ASYMPTOTIC ANALYSIS OF MARKOVIAN RETRIAL QUEUE WITH TWO-WAY COMMUNICATION UNDER LOW RATE OF RETRIALS CONDITION

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KEYWORDS
Retrial queuing system with two-way communication, incoming and outgoing calls, asymptotic analysis method, Gaussian approximation.

ABSTRACT
In this paper we are reviewing the retrial queue with two-way communication and Poisson arrival process. If the server free, incoming call occupies it. The call that finds the server being busy joins an orbit and retries to enter the server after some exponentially distributed time. If the server is idle, it causes the outgoing call from the outside. The outgoing call can find server free, then it starts making an outgoing call in an exponentially distributed time. If the outgoing call finds the server occupied, then it is lost. To research the system in question we have derived first and second order asymptotics of a number of calls in the orbit in an asymptotic condition of a low rate of retrials. Based on found asymptotics we have built the Gaussian approximation of a number of calls in the orbit.

INTRODUCTION
Recently a lot of attention is being paid to the research of the retrial queues such as mathematical models of real call center systems, telecommunication networks, computer networks, economical systems (Artalejo and Gomez-Corral 2008). These systems are characterized by the fact that if the clients (calls, phone calls, messages etc.) couldn’t be served immediately they have to enter the virtual orbit where they wait out some delay before they could access the server for service again (Flajolet and Sedgewick 2009).

As a rule, the ones that are considered are the retrial queues in which arriving calls are either served immediately or join the orbit where they are wait out a random delay before accessing the server again. Recently, however, server is more likely to have the ability to make an outgoing call. The example of that could be the common cellphone that has function of both incoming and outgoing calls. In different call centers operators could receive arriving calls but as soon as they have free time and are in standby mode they could make outgoing calls to advertise, promote and sell packages and services of the centre.

Falin (Falin 1979) derives integral formulas for partial generating functions and some explicit expressions for characteristics of the M|G|1|1 retrial queues with outgoing calls. Choi et al. (Choi et al. 1995) extends Falin’s model for the M/G/1/K retrial queues. Artaelejo and Resing (Artalejo and Resing 2010) have derived first moments for characteristics of the M/G/1/1 retrial queues, in which the times of serving arriving and outgoing calls are different.

Martin and Artalejo (Martin and Artalejo 1995) are considering M|G|1|1 retrial queues with outgoing calls in which calls from an orbit access the server after an exponentially distributed delay in the order of arrival. Artaelejo and Phung-Duc (Artalejo and Tuan 2012) are considering M|M|1|1 retrial queues with outgoing calls and a different service time for incoming and outgoing calls. In their paper the authors have found an explicit solution for two-dimensional probability distribution of a server state and a number of calls in an orbit. Likewise, the factorial moments are found, based on which the proposed numerical and recurrent algorithms may be applied.

In this paper the main method of research is the asymptotic analysis method which allows to find in M|M|1|1 retrial queue with two-way communication type of limit distribution of a number of calls in the orbit in an asymptotic condition of a low rate of retrials and to show that limit distribution is Gaussian.

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This result is achieved by using the original asymptotic analysis method without needing to find the nonlimiting distribution. Furthermore, the discrete distribution is constructed which approximates discrete distribution of a number of calls in an orbit. This distribution will be addressed as Gaussian approximation. Research of retrial queueing system under the asymptotic condition that the retrial rate is extremely low is stated in the following papers (Nazarov and Chernikova 2014) (Nazarov and Izmailova 2016). Furthermore, we have defined conditions of applicability of obtained approximation according to system defining parameters.

The remainder of the paper is presented as follows. In Section “Mathematical Model”, we describe the model in detail and preliminaries for later asymptotic analysis. In Sections “First order asymptotic” and “Second order asymptotic”, we present our main contribution for the model with Poisson input. In Section “Approximation accuracy $P^2(i)$ and its application area” we have defined the conditions of applicability of the obtained approximation depending on values of system-defining parameters. Section “Conclusions” is devoted to concluding remark and future work.

