INFINITE-SERVER QUEUEING TANDEM WITH MMPP ARRIVALS AND RANDOM CAPACITY OF CUSTOMERS

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ABSTRACT

Tandem of two queueing systems with infinite number of servers is considered. Customers arrive at the first stage of the tandem according to Markovian modulated Poisson process, and after a completion of their services, they go to the second stage. Each customer carries some data package with a random value (capacity of the customer). Service time does not depend on the customers' capacities in this study, capacities are used just to fix some additional characteristic of the system evolution. It is shown that two-dimensional probability distribution of total capacities at the stages of the system is two-dimensional Gaussian under the asymptotic condition of a high rate of arrivals. Presented numerical experiments and simulations allow to determine an applicability area of the asymptotic result.

INTRODUCTION

Infinite-server queuing systems are used as relevant models in some fields such as finance, insurance, etc. Furthermore, they may be applicable in case of models with a limited number of server devices as described in (Moiseev and Nazarov 2016a).

Queues with random capacities of customers are also useful for analysis and design problems in information and economic systems (Tikhonenko 1991; Tikhonenko and Klimovich 2001). In the case of information systems, the object under study is data received in the form of random-sized messages. In the case of economic systems, capacity of a customer refers to money that the customer pays to bank account. Also, such models are important in modeling of engineering devices where it is necessary to calculate a sufficient volume of buffer for data storing (Tikhonenko 2005 and 2015).

Results for single-server queues with limited buffer and LIFO service discipline were presented in the papers (Pechinkin 1998; Tikhonenko 2010). Algorithms for calculation of stationary characteristics were obtained for the models.

In the work (Cascone et al. 2011), the system $Geo_k/G/1/\infty$ was studied under a condition of limited total capacity of customers. In that paper, capacities were modeled as discrete random variables that allowed the authors to obtain simple and efficient algorithms for the calculation of basic stationary characteristics of the system evolution.

In the paper (Naumov et al. 2016), a multi-server queue with losses is considered. Losses in the model are caused by the lack of resources required for customers' service. A customer accepted for service takes a random amount of resources of several types according to given distribution functions. Under the assumption of Poisson arrivals and exponential service time, authors derived an asymptotical joint probability distribution of the number of customers in the system and a distribution of vector of occupied resources' volumes. There is an example which illustrates applying of the model for analysis of characteristics of video conference service in a wireless network LTE.

A new trend in the study of queuing systems is analysis of the systems with non-Poisson arrivals and non-exponential service time. So, in the works (Pankratova and Moiseeva 2014 and 2015; Moiseev and Nazarov 2016a), queues and networks with MAP and renewal arrivals are studied under various asymptotic conditions.

Tandem queues (Reich 1957) are the models with sequential processing of customers at the stages of the system. When a customer arrives at the system, it goes to the first stage of the tandem. There, it is serviced during a random period, and when the service is complete, it goes to the next stage, and so on, until its service will be completed at the last stage of the system.

The analytical results for the number of customers at the stages of the system were obtained in the papers (Moiseev and Nazarov 2014 and 2016b) for tandem queues with renewal and MAP arrivals and non-exponential service time. Analytical results about distributions of total capacities of customers at the stages of tandem queues for such models are not obtained yet.

The goal of the paper is to study total capacities' volume at the stages of the tandem system with incoming Markovian modulated Poisson process, two stages with infinite number of servers and non-exponential service time distribution.

MATEMATICAL MODEL

Consider a queue tandem with two stages and infinite number of servers at each stage. Customers arrive according to Markovian modulated Poisson process (MMPP). The process is given by generator matrix

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$\mathbf{Q}=\|q_{ij}\|$ of size $K\times K$ and conditional rates $\lambda_1,...,\lambda_K$ which we compose into a diagonal matrix $\mathbf{\Lambda}=\mathrm{diag}\{\lambda_1,...,\lambda_K\}$. Denote the underlying Markov chain of the MMPP as $k(t)\in 1,2,...,K$. Let each customer has some random capacity $\upsilon>0$ with distribution function $G(y)$. Arriving customer instantly occupies a server at the first stage of the system. Service time at this stage has distribution function $B_1(x)$. When the service is complete, the customer moves to the second stage for the further service. Service time at the second stage has distribution function $B_2(x)$. When the service is complete at the second stage, the customer leaves the system. Customers' capacities, service times are not dependent on each other and are not dependent on epochs of customers' arrivals.

Denote the number of customers at the first and at the second stages of the system at a moment t by $i_1(t)$ and $i_2(t)$, and denote the total capacities of all customers at the first and at the second stages by $V_1(t)$ and $V_2(t)$ respectively. Let us obtain probabilistic characteristics of multi-dimensional process $\{i_1(t), V_1(t), i_2(t), V_2(t)\}$. This process is not Markovian, therefore, we use the dynamic screening method (Moiseev and Nazarov 2016a) for its investigation.

Consider three time axes that are numbered from 0 to 2 (Fig. 1). Let axis 0 show the epochs of customers' arrivals. Axes 1 and 2 correspond to the stages of the system.

