BOUNDS FOR MARKOVIAN QUEUES WITH POSSIBLE CATASTROPHES

Alexander Zeifman
Vologda State University,
Vologda, Russia
IPI FRC CSC RAS;
ISEDT RAS
Email: a_zeifman@mail.ru

Anna Korotysheva,
Yacov Satin
Vologda State University
Vologda, Russia

Ksenia Kiseleva
RUDN University
Moscow, Russia
Vologda State University
Vologda, Russia

Victor Korolev
Moscow State University,
Moscow, Russia
IPI FRC CSC RAS,
Moscow, Russia
Hangzhou Dianzi University,
Hangzhou, China

Sergey Shorgin
Institute of Informatics Problems
of the FRC CSC RAS,
Moscow, Russia

KEYWORDS

Inhomogeneous birth-death processes; queueing models; bounds on the rate of convergence.

ABSTRACT

We consider a general Markovian queueing model with possible catastrophes and obtain new and sharp bounds on the rate of convergence. Some special classes of such models are studied in details, namely, (a) the queueing system with S servers, batch arrivals and possible catastrophes and (b) the queueing model with “attracted” customers and possible catastrophes. A numerical example illustrates the calculations. Our approach can be used in modeling information flows related to high-performance computing.

INTRODUCTION

There is a large number of papers devoted to the research of Markovian queueing models with possible catastrophes, see for instance, [1], [3], [2], [10], [11], [17], [18], [19], [21], [24], [25] and the references therein. Such models are widely used in simulations for hight-performance computing. In particular, in some recent papers the authors deal with more or less special birth-death processes with additional transitions from and to origin [1], [2], [3], [10], [11], [21], [24], [25]. In the present paper we consider a more general class of Markovian queueing models with possible catastrophes and obtain key bounds on the rate of convergence, which allow us to compute the limiting characteristics of the corresponding processes.

Namely, we suppose that the queue-length process is an inhomogeneous continuous-time Markov chain $\{X(t),\ t \geq 0\}$ on the state space $E = \{0, 1, 2 \ldots\}$. All possible transition intensities are assumed to be non-random functions of time and may depend on the state of the process. From any state i the chain can jump to any another state $j > 0$ with transition intensity $q_{ij}(t)$. Moreover, the transition functions from state $i > 0$ to state 0 (catastrophe intensities) are $\beta_i(t)$. Denote by $p_{ij}(s,t) = \mathsf{P}\{X(t) = j\,|X(s) = i\}$, $i, j \geq 0,\ 0 \leq s \leq t$ the probability of transition $X(t)$, and by $p_i(t) = \mathsf{P}\{X(t) = i\}$ the corresponding state probability that $X(t)$ is in state i at the moment t. Let $\mathbf{p}(t) = (p_0(t), p_1(t), \ldots)^T$ be the vector of state probabilities at the moment t.

Throughout the paper we suppose that for any i, j

$$\mathsf{P}(X(t+h) = j|X(t) = i) =$$

$$= \begin{cases} q_{ij}(t)\,h + \alpha_{ij}(t,h), & \text{if } j \neq i, \\ \beta_i(t)\,h + \alpha_{i0}(t,h) = q_{i0}(t) + \alpha_{i0}(t,h), & \text{if } j = 0,\ i > 1, \\ 1 - \sum_{j \neq i} q_{ij}(t)h + \alpha_i(t,h), & \text{if } j = i, \end{cases} \tag{1}$$

where

$$\sup_i |\alpha_i(t,h)| = o(h). \tag{2}$$

Let $Q(t)$ be the corresponding intensity matrix. We suppose that all intensity functions are non-negative and locally integrable on $[0, \infty)$.
Put $a_{ij}(t) = q_{ji}(t)$ for $j \neq i$ and

$$a_{ii}(t) = -\sum_{j \neq i} a_{ji}(t) = -\sum_{j \neq i} q_{ij}(t). \tag{3}$$

Proceedings 31st European Conference on Modelling and Simulation ©ECMS Zita Zoltay Paprika, Péter Horák, Kata Váradi, Péter Tamás Zwierczyk, Ágnes Vidovics-Dancs, János Péter Rádics (Editors)
ISBN: 978-0-9932440-4-9/ ISBN: 978-0-9932440-5-6 (CD)

As in our previous papers [9], [13], [16], [15], we suppose that

$$\sup_i |a_{ii}(t)| = L(t) < \infty, \tag{4}$$

for almost all $t \geq 0$.