MATHEMATICAL MODEL

Let’s consider retrial queue (Figure 1) with Poisson arrival process of incoming calls with rate $\lambda$.

![Retrial queue with two-way communication](image)

Figure 1: Retrial queue with two-way communication

The incoming call finds the server and goes into service for an exponentially distributed time with rate $\mu_1$. If upon entering the system the call finds the server busy the call immediately joins the orbit, where it stays during a random time distributed exponentially with rate $\sigma$.

If the server is idle (empty) it starts making outgoing calls from the outside with rate $\alpha$. If the outgoing call finds the server free the call goes into service for an exponentially distributed time with rate $\mu_2$. If upon entering the system the outgoing call finds the server being busy the call is lost and is not considered in the future. Let’s denote:

$n(t)$ – server state: $0$ – server is free, $1$ – server is busy serving an incoming call, $2$ – server is busy serving an outgoing call.

Let’s consider two-dimensional Markovian process \{i(t), n(t)\} for probability distribution

\[
P(i(t) = i, n(t) = n) = P_n(i, t)
\]

setting up system of Kolmogorov equations

\[
-(\lambda + i\sigma + \alpha)P_0(i) + \mu_1P_1(i) + \mu_2P_2(i) = 0,
\]

\[
-(\lambda + \mu_1)P_1(i) + \lambda P_1(i-1) + P_0(i) + + (i+1)\sigma P_3(i+1) = 0,
\]

\[
-(\lambda + \mu_2)P_2(i) + P_3(i)\alpha + P_2(i-1)\lambda = 0.
\]

Introducing partial characteristic functions (Nazarov and Paul 2016), denoting $j = \sqrt{-1}$,

\[
H_n(u) = \sum_{i=0}^{\infty} e^{ju}P_n(i).
\]

Rewriting system (1) in the following form

\[
-(\lambda + \alpha)H_0(u) + j\sigma \frac{dH_0(u)}{du} + \mu_1H_1(u) + \mu_2H_2(u) = 0,
\]

\[
\left[\lambda e^{ju} - 1\right] - \mu_1H_1(u) + \lambda H_0(u) - j\sigma e^{ju} \frac{dH_0(u)}{du} = 0,
\]

\[
\left[\lambda e^{ju} - 1\right] - \mu_2H_2(u) + \alpha H_0(u) = 0.
\]

Characteristic function $H(u)$ of a number of incoming calls in an orbit and server states probability distribution $r_n$ are relatively easy expressed through partial characteristic functions $H_n(u)$ by the following equations

\[
H(u) = \frac{Me^{ju}}{1 - \frac{\lambda}{\mu_1}H_1(u) + \frac{\mu_1}{\mu_2}H_2(u) + \alpha H_0(u)},
\]

\[
r_n = H_n(0), \quad n = 0, 1, 2.
\]

The task is put to find these characteristics of retrial queue with two-way communication. The main content of this paper is the solution of system (2) by using asymptotic analysis method in limit condition of a low rate of retrials, when $\sigma \rightarrow 0$.

This is due to the fact that for the more complicated queues with an incoming MMPP, the equation system similar to (2) is analytically unsolvable, but a solution by using asymptotic analysis method is allowed.

Application of asymptotic results in prelimit situation is causing the necessity of specifying the area of its applicability, which is obtainable only through comparison of asymptotic and prelimit characteristics and that is relatively easy implemented for the retrial queue in question. For more complex systems prelimit characteristics are usually defined by results of imitational modeling or by using pretty complicated numerical algorithms. The asymptotic analysis method suggested below is implemented by sequential determination of first and second order asymptotics.