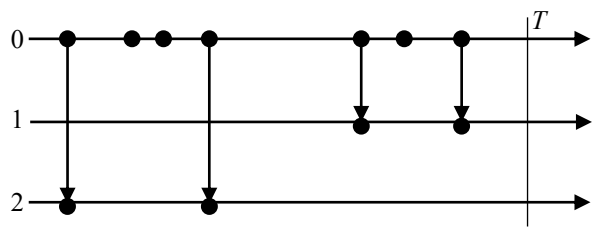


Figure 1: Screening of the Customers' Arrivals

We introduce a set of two functions $S_1(t)$, $S_2(t)$ (dynamic probabilities) that satisfy the conditions

$$0\le S_1(t)\le 1,\quad 0\le S_2(t)\le 1,\quad S_1(t)+S_2(t)\le 1\,.$$

We assume that an epoch t of customer's arrival may be screened to axis 1 with the probability $S_1(t)$, or to axis 2 with the probability $S_2(t)$, or may be not screened anywhere with the probability $S_0(t) = 1 - S_1(t) - S_2(t)$.

Let the system be empty at moment t_0, and let us fix some arbitrary moment T in the future. We will use $S_1(t)$ as the probability that a customer arrived at the moment t will be serviced at the first stage at the moment T, and $S_2(t)$ as the probability that a customer arrived at the moment t will be serviced at the second stage of the system at the moment T. It is shown (Moiseev and Nazarov 2014) that

$$S_1(t) = 1 - B_1(T-t),\quad S_2(t) = B_1(T-t) - B_2^*(T-t)$$

for $t_0 \le t \le T$, where

$$B_2^*(\tau)=(B_1*B_2)(\tau)=\int_0^\tau B_2(\tau-y)dB_1(y)$$

is a convolution of functions $B_1(x)$ and $B_2(x)$.

Denote the number of arrivals screened before the moment t on axes 1 and 2 by $n_1(t)$ and $n_2(t)$, and denote the total capacities of customers screened on axis 1 and 2 by $W_1(t)$ and $W_2(t)$ respectively.

As it is shown in (Moiseev and Nazarov 2016b), the multi-dimensional joint probability distribution of the number of customers at the stages of the tandem system at the moment T coincides with multi-dimensional joint probability distribution of the number of screened arrivals on respective axes:

$$P\{i_1(T)=m_1,i_2(T)=m_2\}=P\{n_1(T)=m_1,n_2(T)=m_2\}$$

for all $m_1, m_2 = 0, 1, 2, \ldots$ It is easy to prove the same property for extended process $\{i_1(t), V_1(t), i_2(t), V_2(t)\}$:

$$P\{i_1(T)=m_1,V_1(T)<z_1,i_2(T)=m_2,V_2(T)<z_2\}=$$
$$P\{n_1(T)=m_1,W_1(T)<z_1,n_2(T)=m_2,W_2(T)<z_2\}\quad(1)$$

for all $m_1, m_2 = 0, 1, 2, \ldots$ and $z_1, z_2 \ge 0$. We use Equalities (1) for investigation of the process $\{i_1(t), V_1(t), i_2(t), V_2(t)\}$ via analysis of the process $\{n_1(t), W_1(t), n_2(t), W_2(t)\}$.

KOLMOGOROV DIFFERENTIAL EQUATIONS

Let us consider the five-dimensional Markovian process $\{k(t), n_1(t), W_1(t), n_2(t), W_2(t)\}$. Denoting the probability distribution of this process by $P(k,n_1,w_1,n_2,w_2,t)=$ $P\{k(t)=k,n_1(t)=n_1,W_1(t)<w_1,n_2(t)=n_2,W_2(t)<w_2\}$ and taking into account the formula of total probability, we can write the following system of Kolmogorov differential equations:

$$\frac{\partial P(k,n_1,w_1,n_2,w_2,t)}{\partial t}=$$
$$-\lambda_k(S_1(t)+S_2(t))P(k,n_1,w_1,n_2,w_2,t)+$$
$$\lambda_k S_1(t)\int_0^{w_1}P(k,n_1-1,w_1-y,n_2,w_2,t)dG(y)+$$
$$\lambda_k S_2(t)\int_0^{w_2}P(k,n_1,w_1,n_2-1,w_2-y,t)dG(y)+$$
$$\sum_{\nu}q_{\nu k}P(\nu,n_1,w_1,n_2,w_2,t)$$

for $k = 1\ldots K$, $n_1, n_2 = 0,1,2,\ldots$, $w_1, w_2 > 0$.

We introduce the partial characteristic function:

$$h(k,u_1,v_1,u_2,v_2,t)=$$
$$\sum_{n_1=0}^{\infty}e^{ju_1n_1}\int_0^{\infty}e^{jv_1w_1}\sum_{n_2=0}^{\infty}e^{ju_2n_2}\int_0^{\infty}e^{jv_2w_2}P(k,n_1,dw_1,n_2,dw_2,t),$$

where $j=\sqrt{-1}$ is an imaginary unit. Then we can write the following equations:

$$\frac{\partial h(k,u_1,v_1,u_2,v_2,t)}{\partial t}=\lambda_k h(k,u_1,v_1,u_2,v_2,t)\cdot$$

$$\left[S_1(t)\left(e^{ju_1}G^*(v_1)-1\right)+S_2(t)\left(e^{ju_2}G^*(v_2)-1\right)\right]+$$
$$\sum_{\nu} q_{\nu k} h(\nu, n_1, w_1, n_2, w_2, t)$$

for $k = 1\ldots K$, where $G^*(v)=\int_0^\infty e^{jvy}dG(y)$.