Then one can write for $X(t)$ the forward Kolmogorov system

$$\frac{d\mathbf{p}(t)}{dt} = A(t)\,\mathbf{p}(t), \tag{5}$$

where $A(t) = Q^T(t)$ is a transposed intensity matrix.

Denote by $\|\cdot\|$ the l_1-norm of vector, $\|\mathbf{x}\| = \sum |x_i|$, $\|B\| = \sup_j \sum_i |b_{ij}|$, if $B = (b_{ij})_{i,j=0}^{\infty}$, and denote by Ω the set of all vectors from l_1 with nonnegative coordinates and unit norm.

We have$\|A(t)\| = 2\sup_k |a_{kk}(t)| \leq 2L(t)$ for almost all $t \geq 0$. Hence the operator function $A(t)$ from l_1 to itself is bounded for almost all $t \geq 0$ and locally integrable on interval $[0;\infty)$.

One can see that assumption (2) implies the equality $\mathbf{p}(t+h) = A(t)\mathbf{p}(t)h + \mathbf{p}(t) + o(h)$, hence the relation (5) can be considered as a differential equation in the space of sequences l_1 and one can apply to (5) the approach of [4].

Denote by $E(t,k) = E\{X(t)\,|X(0) = k\}$ the mathematical expectation (the mean) of $X(t)$ at the moment t if $X(0) = k$.

Definition. A Markov chain $X(t)$ is called *weakly ergodic*, if $\lim_{t\to\infty} \left\|\mathbf{p}^1(t) - \mathbf{p}^2(t)\right\| = 0$ for any initial conditions $\mathbf{p}^1(0) = \mathbf{p}^1 \in \Omega$, $\mathbf{p}^2(0) = \mathbf{p}^2 \in \Omega$. In this situation one can consider *any* $\mathbf{p}^1(t)$ as a *quasi-stationary distribution* of the chain $X(t)$.

Definition. A Markov chain $X(t)$ has the limiting mean $\phi(t)$, if $|E(t;k) - \phi(t)| \to 0$ as $t \to \infty$ for any k.

There are two approaches to the study of the rate of convergence of continuous-time Markov chains.

The first approach is based on the notion of the logarithmic norm of a linear operator function and the respective bounds of Cauchy operator, the detailed discussion see in [6], [9], [14]. Namely, if $B(t)$, $t \geq 0$ is a one-parameter family of bounded linear operators on a Banach space $\mathcal{B}$, then

$$\gamma(B(t))_{\mathcal{B}} = \lim_{h\to+0} \frac{\|I + hB(t)\| - 1}{h} \tag{6}$$

is called the logarithmic norm of the operator $B(t)$. If $\mathcal{B} = l_1$ then the operator $B(t)$ is given by the matrix $B(t) = (b_{ij}(t))_{i,j=0}^{\infty}$, $t \geq 0$, and the logarithmic norm of $B(t)$ can be found explicitly:

$$\gamma(B(t)) = \sup_j \left(b_{jj}(t) + \sum_{i\neq j} |b_{ij}(t)| \right), \quad t \geq 0.$$

Here we apply **the second approach,** see detailed consideration for the situation of finite state space in our recent paper [22], see also [6], [8].

A matrix is called *essentially nonnegative*, if all off-diagonal elements of this matrix are nonnegative.

Let

$$\frac{d\mathbf{x}}{dt} = H(t)\mathbf{x}(t), \tag{7}$$

be a differential equation in the space of sequences l_1 with essentially nonnegative for all $t \geq 0$ countable matrix $H(t) = (h_{ij}(t))$ such that the corresponding operator function on l_1 is bounded for almost all $t \geq 0$ and locally integrable on $[0,\infty)$.

Therefore $\mathbf{x}(s) \geq 0$ implies $\mathbf{x}(t) \geq 0$ for any $t \geq s$.