FIRST ORDER ASYMPTOTIC

We introduce the following notations

\[
\sigma = \varepsilon, \quad u = \varepsilon \omega, \quad H_n(u) = F_n(\omega, \varepsilon),
\]

then we will get this system

\[
P(i(t) = i, n(t) = n) = P_n(i, t)
\]
Theorem 1. (First order asymptotic) Suppose \( i(t) \) is a number of calls in an orbit of stationary \( M|M|1 \) retrial queue with two-way communication, then the following equation is true for a number of calls \( i(t) \) in an orbit of stationary \( M|M|1 \) retrial queue with two-way communication:

\[
\lim_{\sigma \to 0} M_{i(t)} \sigma = e^{\kappa_{1}w},
\]

where the parameter \( \kappa_{1} \) is defined by the following:

\[
\kappa_{1} = \frac{\lambda}{\mu_{2}}\mu_{1} + \alpha(\mu_{1} - \lambda).
\]

Proof. Consider \( \varepsilon \to 0 \), then we will get

\[
-(\lambda + \alpha)F_{0}(w) + \mu_{1}F_{1}(w) + \mu_{2}F_{2}(w) = 0, \quad -\lambda F_{1}(w) + \lambda F_{0}(w) = 0, \quad -\mu_{1}F_{1}(w) + \alpha F_{0}(w) = 0,
\]

by denoting

\[
\lim_{\varepsilon \to 0} F_{0}(w, \varepsilon) = F_{n}(w),
\]

where \( r_{n} \) is the scalar server state probability distribution, and the function \( \Phi(w) \) is defined in the following form

\[
\Phi(w) = \exp\left\{ j\kappa_{1}w \right\}.
\]

First order asymptotic i.e. the proven theorem, only defines the mean asymptotic value \( 1/\sigma \) of a number of calls in an orbit in prelimit situation of nonzero values of \( \sigma \). For more detailed research of a number \( i(t) \) of calls in an orbit let’s consider the second order asymptotic.

SECOND ORDER ASYMPTOTIC

Let’s substitute the following in the system (2)

\[
H_{n}(u) = \exp\left\{ \frac{u}{\sigma} \kappa_{1} \right\} H_{n}^{(2)}(u),
\]

we will get the system

\[
-H_{0}^{(2)}(u)(\lambda + \alpha + \kappa_{1}) + j\sigma \frac{dH_{0}^{(2)}(u)}{du} + \mu_{1}H_{1}^{(2)}(u) + \mu_{2}H_{2}^{(2)}(u) = 0, \quad H_{1}^{(2)}(u)e^{j\mu_{1}u} - \mu_{1}H_{1}^{(2)}(u) + \alpha H_{0}^{(2)}(u)\lambda + \kappa_{1}e^{-j\mu_{1}u}H_{1}^{(2)}(u) - j\sigma e^{-j\mu_{1}u} \frac{dH_{1}^{(2)}(u)}{du} = 0, \quad H_{2}^{(2)}(u)(e^{j\mu_{1}u} - \mu_{1} - \mu_{2} + \mu_{1} + \alpha H_{2}^{(2)}(u) = 0.
\]

Let’s make substitutions as shown below

\[
\sigma = \varepsilon^{2}, \quad u = j\varepsilon w, \quad H_{n}^{(2)}(u) = F_{n}^{(2)}(w, \varepsilon),
\]

then we will get the system

\[
-H_{0}^{(2)}(w, \varepsilon)(\lambda + \alpha + \kappa_{1}) + j\sigma \frac{\partial F_{0}^{(2)}(w, \varepsilon)}{\partial w} + \mu_{1}F_{1}^{(2)}(w, \varepsilon) + \mu_{2}F_{2}^{(2)}(w, \varepsilon) = 0, \quad \left\{ e^{j\mu_{1}w} - \mu_{1} \right\} F_{1}^{(2)}(w, \varepsilon) + \alpha F_{0}^{(2)}(w, \varepsilon) = 0. \]

Let’s denote

\[
\kappa_{1} = \frac{r_{1} + r_{2} \kappa_{2}}{r_{0}}.
\]

By taking into consideration the normalization condition for server state probability distribution and solving the system (6) alongside with the system (8) we will get the values of probabilities \( r_{n} \) and the value of parameter \( \kappa_{1} \).

\[
\begin{align*}
\kappa_{1} & = \frac{r_{1} + r_{2} \kappa_{2}}{r_{0}}, \\
\end{align*}
\]