Let us rewrite this system in the matrix form:

$$\frac{\partial \mathbf{H}(u_1,v_1,u_2,v_2,t)}{\partial t}=\mathbf{H}(u_1,v_1,u_2,v_2,t)\cdot$$
$$\left[\mathbf{\Lambda}\left(S_1(t)\left(e^{ju_1}G^*(v_1)-1\right)+S_2(t)\left(e^{ju_2}G^*(v_2)-1\right)\right)+\mathbf{Q}\right] \quad (2)$$

with the initial condition $\mathbf{H}(u_1,v_1,u_2,v_2,t_0)=\mathbf{r}$, where

$$\mathbf{H}(u_1, v_1, u_2, v_2, t) =$$
$$[h(1, u_1, v_1, u_2, v_2, t), \ldots, h(K, u_1, v_1, u_2, v_2, t)],$$

and $\mathbf{r} = [r(1), \ldots, r(K)]$ is a vector of the stationary distribution of the underlying Markov chain. Vector $\mathbf{r}$ satisfies the following linear system:

$$\begin{cases}\mathbf{rQ}=\mathbf{0},\\ \mathbf{re}=1,\end{cases} \quad (3)$$

where $\mathbf{e}$ is a column vector with all entries equal to 1.

ASYMPTOTIC ANALYSIS

The exact solution of Equation (2) is not possible in general case, but it may be solved under an asymptotic condition. In the paper, we consider the asymptotic condition of an infinitely growing arrivals' rate. Let us substitute $\mathbf{\Lambda} = N\mathbf{\Lambda}_1$ and $\mathbf{Q} = N\mathbf{Q}_1$ into Equation (2), where N is some parameter which is used for the asymptotic analysis ($N \to \infty$ in theoretical studies). Then Equation (2) may be rewritten as follows:

$$\frac{1}{N}\frac{\partial \mathbf{H}(u_1,v_1,u_2,v_2,t)}{\partial t}=\mathbf{H}(u_1,v_1,u_2,v_2,t)[\mathbf{Q}_1+$$
$$\mathbf{\Lambda}_1\left(S_1(t)\left(e^{ju_1}G^*(v_1)-1\right)+S_2(t)\left(e^{ju_2}G^*(v_2)-1\right)\right)] \quad (4)$$

with the initial condition

$$\mathbf{H}(u_1,v_1,u_2,v_2,t_0)=\mathbf{r}\,. \quad (5)$$

We solve Problem (4)–(5) under the asymptotic condition and we obtain a solution in the form of approximations which are named as "first-order asymptotic" $\mathbf{H}(u_1,v_1,u_2,v_2,t)\approx\mathbf{H}^{(1)}(u_1,v_1,u_2,v_2,t)$ and "second-order asymptotic" $\mathbf{H}(u_1,v_1,u_2,v_2,t)\approx\mathbf{H}^{(2)}(u_1,v_1,u_2,v_2,t)$. These approximations have different order of accuracy.

First-order Asymptotic Analysis

We formulate and prove the following statement.

Lemma. The first-order asymptotic characteristic function of the probability distribution of the process $\{k(t), n_1(t), W_1(t), n_2(t), W_2(t)\}$ has the form

$$\mathbf{H}^{(1)}(u_1,v_1,u_2,v_2,t)=\mathbf{r}\exp\left\{N\kappa_1\left[(ju_1+jv_1a_1)\int_{t_0}^{t}S_1(\tau)d\tau+\right.\right.$$
$$\left.\left.(ju_2+jv_2a_1)\int_{t_0}^{t}S_2(\tau)d\tau\right]\right\},$$

where $\kappa_1=\mathbf{r}\mathbf{\Lambda}_1\mathbf{e}$, and $a_1=\int_0^\infty ydG(y)$ is the mean of a customer capacity.

Proof.

Let us perform the substitutions

$$\varepsilon=\frac{1}{N},\ u_1=\varepsilon x_1,\ v_1=\varepsilon y_1\ \ u_2=\varepsilon x_2,\ v_2=\varepsilon y_2,$$
$$\mathbf{H}(u_1,v_1,u_2,v_2,t)=\mathbf{F}_1(x_1,y_1,x_2,y_2,t,\varepsilon)$$

in Expressions (4) and (5). Using such substitutions allows us to exclude a direct influence of an asymptotic parameter from the variables x_1, x_2, y_1, y_2. Then we obtain the following equation

$$\varepsilon\frac{\partial \mathbf{F}_1(x_1,y_1,x_2,y_2,t,\varepsilon)}{\partial t}=\mathbf{F}_1(x_1,y_1,x_2,y_2,t,\varepsilon)\cdot \quad (6)$$
$$\left\{\mathbf{Q}_1+\mathbf{\Lambda}_1\left[S_1(t)\left(e^{j\varepsilon x_1}G^*(\varepsilon y_1)-1\right)+S_2(t)\left(e^{j\varepsilon x_2}G^*(\varepsilon y_2)-1\right)\right]\right\}$$

with initial condition

$$\mathbf{F}_1(x_1,y_1,x_2,y_2,t_0,\varepsilon)=\mathbf{r}\,. \quad (7)$$

Let us find the asymptotic solution of Problem (6)–(7) $\mathbf{F}_1(x_1,y_1,x_2,y_2,t)=\lim_{\varepsilon\to 0}\mathbf{F}_1(x_1,y_1,x_2,y_2,t,\varepsilon)$ in two steps.