Put

$$h^*(t) = \sup_j \sum_i h_{ij}(t),\; h_*(t) = \inf_j \sum_i h_{ij}(t). \tag{8}$$

Let $\mathbf{x}(0) \geq 0$. Then $\mathbf{x}(t) \geq 0$ if $t \geq 0$ and $\|\mathbf{x}(t)\| = \sum_i x_i(t)$. Hence (7) implies the inequality

$$\frac{d\mathbf{x}(t)}{dt} = \frac{d\sum_i x_i(t)}{dt} = \sum_i \left(\sum_j h_{ij} x_j \right) =$$

$$= \sum_j \left(\sum_i h_{ij} \right) x_j \leq h^*(t) \sum_j x_j = h^*(t)\|\mathbf{x}\|.$$

Then

$$\|\mathbf{x}(t)\| \leq e^{\int_0^t h^*(\tau)d\tau} \|\mathbf{x}(0)\|, \tag{9}$$

if $\mathbf{x}(0) \geq \mathbf{0}$. Let now $\mathbf{x}(0)$ be arbitrary vector from l_1. Put $x_i^+(0) = \sup(x_i(0),0)$, $\mathbf{x}^+(0) = \left(x_1^+(0), x_2^+(0), \cdots\right)^T$ and $\mathbf{x}^-(0) = \mathbf{x}^+(0) - \mathbf{x}(0)$. Then $\mathbf{x}^+(0) \geq \mathbf{0}$, $\mathbf{x}^-(0) \geq \mathbf{0}$, $\mathbf{x}(0) = \mathbf{x}^+(0) - \mathbf{x}^-(0)$, hence $\|\mathbf{x}(0)\| = \|\mathbf{x}^+(0)\| + \|\mathbf{x}^-(0)\|$.

Finally we obtain the upper bound

$$\begin{aligned} &\|\mathbf{x}(t)\| = \\ &= \|\mathbf{x}^+(t) - \mathbf{x}^-(t)\| \leq \|\mathbf{x}^+(t)\| + \|\mathbf{x}^-(t)\| \leq \\ &\leq e^{\int_0^t h^*(\tau)d\tau} \left(\|\mathbf{x}^+(0)\| + \|\mathbf{x}^-(0)\|\right) = \\ &= e^{\int_0^t h^*(\tau)d\tau} \|\mathbf{x}(0)\|. \end{aligned} \tag{10}$$

for any initial condition.

On the other hand, if $\mathbf{x}(0) \geq 0$, then

$$\frac{d\mathbf{x}(t)}{dt} = \sum_j \left(\sum_i h_{ij} \right) x_j \geq h_*(t) \sum_j x_j = h_*(t)\|\mathbf{x}\|,$$

and we obtain the following lower bound

$$\|\mathbf{x}(t)\| \geq e^{\int_0^t h_*(\tau)d\tau}\|\mathbf{x}(0)\|, \tag{11}$$

for any nonnegative initial condition.

On sharpness of bounds.

Let us note that if the matrix of system (7) is essentially nonnegative for any t, then one can see that the logarithmic norm of this matrix is equal to our new characteristic, $\gamma(H(t)) = h^*(t)$.

Let $\{d_i\}$, $i \geq 0$ be a sequence of positive numbers such that $\inf_i d_i = d > 0$. Let $\mathsf{D} = diag(d_0, d_1, d_2, \dots)$ be the corresponding diagonal matrix and $l_{1\mathsf{D}}$ be a space of vectors

$$l_{1\mathsf{D}} = \{\mathbf{x} = (x_0, x_1, x_2, \dots)/\|\mathbf{x}\|_{1\mathsf{D}} = \|\mathsf{D}\mathbf{x}\|_1 < \infty\}. \tag{12}$$

Put $\mathbf{z}(t) = \mathsf{D}\mathbf{x}(t)$, then (7) implies the equation

$$\frac{d\mathbf{z}}{dt} = H_{\mathsf{D}}(t)\mathbf{z}(t), \tag{13}$$

where $H_{\mathsf{D}}(t) = \mathsf{D}H(t)\mathsf{D}^{-1}$ with entries $h_{ij\mathsf{D}}(t) = \frac{d_i}{d_j}h_{ij}(t)$ is also essentially nonnegative for any $t \geq 0$. If one can find a sequence $\{d_i\}$ such that

$$h^*_{\mathsf{D}}(t) = \sup_j \sum_i \frac{d_i}{d_j}h_{ij}(t) = \inf_j \sum_i \frac{d_i}{d_j}h_{ij}(t), \tag{14}$$

then the following *equality* holds

$$\|\mathbf{x}(t)\|_{\mathsf{D}} = e^{\int_0^t h^*_{\mathsf{D}}(\tau)d\tau}\|\mathbf{x}(0)\|_{\mathsf{D}}, \tag{15}$$

for any nonnegative initial condition. Therefore, the bound

$$\|\mathbf{x}(t)\|_{\mathsf{D}} \leq e^{\int_0^t h^*_{\mathsf{D}}(\tau)d\tau}\|\mathbf{x}(0)\|_{\mathsf{D}}, \tag{16}$$

which is correct for any initial condition, is *sharp*.