First order asymptotic i.e. the proven theorem, only defines the mean asymptotic value \( \kappa_{1}/\sigma \) of a number of calls in an orbit in prelimit situation of nonzero values of \( \sigma \). For more detailed research of a number \( i(t) \) of calls in an orbit let’s consider the second order asymptotic.
Theorem 2. (Second order asymptotic) In the context of Theorem 1 the following equation is true
\[
\lim_{\varepsilon \to 0} Me = e^{-\frac{\varepsilon}{2} \kappa_2},
\]
where parameter \( \kappa_2 \) is defined by the following expression
\[
\kappa_2 = \frac{\lambda^3 + \lambda^2 \alpha + \lambda \mu_1^2 - 3 \lambda^2 \alpha \mu_1}{\mu_2^2 (\mu_1 - \lambda)^2} + \kappa_1.
\]

Proof. Let's substitute the following expansion into the system (5)
\[
F_2^{(2)}(w, \varepsilon) = \Phi_2(w) \left[ r_0 + j \varepsilon w f'_0 \right] + o(\varepsilon^2),
\]
then we will get
\[
-\Phi_2^{(2)}(w) \left[ r_0 + j \varepsilon w f'_0 \right] (\lambda + \alpha + \kappa_1) + j \varepsilon w \Phi_2^{(2)}(w) r_0 + \mu_1 \Phi_2^{(2)}(w) \left[ \mu_1 + j \varepsilon w f'_1 \right] + j \varepsilon w \Phi_2^{(2)}(w) \left[ \mu_1 + j \varepsilon w f'_1 \right] = o(\varepsilon^2).
\]

By summing equations of the system (11) we have
\[
\Phi_2^{(2)}(w) \left[ r_0 + j \varepsilon w f'_0 \right] \frac{j \varepsilon w}{\Phi_2^{(2)}(w) \left[ r_0 + j \varepsilon w f'_0 \right]} = o(\varepsilon^2).
\]

Then
\[
\Phi_2^{(2)}(w) \left[ r_0 + j \varepsilon w f'_0 \right] = o(\varepsilon^2),
\]
\[
-j \varepsilon w \Phi_2^{(2)}(w) \left[ \mu_1 + j \varepsilon w f'_1 + r_2 \lambda + \alpha f'_2 \right] = o(\varepsilon^2).
\]

Let's divide the equation of the system by \( \varepsilon \), we will get
\[
\Phi_2^{(2)}(w) \left[ r_0 - \lambda - \alpha - \kappa_1 + \mu_1 f_1 + \mu_2 f_2 \right] + j \varepsilon w \Phi_2^{(2)}(w) \left[ r_0 + j \varepsilon w f'_0 \right] = 0,
\]
\[
\Phi_2^{(2)}(w) \left[ r_0 - \lambda - \alpha - \kappa_1 + \mu_1 f_1 + \mu_2 f_2 \right] = o(\varepsilon^2),
\]
\[
-j \varepsilon w \Phi_2^{(2)}(w) \left[ \mu_1 f_1 + r_2 \lambda + \alpha f'_2 \right] = o(\varepsilon^2).
\]

Then
\[
\Phi_2^{(2)}(w) \left[ r_0 - \lambda - \alpha - \kappa_1 + \mu_1 f_1 + \mu_2 f_2 \right] = o(\varepsilon^2),
\]
\[
-j \varepsilon w \Phi_2^{(2)}(w) \left[ \mu_1 f_1 + r_2 \lambda + \alpha f'_2 \right] = o(\varepsilon^2).
\]

Let's divide the equation of the system by \( \varepsilon \), we will get
\[
\Phi_2^{(2)}(w) \left[ r_0 - \lambda - \alpha - \kappa_1 + \mu_1 f_1 + \mu_2 f_2 \right] = o(\varepsilon^2),
\]
\[
-j \varepsilon w \Phi_2^{(2)}(w) \left[ \mu_1 f_1 + r_2 \lambda + \alpha f'_2 \right] = o(\varepsilon^2).
\]