Step 1. Substituting $\varepsilon = 0$ in (6), we obtain

$$\mathbf{F}_1(x_1,y_1,x_2,y_2,t)\mathbf{Q}_1=\mathbf{0}\,.$$

Comparing this equation with the first one in (3), we can conclude that $\mathbf{F}_1(x_1,y_1,x_2,y_2,t)$ can be expressed as

$$\mathbf{F}_1(x_1,y_1,x_2,y_2,t)=\mathbf{r}\Phi_1(x_1,y_1,x_2,y_2,t), \quad (8)$$

where $\Phi_1(x_1,y_1,x_2,y_2,t)$ is some scalar function which satisfies the condition

$$\Phi_1(x_1,y_1,x_2,y_2,t_0)=1\,. \quad (9)$$

Step 2. Let us multiply (6) by vector $\mathbf{e}$, substitute (8), divide the results by ε and perform the asymptotic transition $\varepsilon \to 0$. Then taking into account $\mathbf{Q}_1\mathbf{e} = \mathbf{0}$ and $\mathbf{re} = 1$, we obtain the following differential equation for the function $\Phi_1(x_1,y_1,x_2,y_2,t)$

$$\frac{\partial \Phi_1(x_1,y_1,x_2,y_2,t)}{\partial t}=\Phi_1(x_1,y_1,x_2,y_2,t)\cdot$$
$$\kappa_1\left[S_1(t)(jx_1+jy_1a_1)+S_2(t)(jx_2+jy_2a_1)\right]. \quad (10)$$

The solution of Problem (9)–(10) is as follows:

$$\Phi_1(x_1,y_1,x_2,y_2,t)=\exp\left\{\kappa_1\left[(jx_1+jy_1a_1)\int_{t_0}^{t}S_1(\tau)d\tau+\right.\right.$$

$$\left.\left.(jx_2 + jy_2 a_1)\int_{t_0}^{t} S_2(\tau)d\tau\right]\right\}.$$

Substituting this expression into (8) and making inverse substitutions, we obtain

$$\mathbf{H}(u_1,v_1,u_2,v_2,t) \approx \mathbf{H}^{(1)}(u_1,v_1,u_2,v_2,t) =$$
$$\mathbf{r}\exp\left\{\kappa_1\left[(jx_1 + jy_1 a_1)\int_{t_0}^{t} S_1(\tau)d\tau + \right.\right.$$
$$\left.\left.(jx_2 + jy_2 a_1)\int_{t_0}^{t} S_2(\tau)d\tau\right]\right\}.$$

Thus, the proof is complete.

Second-order Asymptotic Analysis

The main result is the following theorem.

Theorem. The second-order asymptotic characteristic function of the probability distribution of the process $\{k(t), n_1(t), W_1(t), n_2(t), W_2(t)\}$ has the form

$$\mathbf{H}^{(2)}(u_1,v_1,u_2,v_2,t) = \mathbf{r}\exp\left\{N\kappa_1(ju_1 + jv_1 a_1)\int_{t_0}^{t} S_1(\tau)d\tau + \right.$$
$$N\kappa_1(ju_2 + jv_2 a_1)\int_{t_0}^{t} S_2(\tau)d\tau +$$
$$\frac{(ju_1)^2}{2}\left(N\kappa_1\int_{t_0}^{t} S_1(\tau)d\tau + N\kappa_2\int_{t_0}^{t} S_1^{\,2}(\tau)d\tau\right) +$$
$$\frac{(jv_1)^2}{2}\left(N\kappa_1 a_2\int_{t_0}^{t} S_1(\tau)d\tau + N\kappa_2 a_1^{\,2}\int_{t_0}^{t} S_1^{\,2}(\tau)d\tau\right) +$$
$$ju_1\, jv_1\left(N\kappa_1 a_1\int_{t_0}^{t} S_1(\tau)d\tau + N\kappa_2 a_1\int_{t_0}^{t} S_1^{\,2}(\tau)d\tau\right) + \quad (11)$$
$$\frac{(ju_2)^2}{2}\left(N\kappa_1\int_{t_0}^{t} S_2(\tau)d\tau + N\kappa_2\int_{t_0}^{t} S_2^{\,2}(\tau)d\tau\right) +$$
$$\frac{(jv_2)^2}{2}\left(N\kappa_1 a_2\int_{t_0}^{t} S_2(\tau)d\tau + N\kappa_2 a_1^{\,2}\int_{t_0}^{t} S_2^{\,2}(\tau)d\tau\right) +$$
$$ju_2\, jv_2\left(N\kappa_1 a_1\int_{t_0}^{t} S_2(\tau)d\tau + N\kappa_2 a_1\int_{t_0}^{t} S_2^{\,2}(\tau)d\tau\right) +$$
$$ju_1\, ju_2\, N\kappa_1\int_{t_0}^{t} S_1(\tau)S_2(\tau)d\tau +$$
$$jv_1\, jv_2\, N\kappa_2 a_1^{\,2}\int_{t_0}^{t} S_1(\tau)S_2(\tau)d\tau +$$
$$\left.(ju_1\, jv_2 + ju_2\, jv_1)N\kappa_2 a_1\int_{t_0}^{t} S_1(\tau)S_2(\tau)d\tau\right\},$$