Note that the construction of such sequences for homogeneous birth-death processes has been studied previously in [5], [6], [7], [8], [12], [13].

ERGODICITY BOUNDS

Put

$$\beta_*(t) = \inf_i \beta_i(t), \quad \beta^*(t) = \sup_i \beta_i(t). \tag{17}$$

Theorem 1. Let catastrophe rates be essential, i. e. let

$$\int_0^\infty \beta_*(t)\, dt = +\infty. \tag{18}$$

Then the queue-length process $X(t)$ is weakly ergodic in the uniform operator topology and the following bound hold

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq$$
$$\leq e^{-\int_0^t \beta_*(\tau)\,d\tau}\|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\| \leq 2e^{-\int_0^t \beta_*(\tau)\,d\tau}, \tag{19}$$

for any initial conditions $\mathbf{p}^*(0), \mathbf{p}^{**}(0)$ and any $t \geq 0$.

Moreover, if $\mathbf{p}^*(0) - \mathbf{p}^{**}(0) \geq \mathbf{0}$, then

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \geq$$
$$\geq e^{-\int_0^t \beta_*(\tau)\,d\tau}\|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|, \tag{20}$$

for any any $t \geq 0$.

Proof. Rewrite the forward Kolmogorov system (5) as

$$\frac{d\mathbf{p}}{dt} = A^*(t)\mathbf{p} + \mathbf{g}(t), \quad t \geq 0. \tag{21}$$

Here $\mathbf{g}(t) = (\beta_*(t), 0, 0, \dots)^T$, $A^*(t) = \left(a^*_{ij}(t)\right)_{i,j=0}^\infty$, and

$$a^*_{ij}(t) = \begin{cases} a_{0j}(t) - \beta_*(t), & \text{if } i = 0, \\ a_{ij}(t), & \text{otherwise}. \end{cases} \tag{22}$$

The solution of this equation can be written in the form

$$\mathbf{p}(t) = U^*(t,0)\mathbf{p}(0) + \int_0^t U^*(t,\tau)\mathbf{g}(\tau)\, d\tau, \tag{23}$$

where $U^*(t,s)$ is the Cauchy operator of the corresponding homogeneous equation

$$\frac{d\mathbf{z}}{dt} = A^*(t)\mathbf{z}. \tag{24}$$

All off-diagonal elements of matrix $A^*(t)$ are nonnegative for any $t \geq 0$. Hence we can apply the approach of previous Section with $H(t) = A^*(t)$. Then we have

$$h^*(t) = \sup_i \left(a^*_{ii}(t) + \sum_{j \neq i} a^*_{ji}(t) \right) = -\beta_*(t), \tag{25}$$

hence, $\|U^*(t,s)\| \leq e^{-\int_s^t \beta_*(\tau)\,d\tau}$, and we obtain

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq \|U^*(t,0)\|\,\|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\| \leq \tag{26}$$
$$\leq e^{-\int_0^t \beta_*(\tau)\,d\tau}\|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\| \leq 2e^{-\int_0^t \beta_*(\tau)\,d\tau},$$

for any initial conditions $\mathbf{p}^*(0), \mathbf{p}^{**}(0)$ and any $t \geq 0$.

On the other hand, bound (20) follows from the inequality

$$h_*(t) = \inf_i \left(a^*_{ii}(t) + \sum_{j \neq i} a^*_{ji}(t) \right) = -\beta_*(t). \tag{27}$$

Now consider bounds in "weighted" norms. Let $\{d_i\}$, $1 = d_0 \leq d_1 \leq \dots$ be a non-decreasing sequence, and $\mathsf{D} = diag\,(d_0, d_1, d_2, \dots)$ be the corresponding diagonal matrix. Let $l_{1\mathsf{D}}$ be the space of vectors such that (12) holds.