Consider \( \varepsilon \to 0 \)
\[
\frac{d\Phi_2^{(2)}(w)}{dw} r_0 = w \Phi_2^{(2)}(w) \left[ \mu_1 (f_0 - f_0) - (f_1 + f_2) \lambda \right],
\]
then
\[
\frac{d\Phi_2^{(2)}(w)}{\Phi_2^{(2)}(w)} r_0 = w \Phi_2^{(2)}(w) \left[ \mu_1 (f_0 - f_0) - (f_1 + f_2) \lambda \right] = j^2 \kappa_2,
\]
where
Then, considering that \( \Phi_2(0) = 1 \) we have
\[
\Phi_2(w) = \exp \left( -\frac{(jw)^2}{2} \kappa_2 \right).
\]

Let’s find \( \kappa_2 \), by expressing
\[
f_1 = f_0 \frac{\lambda + \kappa_1}{\mu_1} - \frac{(\kappa_1 - \kappa_2)^2}{\mu_1} + \frac{r_1 \lambda}{\mu_1},
\]
\[
f_2 = \frac{\alpha}{\mu_2} f_0 + r_2 \frac{\lambda}{\mu_1}.
\]
Then
\[
(f_1 + f_2) \kappa = -\lambda \kappa + \kappa_0 f_0 = \frac{f_0}{\mu_1} \left( \frac{\lambda + \kappa_1}{\mu_1} + \frac{\alpha \lambda}{\mu_2} - \kappa_1 f_0 \right) = \frac{\lambda (\lambda \mu_2 + \alpha \mu_1) - \kappa_1 \mu_2 (\mu_1 - \lambda)}{\mu_1 \mu_2} = \frac{\lambda (\lambda \mu_2 + \alpha \mu_1) - \lambda (\lambda \mu_2 + \alpha \mu_1)}{\mu_2} = 0.
\]

Then
\[
(\kappa_2 - \kappa_1) f_0 \left( \frac{\mu_1 - \lambda}{\mu_1} = \frac{r_1 \lambda^2}{\mu_1} + \frac{r_2 \lambda^2}{\mu_2},
\]
and
\[
\kappa_3 = \frac{1}{\mu_1 - \lambda} \left[ \frac{r_1 \lambda^2}{\mu_1} + \frac{r_2 \lambda^2}{\mu_2} \right] + \kappa_1 = \frac{\lambda^2}{\mu_1} + \frac{\lambda^2 \mu_2}{\mu_2} + \frac{\alpha \mu_2}{\mu_2} + \frac{\alpha \mu_1}{\mu_2} + \kappa_1.
\]

We have found that the parameter \( \kappa_3 \) equals
\[
\kappa_3 = \lambda^3 + \lambda^2 \frac{\lambda \mu_2}{\mu_2} + \lambda \frac{\lambda \alpha \mu_2}{\mu_2} + \lambda \frac{\alpha \mu_1}{\mu_2} + \kappa_1.
\]

Second order asymptotic i.e. the proven theorem 2, shows that the asymptotic probability distribution of a number \( i(t) \) of calls in an orbit is Gaussian with mean asymptotic \( \kappa_i / \sigma \) and dispersion \( \kappa_i / \sigma \). Then, with the following priori distribution in mind
\[
P(i) = P_0(i) + P_1(i) + P_2(i), i \geq 0,
\]
we could build an approximation for said distribution and in particular the \( P^{(2)}(i) \) approximation
\[
P^{(2)}(i) = \left( L(i + 0.5) - L(i - 0.5) \right) (1 - L(-0.5))^{-1},
\]
where \( L(x) \) is the normal distribution function with parameters \( \kappa_i / \sigma \) and \( \kappa_i / \sigma \).

Gaussian approximation (15), as will be shown below, is fairly applicable at low values \( \sigma < 0.05 \) and gives relative error at \( \sigma > 0.05 \). Moreover, prelimit distribution (14) is asymmetrical whilst the Gaussian approximation (15) is built upon the basis of symmetrical normal distribution.