where $\kappa_2 = 2\mathbf{g}(\mathbf{\Lambda}_1 - \kappa_1\mathbf{I})\mathbf{e}$, $a_2 = \int_0^{\infty} y^2 dG(y)$, and a row vector $\mathbf{g}$ satisfies the linear matrix system

$$\begin{cases}\mathbf{g}\mathbf{Q}_1 = \mathbf{r}(\kappa_1\mathbf{I} - \mathbf{\Lambda}_1),\\ \mathbf{g}\mathbf{e} = 1.\end{cases}$$

Proof.

Let $\mathbf{H}_2(x_1,y_1,x_2,y_2,t)$ be a vector function that satisfies the equation

$$\mathbf{H}(u_1,v_1,u_2,v_2,t) = \mathbf{H}_2(u_1,v_1,u_2,v_2,t)\cdot \quad (12)$$
$$\exp\left\{\kappa_1\left[(jx_1 + jy_1 a_1)\int_{t_0}^{t} S_1(\tau)d\tau + (jx_2 + jy_2 a_1)\int_{t_0}^{t} S_2(\tau)d\tau\right]\right\}.$$

Substituting this expression into (4) and (5), we obtain the following problem:

$$\frac{1}{N}\frac{\partial\mathbf{H_2}(u_1,v_1,u_2,v_2,t)}{\partial t} + \mathbf{H_2}(u_1,v_1,u_2,v_2,t)\kappa_1\cdot$$
$$[(ju_1 + jv_1 a_1)S_1(t) + (ju_2 + jv_2 a_1)S_2(t)] =$$
$$\mathbf{H_2}(u_1,v_1,u_2,v_2,t)\cdot \quad (13)$$
$$\left[\mathbf{\Lambda}_1\left(S_1(t)\left(e^{ju_1}G^*(v_1) - 1\right) + S_2(t)\left(e^{ju_2}G^*(v_2) - 1\right)\right) + \mathbf{Q}_1\right]$$

with the initial condition

$$\mathbf{H_2}(u_1,v_1,u_2,v_2,t_0) = \mathbf{r}\,. \quad (14)$$

Let us make the substitutions

$$\varepsilon^2 = \frac{1}{N},\ u_1 = \varepsilon x_1,\ w_1 = \varepsilon y_1,\ u_2 = \varepsilon x_2,\ w_2 = \varepsilon y_2,$$
$$\mathbf{H_2}(u_1,v_1,u_2,v_2,t) = \mathbf{F_2}(u_1,v_1,u_2,v_2,t,\varepsilon)\,. \quad (15)$$

Using these notations, Problem (13)–(14) can be rewritten in the form

$$\varepsilon^2\frac{\partial\mathbf{F_2}(x_1,y_1,x_2,y_2,t,\varepsilon)}{\partial t} + \mathbf{F_2}(x_1,y_1,x_2,y_2,t,\varepsilon)\kappa_1\cdot$$
$$[S_1(t)(j\varepsilon x_1 + j\varepsilon y_1 a_1) + S_2(t)(j\varepsilon x_1 + j\varepsilon y_1 a_1)] =$$
$$\mathbf{F_2}(x_1,y_1,x_2,y_2,t,\varepsilon)\cdot \quad (16)$$
$$\left[\mathbf{\Lambda}_1\left(S_1(t)\left(e^{j\varepsilon x_1}G^*(\varepsilon y_1) - 1\right) + S_2(t)\left(e^{j\varepsilon x_2}G^*(\varepsilon y_2) - 1\right)\right) + \mathbf{Q}_1\right]$$

with the initial condition

$$\mathbf{F_2}(x_1,y_1,x_2,y_2,t_0,\varepsilon) = \mathbf{r}\,. \quad (17)$$

Let us find the asymptotic solution of this problem $\mathbf{F}_2(x_1,y_1,x_2,y_2,t) = \lim_{\varepsilon\to 0}\mathbf{F}_2(x_1,y_1,x_2,y_2,t,\varepsilon)$ in three steps.

