the Ball & Plate system has integrating properties, the output of the controller is zero when the error is zero (although this would not be true if an unmeasurable load disturbance is present in the process – e.g. errors of motors).

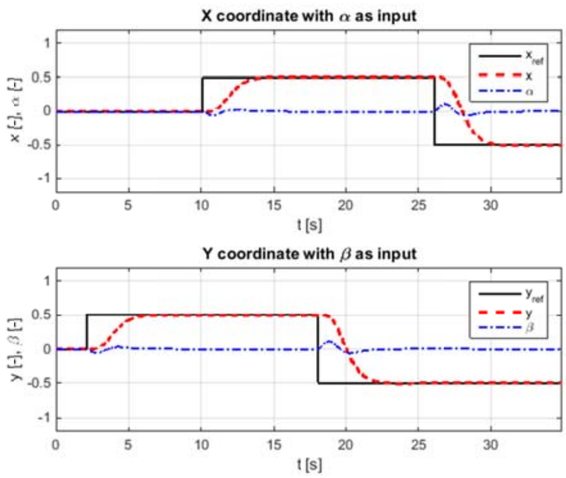

Figure 3: Step reference tracking

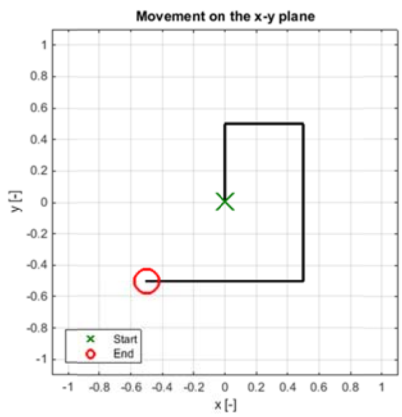

Figure 4: Step reference tracking on x-y plane

Figure 5 and Figure 6 show ability of controller to reject disturbances. Introduced disturbances were in the form of steps and it can be seen that the controller swiftly reacts to the disturbance and stabilizes the ball. Faster responses could lead to large changes in controller output, which is not appropriate in this kind of system.