**NUMERICAL ALGORITHM FOR SOLVING SYSTEM (1)**

Let’s write down system (1) at \( i = 0, i = 1 \) and \( i \geq 2 \), then we will have three systems
\[
- (\lambda + \alpha) P_0(i) + \mu_1 P_1(i) + \mu_2 P_2(i) = 0,
\]
\[
- (\lambda + \alpha) P_0(i) + \mu_1 P_1(i) + \sigma P_0(i) = 0,
\]
\[
- (\lambda + \mu_1) P_1(i) + P_2(i) + \mu P_0(i) = 0.
\]
(16)

Let’s consider \( P_0(i) = 1 \). Using the third and the first equations of the system (16) we could write down
\[
P_2(i) = \frac{1}{\frac{\alpha}{\lambda + \mu_2}}, \quad P_1(i) = \frac{1}{\mu_1} (\lambda + \alpha) \mu P_0(i) - \mu_2 P_0(i)).
\]
(17)

Using the second equation of the system (16) we could write down
\[
P_0(i) = \frac{1}{\sigma} \left[ (\lambda + \mu_1) P_0(i) - \lambda \mu P_0(i) \right].
\]
(18)

Using the third and the first equations of the system (17) we could write down
\[
P_2(i) = \frac{1}{\frac{\sigma}{\frac{\lambda}{\lambda + \mu_2}}} \left[ (\lambda + \alpha) \mu P_0(i) + \lambda \mu P_0(i) \right],
\]
\[
P_1(i) = \frac{1}{\frac{1}{\mu_1}} (\lambda + \alpha + \mu) P_0(i) - \mu_2 P_0(i)).
\]

Further at \( 0 \leq i \leq N \) the recurrent procedure is implemented by the following equations
\[
P_0(i) = \frac{1}{\frac{1}{\mu_1}} (\lambda + \mu_1) P_0(i - 1) - \lambda P_0(i),
\]
\[
P_1(i) = \frac{1}{\frac{1}{\mu_1}} (\lambda + \alpha + \mu) P_0(i - 1) - \mu_2 P_0(i - 1)).
\]

By normalizing the obtained results we have found the solution \( P_0(i) \) of system (1) for all \( 0 \leq i \leq N \). Suggested numerical algorithm is fairly effective as it allows finding the solution \( P_0(i) \) for large values (up to thousands) of \( N \).

**APPROXIMATION ACCURACY \( P^{(2)}(i) \) AND ITS APPLICATION AREA**

Approximation accuracy \( P^{(2)}(i) \) will be defined by using Kolmogorov equation
\[
\Delta = \max \left\{ \sum_{i=0}^{N} \left| P(i) - P^{(2)}(i) \right| \right\}
\]

For range between distributions \(P(i)\) and \(P^{(2)}(i)\), where distribution \(P(i)\) is defined by using numerical algorithm and the approximation \(P^{(2)}(i)\) is built upon the basis of the second asymptotic and the obtained Gaussian distribution. Tables 1-5 contain values for this range \(\Delta\) for various values of rate \(\lambda\) and \(\sigma\). We consider \(\mu_1 = 1\) and \(\mu_2 = 2\) for all Tables. Let’s consider \(\alpha = 1\).

Table 1: Kolmogorov range

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>(\sigma)</th>
<th>(\lambda = 0.5)</th>
<th>(\lambda = 0.6)</th>
<th>(\lambda = 0.7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma = 1)</td>
<td>0.092</td>
<td>0.108</td>
<td>0.123</td>
<td></td>
</tr>
<tr>
<td>(\sigma = 0.5)</td>
<td>0.066</td>
<td>0.079</td>
<td>0.092</td>
<td></td>
</tr>
<tr>
<td>(\sigma = 0.1)</td>
<td>0.064</td>
<td>0.039</td>
<td>0.045</td>
<td></td>
</tr>
<tr>
<td>(\sigma = 0.05)</td>
<td>0.026</td>
<td>0.028</td>
<td>0.032</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Kolmogorov range