Step 1. Substituting $\varepsilon = 0$ in (16)–(17), we obtain the following system of equations:

$$\begin{cases}\mathbf{F_2}(x_1,y_1,x_2,y_2,t)\mathbf{Q}_1 = \mathbf{0},\\ \mathbf{F_2}(x_1,y_1,x_2,y_2,t_0) = \mathbf{r}.\end{cases}$$

Therefore, taking into account (3), we can write

$$\mathbf{F}_2(x_1,y_1,x_2,y_2,t) = \mathbf{r}\Phi_2(x_1,y_1,x_2,y_2,t), \quad (18)$$

where $\Phi_2(x_1,y_1,x_2,y_2,t)$ is some scalar function which satisfies the condition

$$\Phi_2(x_1,y_1,x_2,y_2,t_0) = 1\,. \quad (19)$$

Step 2. Using (18), the function $\mathbf{F}_2(x_1, y_1, x_2, y_2, t)$ can be represented in the expansion form

$$\mathbf{F}_2(x_1, y_1, x_2, y_2, t, \varepsilon) = \Phi_2(x_1, y_1, x_2, y_2, t)[\mathbf{r} + \mathbf{g}(S_1(t)(j\varepsilon x_1 + j\varepsilon y_1 a_1) + S_2(t)(j\varepsilon x_2 + j\varepsilon y_2 a_1))] + \mathbf{O}(\varepsilon^2), \quad (20)$$

where $\mathbf{g}$ is the row vector that satisfies the condition $\mathbf{ge} = 1$, and $\mathbf{O}(\varepsilon^2)$ is a row vector of infinitesimals of the order ε^2. Let us use substitution (20) and expansion $e^{j\varepsilon x} = 1 + j\varepsilon x + O(\varepsilon^2)$ in Equation (16). Taking into account (3) and making a transition $\varepsilon \to 0$, we obtain matrix equation for the vector $\mathbf{g}$

$$\mathbf{gQ}_1 = \mathbf{r}(\kappa_1 \mathbf{I} - \mathbf{\Lambda}_1),$$

where $\mathbf{I}$ is an identity matrix.

Step 3. We multiply Equation (16) by vector $\mathbf{e}$ and use Expression (20) and the second-order expansion

$$e^{j\varepsilon x} = 1 + j\varepsilon x + \frac{(j\varepsilon x)^2}{2} + O(\varepsilon^3).$$

After some transformations, using the notation

$$\kappa_2 = 2\mathbf{g}(\mathbf{\Lambda}_1 - \kappa_1 \mathbf{I})\mathbf{e},$$

we obtain the following differential equation for the function $\Phi_2(x_1, y_1, x_2, y_2, t)$

$$\frac{\partial \Phi_2(x_1, y_1, x_2, y_2, t)}{\partial t} = \Phi_2(x_1, y_1, x_2, y_2, t) \cdot$$
$$\left[\frac{(jx_1)^2}{2}\left(\kappa_1 S_1(t) + \kappa_2 S_1^2(t)\right) + \frac{(jy_1)^2}{2}\left(\kappa_1 a_2 S_1(t) + \kappa_2 a_1^2 S_1^2(t)\right) \right.$$
$$+ jx_1\, jy_1\left(\kappa_1 a_1 S_1(t) + \kappa_2 a_1 S_1^2(t)\right) +$$
$$jx_2\, jy_2\left(\kappa_1 a_1 S_2(t) + \kappa_2 a_1 S_2^2(t)\right) +$$
$$jx_1\, jx_2\, \kappa_1 S_1(t) S_2(t) + jy_1\, jy_2\, \kappa_2 a_1^2 S_1(t) S_2(t) +$$
$$(jx_1\, jy_2 + jx_2\, jy_1)\kappa_2 a_1 S_1(t) S_2(t) + \frac{(jx_2)^2}{2}\left(\kappa_1 S_2(t) + \kappa_2 S_2^2(t)\right)$$
$$\left. + \frac{(jy_2)^2}{2}\left(\kappa_1 a_2 S_2(t) + \kappa_2 a_1^2 S_2^2(t)\right) \right].$$

The solution of this equation with initial condition (19) is as follows:

$$\Phi_2(x_1, y_1, x_2, y_2, t) =$$
$$\exp\left\{ \frac{(jx_1)^2}{2}\left(\kappa_1 \int_{t_0}^{t} S_1(\tau)d\tau + \kappa_2 \int_{t_0}^{t} S_1^2(\tau)d\tau\right) + \right.$$
$$\frac{(jy_1)^2}{2}\left(\kappa_1 a_2 \int_{t_0}^{t} S_1(\tau)d\tau + \kappa_2 a_1^2 \int_{t_0}^{t} S_1^2(\tau)d\tau\right) +$$
$$jx_1\, jy_1\left(\kappa_1 a_1 \int_{t_0}^{t} S_1(\tau)d\tau + \kappa_2 a_1 \int_{t_0}^{t} S_1^2(\tau)d\tau\right) +$$
$$\frac{(jx_2)^2}{2}\left(\kappa_1 \int_{t_0}^{t} S_2(\tau)d\tau + \kappa_2 \int_{t_0}^{t} S_2^2(\tau)d\tau\right) +$$
$$\frac{(jy_2)^2}{2}\left(\kappa_1 a_2 \int_{t_0}^{t} S_2(\tau)d\tau + \kappa_2 a_1^2 \int_{t_0}^{t} S_2^2(\tau)d\tau\right) +$$
$$jx_2\, jy_2\left(\kappa_1 a_1 \int_{t_0}^{t} S_2(\tau)d\tau + \kappa_2 a_1 \int_{t_0}^{t} S_2^2(\tau)d\tau\right) +$$
$$jx_1\, jx_2\, \kappa_1 \int_{t_0}^{t} S_1(\tau) S_2(\tau)d\tau +$$
$$jy_1\, jy_2\, \kappa_2 a_1^2 \int_{t_0}^{t} S_1(\tau) S_2(\tau)d\tau +$$
$$\left. (jx_1\, jy_2 + jy_2\, jx_1)\kappa_2 a_1 \int_{t_0}^{t} S_1(\tau) S_2(\tau)d\tau \right\}.$$

Substituting this expression in Formula (18) and performing the substitutions that are inverse to (15) and (12), we obtain Expression (11) for the asymptotic characteristic function of the process $\{k(t), n_1(t), W_1(t), n_2(t), W_2(t)\}$. The proof is complete.