Put

$$\beta_{**}(t) = \inf_i \left(|a^*_{ii}(t)| - \sum_{j\neq i} \frac{d_j}{d_i} a^*_{ji}(t) \right), \qquad (28)$$

and

$$\beta^{**}(t) = \sup_i \left(|a^*_{ii}(t)| - \sum_{j\neq i} \frac{d_j}{d_i} a^*_{ji}(t) \right). \qquad (29)$$

Consider (21) as a differential equation in the space of sequences $l_{1\mathsf{D}}$. We have

$$\|A^*(t)\|_{1\mathsf{D}} = \|\mathsf{D}A^*(t)\mathsf{D}^{-1}\| =$$
$$= \sup_i \left(|a^*_{ii}(t)| + \sum_{j\neq i} \frac{d_j}{d_i} a^*_{ji}(t) \right) \leq \qquad (30)$$
$$\beta_{**}(t) + 2\sup_i |a^*_{ii}(t)| \leq \beta_{**}(t) + 2L(t),$$

and $\|\mathbf{g}(t)\|_{1\mathsf{D}} = \beta_*(t)$, hence we can apply the same approach to equation (21) in the space $l_{1\mathsf{D}}$, and the equality

$$\gamma\left(A^*(t)\right)_{1\mathsf{D}} = \gamma\left(\mathsf{D}A^*(t)\mathsf{D}^{-1}\right) =$$
$$= \sup_i \left(a^*_{ii}(t) + \sum_{j\neq i} \frac{d_j}{d_i} a^*_{ji}(t) \right) = -\beta_{**}(t), \qquad (31)$$

implies the following statement.

Theorem 2. Let $\{d_i\}$, $1 = d_0 \leq d_1 \leq \dots$ be a non-decreasing sequence such that,

$$\int_0^\infty \beta_{**}(t)\,dt = +\infty. \qquad (32)$$

Then the following bound on the rate of convergence holds:

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1\mathsf{D}} \leq$$
$$\leq e^{-\int\limits_0^t \beta_{**}(\tau)\,d\tau} \|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1\mathsf{D}}, \qquad (33)$$

for any initial conditions $\mathbf{p}^*(0), \mathbf{p}^{**}(0)$ and any $t \geq 0$. Moreover, if $\mathbf{p}^*(0) - \mathbf{p}^{**}(0) \geq \mathbf{0}$, then

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1\mathsf{D}} \geq$$
$$\geq e^{-\int\limits_0^t \beta^{**}(\tau)\,d\tau} \|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1\mathsf{D}}, \qquad (34)$$

for any $t \geq 0$.

Let $l_{1E} = \{\mathbf{z} = (p_1, p_2, \dots)\}$ be a space of sequences such that $\|\mathbf{z}\|_{1E} = \sum_{k\geq 1} k|p_k| < \infty$. Put $W = \inf_{k\geq 1} \frac{d_k}{k}$. Then $W\|\mathbf{z}\|_{1E} \leq \|\mathbf{z}\|_{1\mathsf{D}}$.

Corollary 1. Let a sequence $\{d_i\}$ be such that (32) holds, and, let moreover $W > 0$. Then $X(t)$ has the limiting mean, say $\phi(t) = E(t, 0)$, and the following bound holds:

$$|E(t,j) - E(t,0)| \leq \frac{1+d_j}{W} e^{-\int\limits_0^t \beta_{**}(\tau)\,d\tau}, \qquad (35)$$

for any j and any $t \geq 0$.

We can use this approach and formula (23) for obtaining the bounds of state probabilities in the following way. Consider again the space of sequences $l_{1\mathsf{D}}$, and put $X(0) = 0$. Then $\mathbf{p}(0) = \mathbf{0}$ and we obtain

$$\sum_i d_i p_i(t) = \|\mathbf{p}\,(t)\,\| \leq$$
$$\leq \int_0^t \|U^*\,(t,\tau)\,\mathbf{g}\,(\tau)\,\|\,d\tau \leq \int_0^t \beta_*(\tau) e^{-\int\limits_\tau^t \beta_{**}(\tau)\,d\tau}, \quad (36)$$

in the $1\mathsf{D}$-norm. Hence

$$d_N \sum_{i\geq N} p_i(t) \leq \|\mathbf{p}\,(t)\,\| \leq \int_0^t \beta_*(\tau) e^{-\int\limits_\tau^t \beta_{**}(\tau)\,d\tau}, \quad (37)$$

$$\sum_{i\geq N} p_i(t) \leq d_N^{-1} \int_0^t \beta_*(\tau) e^{-\int\limits_\tau^t \beta_{**}(\tau)\,d\tau}, \qquad (38)$$

and we obtain the following statement.

Corollary 2. Let sequence $\{d_i\}$ be such that (32) holds. Then the following bound holds:

$$\sum_{i<N} p_i(t) \geq 1 - d_N^{-1} \int_0^t \beta_*(\tau) e^{-\int\limits_\tau^t \beta_{**}(\tau)\,d\tau}, \qquad (39)$$

if $X(0) = 0$ and any $t \geq 0$.