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>(\sigma)</th>
<th>(\lambda = 0.8)</th>
<th>(\lambda = 0.9)</th>
<th>(\lambda = 0.95)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma = 1)</td>
<td>0.116</td>
<td>0.163</td>
<td>0.174</td>
<td></td>
</tr>
<tr>
<td>(\sigma = 0.5)</td>
<td>0.106</td>
<td>0.123</td>
<td>0.131</td>
<td></td>
</tr>
<tr>
<td>(\sigma = 0.1)</td>
<td>0.052</td>
<td>0.060</td>
<td>0.064</td>
<td></td>
</tr>
<tr>
<td>(\sigma = 0.05)</td>
<td>0.037</td>
<td>0.042</td>
<td>0.045</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Kolmogorov range

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>(\sigma)</th>
<th>(\lambda = 0.5)</th>
<th>(\lambda = 0.6)</th>
<th>(\lambda = 0.7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma = 1)</td>
<td>0.053</td>
<td>0.063</td>
<td>0.072</td>
<td></td>
</tr>
<tr>
<td>(\sigma = 0.5)</td>
<td>0.039</td>
<td>0.046</td>
<td>0.053</td>
<td></td>
</tr>
<tr>
<td>(\sigma = 0.1)</td>
<td>0.018</td>
<td>0.021</td>
<td>0.024</td>
<td></td>
</tr>
<tr>
<td>(\sigma = 0.05)</td>
<td>0.013</td>
<td>0.014</td>
<td>0.017</td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Kolmogorov range

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>(\sigma)</th>
<th>(\lambda = 0.8)</th>
<th>(\lambda = 0.9)</th>
<th>(\lambda = 0.95)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma = 1)</td>
<td>0.083</td>
<td>0.094</td>
<td>0.100</td>
<td></td>
</tr>
<tr>
<td>(\sigma = 0.5)</td>
<td>0.060</td>
<td>0.068</td>
<td>0.072</td>
<td></td>
</tr>
<tr>
<td>(\sigma = 0.1)</td>
<td>0.027</td>
<td>0.030</td>
<td>0.032</td>
<td></td>
</tr>
<tr>
<td>(\sigma = 0.05)</td>
<td>0.019</td>
<td>0.021</td>
<td>0.023</td>
<td></td>
</tr>
</tbody>
</table>

Table 5: Kolmogorov range

<table>
<thead>
<tr>
<th>(\sigma)</th>
<th>(\alpha = 1)</th>
<th>(\alpha = 3)</th>
<th>(\alpha = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma = 0.2)</td>
<td>0.072</td>
<td>0.052</td>
<td>0.028</td>
</tr>
<tr>
<td>(\alpha = 3)</td>
<td>0.058</td>
<td>0.041</td>
<td>0.022</td>
</tr>
<tr>
<td>(\alpha = 5)</td>
<td>0.049</td>
<td>0.035</td>
<td>0.019</td>
</tr>
</tbody>
</table>

Analysis of values tabulated in tables 3-5 shows that the accuracy of Gaussian approximation greatly increases while increasing \(\alpha\), and therefore the area of applicability increases too. The area of applicability doubles in size and is applicable at \(\alpha \leq 0.05\). Density diagrams of probability distributions and distribution function diagrams of a number of calls in an orbit are shown in figures 2-4. The dotted line represents designated density of asymptotical distribution probabilities.
CONCLUSIONS

In this paper we have considered retrial queue with two-way communication. To research the system in question we have found first and second order asymptotics of a number of calls in an orbit in asymptotic condition of a low rate of retrials. Based on the obtained asymptotics we have built the Gaussian approximation of a probability distribution of a number of calls in an orbit. We have defined the conditions of applicability of the obtained approximation depending on values of system-defining parameters. As criteria we have chosen the Kolmogorov range assuming that the allowed approximation error is less than 0.05. By analyzing the obtained results we can make the conclusion that the accuracy of Gaussian approximation increases while decreasing values of parameter, increasing values of parameter and/or increasing values of parameter

The results obtained in this paper are planned to be generalized for the case of correlated incoming flow and random time of serving in retrial queues with two-way communication.

REFERENCES


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