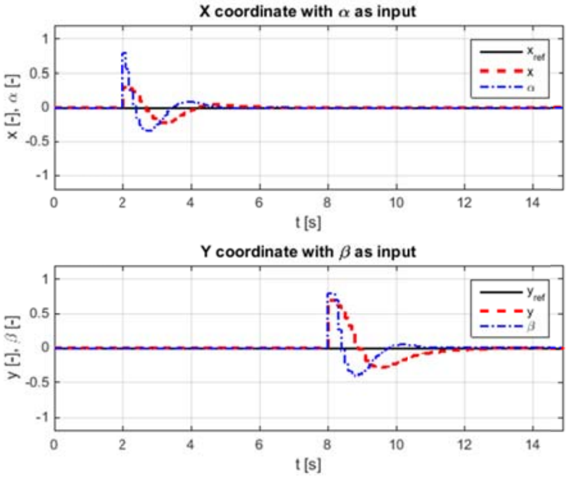

Figure 5: Step disturbance rejection

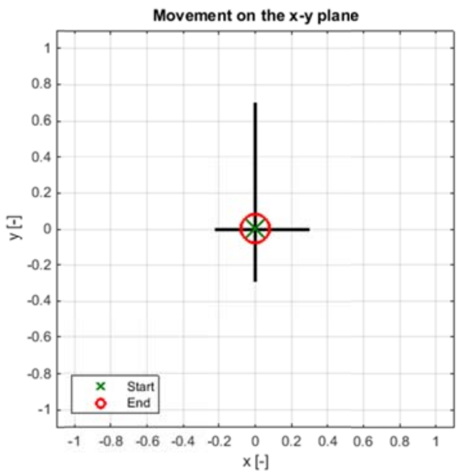

Figure 6: Step disturbance rejection on x-y plane

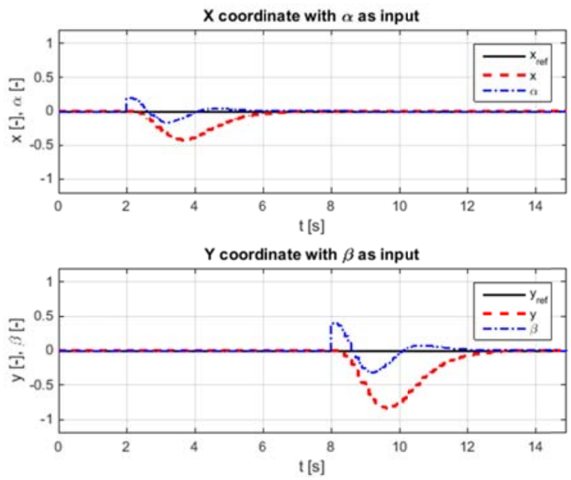

Figure 7: Step load disturbance rejection

Figure 7 and Figure 8 show rejection of disturbances applied directly to the output of the controller instead of ball's position, which simulates errors of controller. These load disturbances are also introduced as steps.

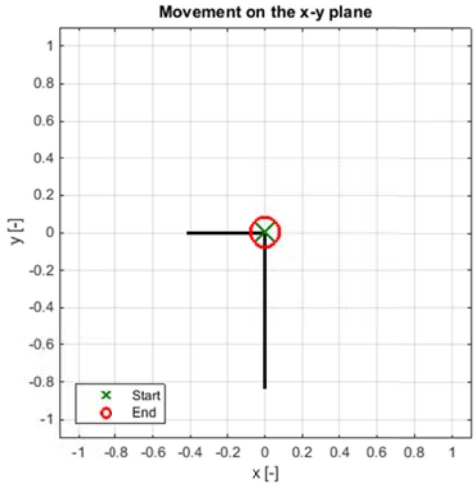

Figure 8: Step load disturbance rejection on x-y plane

Figure 9 and Figure 11 show circular reference tracking with controller designed for harmonic reference signal in (23). If the controller was designed only for step-changing reference signal, a phase lag would be present between reference harmonic signal and ball's position. Although the ball would still follow circular path for reference signal with low frequency. It would experience an amplitude reduction for higher frequencies of reference signal and the circular path would have smaller radius than desired.

A simple graphical user interface (GUI) was designed to provide more user-friendly control over the nonlinear model while testing control algorithms (Figure 10). It allows to choose the type of ball (sphere or spherical shell) and the type of reference value (manual point, circle, maze reference or custom). Relevant information is displayed in plots, which speeds up the design process and testing.