SPECIFIC QUEUEING SYSTEMS

1. Consider firstly the queueing system with S servers, batch arrivals and possible catastrophes, and suppose that the corresponding rate functions are the following:

$\lambda_k(t)$ is the intensity of arrival of a group of k customers to the queue,

$\mu_k(t) = \mu(t)\min(k, S)$ is the intensity of service of a customer if the current number of customers in the queue is k,

finally, $\beta_k(t)$ is the intensity of catastrophes if the current number of customers in the queue is k.

To simplify the formulas, we will assume all intensities $1-$periodic.

Firstly, if assumption (18) is fulfilled, then the queue-length process $X(t)$ is weakly ergodic in the uniform operator topology and bound of the rate of convergence (19) holds.

Bounds in weighted norms seem essentially more interesting.

1a. Let arrival rates be exponentially decreasing in k, namely, let there exist $r > 1$ such that $\lambda_k(t) = r^{-k}\lambda(t)$. Put $d_k = \delta^k$, where $1 < \delta < r$.

Then $\beta_{**}(t) = \inf \alpha_i(t)$, where $\alpha_i(t) = |a^*_{ii}(t)| - \sum_{j\neq i} \frac{d_j}{d_i} a^*_{ji}(t)$. We have

$$\alpha_0(t) = \beta_*(t) - \lambda(t)\frac{r(\delta-1)}{(r-1)(r-\delta)},$$

and

$$\alpha_i(t) = \beta_*(t) + \left(1-\delta^{-k}\right)(\beta_k(t) - \beta_*(t)) +$$

$$+ \left(1-\delta^{-1}\right)\mu_k(t) - \lambda(t)\frac{r(\delta-1)}{(r-1)(r-\delta)} \geq \alpha_0(t),$$

hence

$$\beta_{**}(t) = \alpha_0(t) = \beta_*(t) - \lambda(t)\frac{r(\delta-1)}{(r-1)(r-\delta)},$$

and $\int_0^1 \beta_{**}(t)\,dt > 0$ for sufficiently small $0 < \delta - 1$.

Finally, in this situation $X(t)$ is weakly ergodic in the corresponding $l_{1\mathsf{D}}$-norm and has the limiting mean for any service rate $\mu(t)$ and any S.

1b. Let arrival rates be decreasing in k more slowly, and let, however,

$$\sum_k k\lambda_k(t) \leq Q < \infty, \tag{40}$$

for any $t \in [0,1]$.

Put $d_k = \frac{N+k}{N}$, $k \geq 0$, where N is sufficiently large.

Then also $\beta_{**}(t) = \inf \alpha_i(t)$, where $\alpha_i(t) = |a^*_{ii}(t)| - \sum_{j\neq i} \frac{d_j}{d_i} a^*_{ji}(t)$. We have

$$\alpha_0(t) = \beta_*(t) - \lambda(t)\sum_k \left(\frac{N+k}{N} - 1\right) =$$

$$= \beta_*(t) - \sum_k \frac{k}{N}\lambda(t) \geq \beta_*(t) - \frac{Q}{N},$$

and

$$\alpha_i(t) \geq \alpha_0(t).$$

Therefore

$$\beta_{**}(t) = \alpha_0(t) \geq \beta_*(t) - \frac{Q}{N},$$

and $\int_0^1 \beta_{**}(t)\,dt > 0$ for sufficiently large N.

Finally, in this situation $X(t)$ is weakly ergodic in the corresponding $l_{1\mathsf{D}}$-norm and has the limiting mean for any service rate $\mu(t)$ and any S.

2. Consider now the queueing model with "attracted" customers and possible catastrophes.

Namely, we consider an analog of an inhomogeneous $M|M|S$ queue with catastrophes where customers may arrive to the queue only in groups of $k+1$ customers with intensity $a_{k,2k+1} = \lambda(t)$, if the length of the queue at this moment equals k,
$\mu_k(t) = \mu(t)\min(k,S)$ is the intensity of service of a customer, if the current number of customers in the queue is k,
finally, $\beta_k(t)$ is the intensity of catastrophes, if the current number of customers in the queue is k.

To simplify the formulas we will suppose all intensities 1- periodic.

Certainly, if assumption (18) is fulfilled, then the queue-length process $X(t)$ is weakly ergodic in the uniform operator topology and bound (19) of the rate of convergence holds.