Corollary 1. Assuming $t = T$ and $t_0 \to -\infty$ and using Equalities (1), we obtain the steady-state characteristic function of the process under study $\{i_1(t), V_1(t), i_2(t), V_2(t)\}$:

$$h(u_1, v_1, u_2, v_2) = \exp\{N\kappa_1(ju_1 + jv_1 a_1)b_{11} +$$
$$N\kappa_1(ju_2 + jv_2 a_1)b_{21} + \frac{(ju_1)^2}{2}(N\kappa_1 b_{11} + N\kappa_2 b_{12}) +$$
$$\frac{(jv_1)^2}{2}\left(N\kappa_1 a_2 b_{11} + N\kappa_2 a_1^2 b_{12}\right) +$$
$$ju_1\, jv_1(N\kappa_1 a_1 b_{11} + N\kappa_2 a_1 b_{12}) + \frac{(ju_2)^2}{2}(N\kappa_1 b_{21} + N\kappa_2 b_{22}) +$$
$$\frac{(jv_2)^2}{2}\left(N\kappa_1 a_2 b_{21} + N\kappa_2 a_1^2 b_{22}\right) + \quad (21)$$
$$ju_2\, jv_2(N\kappa_1 a_1 b_{21} + N\kappa_2 a_1 b_{22}) +$$
$$ju_1\, ju_2\, N\kappa_1 b + jv_1\, jv_2\, N\kappa_2 a_1^2 b +$$
$$(ju_1\, jv_2 + ju_2\, jv_1)N\kappa_2 a_1 b\},$$

where

$$b_{11} = \int_0^\infty (1 - B_1(\tau))d\tau, \; b_{12} = \int_0^\infty (1 - B_1(\tau))^2 d\tau,$$

$$b_{21} = \int_0^\infty \left(B_1(\tau) - B_2^*(\tau)\right)d\tau, \; b_{22} = \int_0^\infty \left(B_1(\tau) - B_2^*(\tau)\right)^2 d\tau,$$

$$b = \int_0^\infty (1 - B_1(\tau))\left(B_1(\tau) - B_2^*(\tau)\right)d\tau.$$

From the form of characteristic function (21), it is clear that the probability distribution of four-dimensional process $\{i_1(t), V_1(t), i_2(t), V_2(t)\}$ is asymptotically Gaussian.

Corollary 2. The steady-state joint probability distribution of two-dimensional process of the total capacity at the stages of the system is asymptotically Gaussian with a vector of means

$$\mathbf{a} = N \cdot \begin{bmatrix} \kappa_1 a_1 b_{11} & \kappa_1 a_1 b_{21} \end{bmatrix}$$

and a covariance matrix

$$\mathbf{K} = N \cdot \begin{bmatrix} \kappa_1 a_2 b_{11} + \kappa_2 a_1^{\,2} b_{12} & \kappa_2 a_1^{\,2} b \\ \kappa_2 a_1^{\,2} b & \kappa_1 a_2 b_{21} + \kappa_2 a_1^{\,2} b_{22} \end{bmatrix}.$$

NUMERICAL EXAMPLE

Result (21) is obtained under the asymptotic condition $N \to \infty$. Therefore, the result may be used just as an approximation and it is applicable when N is great enough. So, we need in determining of a low boundary of parameter N which cause the approximation (21) be applicable. To do this we make series of simulation experiments and compare asymptotic distributions with empiric ones by using the Kolmogorov distance

$$\Delta = \max_x \left| F(x) - A(x) \right| \qquad (22)$$

as an accuracy. Here $F(x)$ is a cumulative distribution function of total capacity of customers at a stage of the tandem which is constructed on the basis of simulation results, and $A(x)$ is a Gaussian cumulative distribution function with respective mean and variance from Expression (21). Increasing value of parameter N step by step from one experiment to another, we can find the value of N at which the accuracy (22) is small enough.

Consider the following numerical example. The MMPP is given by parameters $\mathbf{Q} = N\mathbf{Q}_1$ and $\mathbf{\Lambda} = N\mathbf{\Lambda}_1$ where

$$\mathbf{Q}_1 = \begin{bmatrix} -0{,}8 & 0{,}4 & 0{,}4 \\ 0{,}3 & -0{,}6 & 0{,}3 \\ 0{,}4 & 0{,}4 & -0{,}8 \end{bmatrix}, \ \mathbf{\Lambda}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Fundamental rate of arrivals is $N\kappa_1 = N\mathbf{r}\mathbf{\Lambda}_1\mathbf{e} = 3 \cdot N$. Capacities of customers have uniform distribution in the range [0; 1]. Service time has gamma distribution with shape and inverse scale parameters $\alpha_1 = 1{,}5$ and $\beta_1 = 2$ at the first stage of the system, and $\alpha_2 = 0{,}5$ and $\beta_2 = 1{,}5$ at the second stage. So, the fundamental rate of arrivals exceeds exactly by N times the service rate at the second stage, therefore, we consider marginal distributions of the total capacity at this stage.