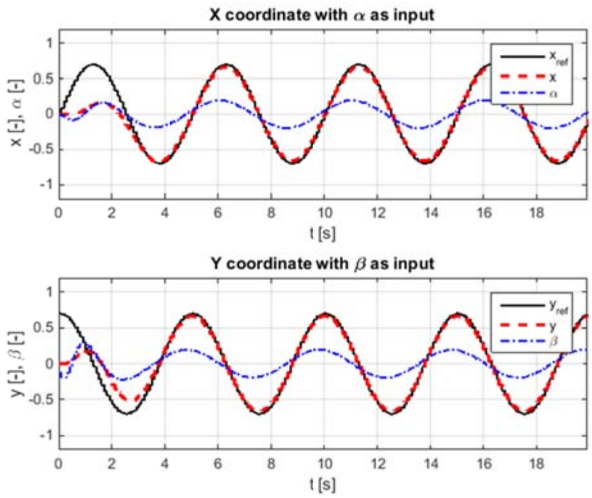

Figure 9: Circular reference tracking

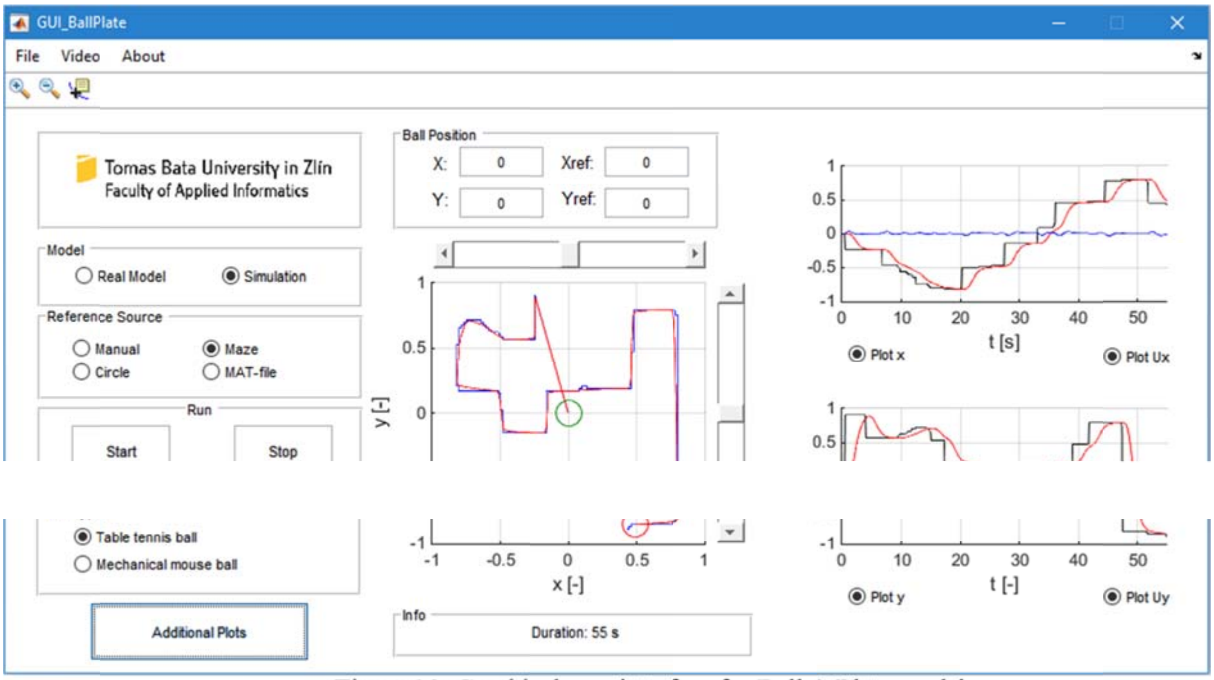

Figure 10: Graphical user interface for Ball & Plate model

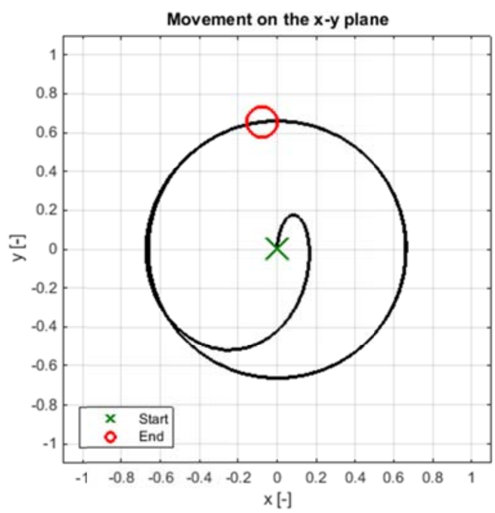

Figure 11: Circular reference tracking on x-y plane

CONCLUSION

The paper deals with design of linear quadratic (LQ) 2DOF controller for the Ball & Plate model. The controller was designed based on linearized mathematical model and polynomial approach for input/output form of the model. The presented method has been tested on computer simulation of nonlinear model of Ball & Plate structure. This model is quite sensitive to large changes in plate inclination (controller output). As a countermeasure, user-defined poles were placed in polynomial method algorithm near the unit circle, which resulted in subtle changes in plate inclination, but slowed the whole process. The minimization of LQ criterion provided rest of poles in an optimal solution, which successfully compensated system dynamics. The controller was designed for step-changing and also harmonic reference signal. It is able to reject disturbances acting on the system and successfully track desired reference value. A simple graphical user interface (GUI) was designed to act as a middlefinger between MATLAB/Simulink environment and the user.

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