Moreover, the limiting mean for this queue-length process exists under the simple additional assumption

$$\int_0^1 (\beta_*(t) - \lambda(t))\,dt > 0. \tag{41}$$

To check this claim put $d_k = k+1,\ k \geq 0$.

Then also $\beta_{**}(t) = \inf \alpha_i(t)$, where $\alpha_i(t) = |a^*_{ii}(t)| - \sum_{j\neq i} \frac{d_j}{d_i} a^*_{ji}(t)$. We have $\alpha_0(t) = \beta_*(t) - \lambda(t)$, and $\alpha_i(t) \geq \alpha_0(t)$, for any $i \geq 1$.

Therefore $\beta_{**}(t) = \alpha_0(t) = \beta_*(t) - \lambda(t)$, and $\int_0^1 \beta_{**}(t)\,dt > 0$ if (41) holds. Finally, in this situation $X(t)$ is weakly ergodic in the respective $l_{1\mathsf{D}}$-norm and has the limiting mean for any service rate $\mu(t)$ and any S.

EXAMPLE

Consider now a simple special model. Let $\lambda_k(t) = \frac{(3+\sin 2\pi t)}{4^k}$, $\mu_k(t) = (1+\cos 2\pi t)\min(k,5)$, $\beta_k(t) = \beta_*(t) = 2 - \sin 2\pi t$.

Put $\delta = 4/3$ and $d_k = \delta^k$. Then we have

$$\beta_{**}(t) = \alpha_0(t) = \beta_*(t) - \lambda(t)/6 = 1.5 - \frac{7}{6}\sin 2\pi t,$$

and $\int_0^1 \beta_{**}(t)\,dt = \beta^0_{**} = 1.5$.

Hence Theorem 2 implies the bound

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1\mathsf{D}} \leq Me^{-1.5t}\,\|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1\mathsf{D}}, \tag{42}$$

where $M \leq e^{\int_0^1 \frac{7}{6}|\sin 2\pi t|\,dt} \leq 2$.

Hence there exists a limiting 1-periodic regime, say $\pi(t)$. We can apply inequality (36) and obtain the following bounds for the solution of the system (21) with zero initial condition:

$$\|\pi(t)\|_{1\mathsf{D}} \leq \int_0^t \beta_*(\tau)e^{-\int\limits_\tau^t \beta_{**}(\tau)\,d\tau} \leq$$

$$\leq 6\int_0^t e^{-1.5(t-\tau)\,d\tau} \leq 4, \tag{43}$$

for any $t \geq 0$.

Therefore, $\limsup_{t\to\infty} \|\pi\,(t)\,\|_{1\mathsf{D}} \;\leq\; 4$, and the 1−periodicity of the limit regime implies the inequality $\|\pi\,(t)\,\|_{1\mathsf{D}} \leq 4$ for any t and *any* initial condition.

Now from (42) we have

$$\|\mathbf{p}^*(t) - \pi(t)\|_{1\mathsf{D}} \leq 2e^{-1.5t}\left(\|\mathbf{p}^*(0)\|_{1\mathsf{D}} + 4\right), \quad (44)$$

and particularly

$$\|\mathbf{p}^*(t) - \pi(t)\|_{1\mathsf{D}} \leq 2e^{-1.5t}\left(\left(\frac{4}{3}\right)^k + 4\right), \quad (45)$$

if $X(0) = k$.

Finally we can apply the approach of [20], [23] and find the appropriate truncations for $X(t)$. The corresponding plots of the limiting characteristics for the queue-length process are shown here.

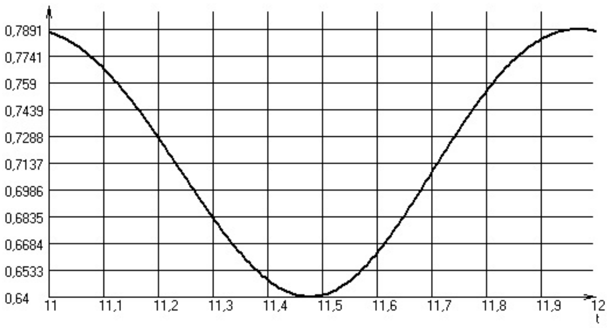

Fig. 1. Approximation of the limiting probability of empty queue $\mathsf{P}\{X(t) = 0|X(0) = 0\}$ on $[11, 12]$.