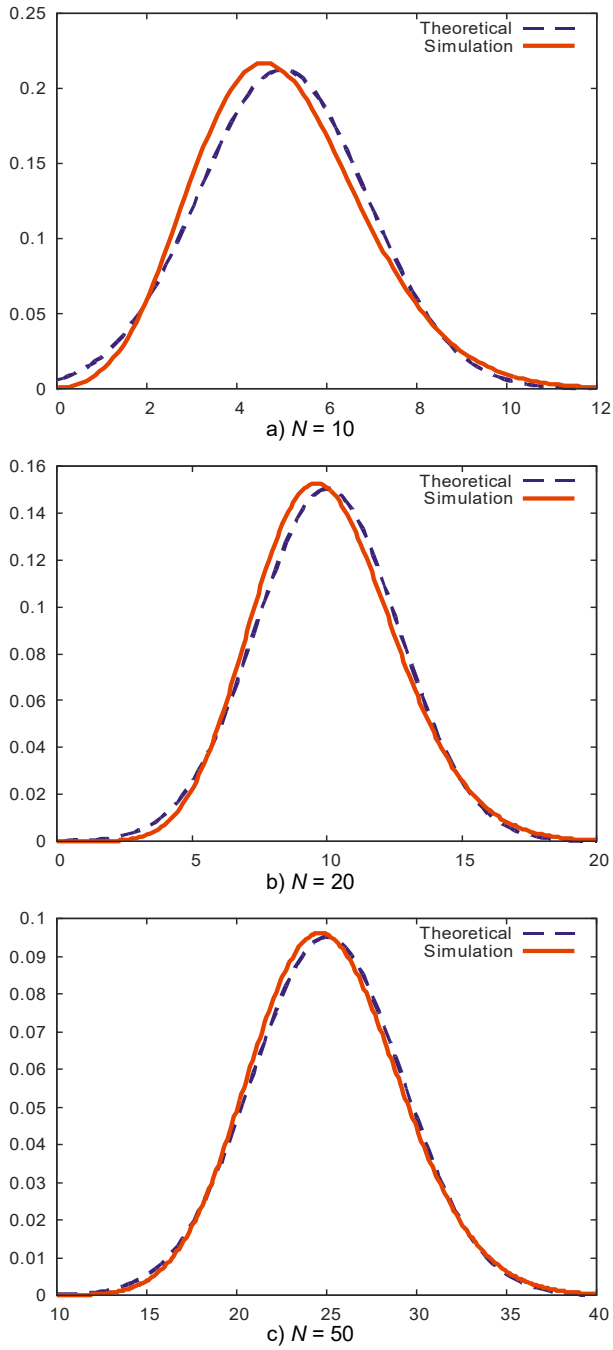
A vector of means and a covariance matrix of the Gaussian approximation for this example are as follows:

$$\mathbf{a} = N \cdot \begin{bmatrix} 0{,}125 & 0{,}5 \end{bmatrix}, \ \mathbf{K} = N \cdot \begin{bmatrix} 0{,}921 & 0{,}039 \\ 0{,}039 & 0{,}352 \end{bmatrix}.$$

So, in Formula (22) $F(x)$ will be a cumulative distribution function of total capacity of customers at the second stage of the system constructed on simulation results, and $A(x)$ will be a Gaussian cumulative distribution function with mean and variance equal to $0{,}5N$ and $0{,}352N$ respectively. Values of the Kolmogorov distance for increasing values of parameter N are presented in Table 1. We can notice that the asymptotic results become more accurate while a value of the parameter N (fundamental rate of arrivals) is increasing. Figures 2 show probability densities of asymptotic and empiric distributions at the second stage of the system and they confirm the effect.

Table 1: Kolmogorov Distances between Simulation and Asymptotic Results for the Total Capacity

N	5	10	20	50	100
Δ	0,056	0,035	**0,027**	**0,015**	**0,009**



Figures 2: Probability Densities of Asymptotic (Marked as "Theoretical") and Empiric (Marked as "Simulation") Distributions of the Total Capacity

We suppose that an approximation is applicable if its Kolmogorov distance less than 0,03. Then we can draw

a conclusion that the asymptotic results are applicable for values of the parameter N equal to 20 or more (marked by boldface in Table 1).

CONCLUSION

In the paper, the queue tandem with MMPP arrivals, infinite number of servers and non-exponential service time is considered. The problem under study is a capacity which each customer brings to the system. The analysis is performed under the asymptotic condition of high rate of arrivals (high values of a fundamental rate of the MMPP). It is shown that two-dimensional probability distribution of total capacities at the stages of the system is two-dimensional Gaussian under this asymptotic condition. Numerical results show that asymptotic results have enough accuracy for marginal distribution of total capacity at a stage of the system when a fundamental rate of arrivals exceeds service rate at the stage by 20 times or more.

Future studies may be devoted to analysis of customers' capacities in queueing tandems with MAP arrivals and systems in random environment.

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