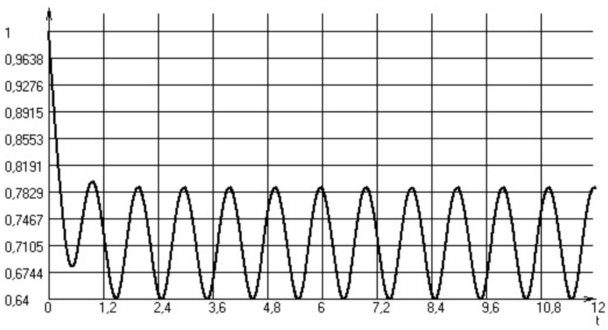

Fig. 2. Approximation of the probability of empty queue $\mathsf{P}\{X(t) = 0|X(0) = 0\}$ on $[0, 12]$.

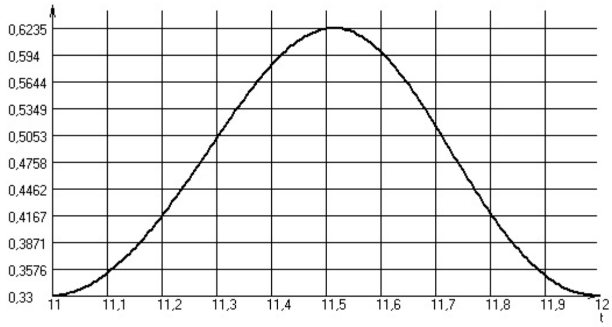

Fig. 3. Approximation of the limiting mean $E(t, 0)$ on $[11, 12]$.

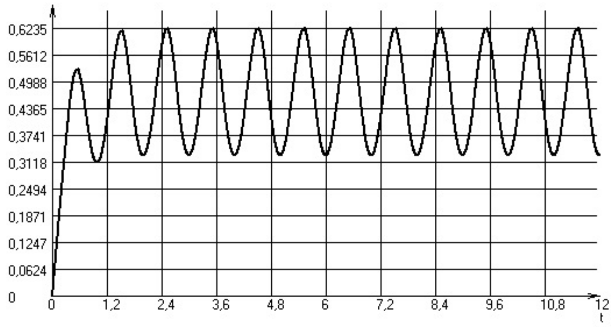

Fig. 4. Approximation of the mean $E(t, 0)$ on $[0, 12]$.

CONCLUSION

We consider a general Markovian queueing model with possible catastrophes and obtain new and sharp bounds on the rate of convergence. Some special classes of such models are studied in details, namely, (a) the queueing system with S servers, batch arrivals and possible catastrophes and (b) the queueing model with “attracted” customers and possible catastrophes. A numerical example illustrates the calculations. Our approach can be used in modeling information flows related to high-performance computing. Perturbation bounds and estimation of error of truncations will be studied in the next paper.

ACKNOWLEDGEMENTS

The work was supported by the Ministry of Education of the Russian Federation (the Agreement number 02.a03.21.0008 of 24 June 2016), by the Russian Foundation for Basic Research, project no. 15-01-01698, and by the RAS Presidium Program No. I.33P (project 0063-2016-0015).

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AUTHOR BIOGRAPHIES

ALEXANDER ZEIFMAN is Doctor of Science in physics and mathematics; professor, Heard of Department of Applied Mathematics, Vologda State University; senior scientist, Institute of Informatics Problems, Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences; principal scientist, Institute of Socio-Economic Development of Territories, Russian Academy of Sciences. His email is `a_zeifman@mail.ru`.

ANNA KOROTYSHEVA is Candidate of Science (PhD) in physics and mathematics, associate professor, Vologda State University. Her email is `a_korotysheva@mail.ru`.

YACOV SATIN is Candidate of Science (PhD) in physics and mathematics, associate professor, Vologda State University. His email is `yacovi@mail.ru`.

KSENIA KISELEVA is scientific researcher, RUDN University, Moscow, Russia; PhD student, Vologda State University. Her email is `ksushakiseleva@mail.ru`.

VICTOR KOROLEV is Doctor of Science in physics and mathematics, professor, Head of Department of Mathematical Statistics, Faculty of Computational Mathematics and Cybernetics, M.V. Lomonosov Moscow State University; leading scientist, Institute of Informatics Problems, Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences; professor, Hangzhou Dianzi University, Hangzhou, China. His email is `victoryukorolev@yandex.ru`.

SERGEY SHORGIN is Doctor of Science in physics and mathematics, professor, Deputy Director of the Institute of Informatics Problems of the Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences. His email is `sshorgin@ipiran.ru`.