OPTIONS WITH STOCHASTIC STRIKE PRICES

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ABSTRACT

In the option pricing theory, the exercise price is constant by definition. The early generalizations of the Black-Scholes formula aimed to get rid of the constant nature of some parameters (the risk-free interest rate or the volatility). The focus of this paper is on generalizing the basic option pricing techniques in another direction: by allowing the strike price to be a random variable or change across time. For this purpose, we will examine a European call option with binomial random strike price and an American put option with time- and state-varying strike price. Taking the exercise price as a random variable might be considered as a bridge between the Balck-Scholes and the Margrabe-model.

INTRODUCTION

Option pricing is one of the favourite topics in quantitative finance. The basic models such as that of Black and Scholes (1973) and Merton (1973) are well-known and used all over the world. In the decades following after their publication, the original framework was extended and generalized in a few directions. One fruitful direction is examining options with various underlying assets. The original model assumed a stock with no dividend, but the underlying might be a currency or an index with continuous dividend yield as well. The famous Black (1976) model is an extension for pricing options on futures or valuing interest rate caps and floors.

Another popular way of improving the model is relaxing the assumption that the volatility of the underlying is constant. One of the most famous stochastic volatility models is that of Heston (1993).

In this paper, we deal with a direction where the strike price is not constant. We show two cases how this feature might appear: the first case is a European option with a binomial strike price, the second is an American option with time- and state-varying strike price. Throughout the paper, we assume that the reader is familiar with the basic Black-Scholes-Merton and binomial option pricing framework.

EXCHANGE OPTIONS

There are many extensions for the BS-model, from our point of view the most important is that of Margrabe (1978) who provided a closed form solution for the price of exchange options. These products grant the holder the right to exchange one risky asset to another risky asset at maturity. It is assumed that both risky assets follow geometric Brownian motion, that is under the risk-neutral Q-measure:

$$dS_{\mathrm{A}} = rS_{\mathrm{A}}dt + \sigma_{\mathrm{A}}S_{\mathrm{A}}dW_{\mathrm{A}} \text{ and}$$

$$dS_{\mathrm{B}} = rS_{\mathrm{B}}dt + \sigma_{\mathrm{B}}S_{\mathrm{B}}dW_{\mathrm{B}},$$

where, W_{A} and W_B are standard Wiener processes under Q, with correlation ρ.

It can be shown (for a derivation see for example Medvegyev–Vidovics-Dancs–Illés, 2015) that the price of the exchange option with payout function $\max(\mathrm{S_{AT}} - \mathrm{S_{BT}}; 0)$ is the following:

$$S_{A0}N(d_1) - S_{B0}N(d_2)\ , \tag{1}$$

$$d_1 = \frac{\ln\left(\frac{S_{\mathrm{A0}}}{S_{\mathrm{B0}}}\right)+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}} \text{ and } d_2 = \frac{\ln\left(\frac{S_{\mathrm{A0}}}{S_{\mathrm{B0}}}\right)-\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

$$\sigma = \sqrt{\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho},$$

and N stands for the cumulative distribution function of the standard normal distribution. The formula (1) is often called Margrabe-formula.

It is easy to see that Margrabe's model for pricing the exchange option is a generalization of the Black-Sholes-Merton model. A simple call option is a special exchange option, where S_B is deterministic (σ_B=0) and the value of S_B at maturity is the strike price. On the other hand, a simple put option is a special exchange option as well, where S_A is deterministic (σ_A=0) and the value of S_A at maturity is the strike price. Substituting these into Margrabe's formula leads back to the Black-Scholes formula (Figure 1a and 1b).

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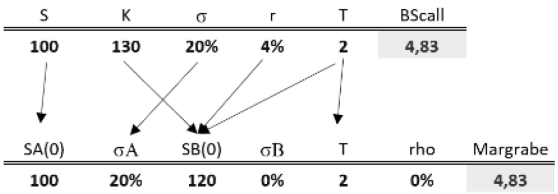

Figure 1: Connection between Black-Scholes and Margrabe-formulas

What we would like to do in this paper is to build a bridge between the Black-Scholes-Merton and the Margrabe-model. Hence, we will analyse options the strike price of which is stochastic or time-varying, but will not replace it with another traded asset (i.e. with another geometric Brownian motion) as Margrabe did. With other words, our analysis can be interpreted as another (not "Margrabe-style") extension of the famous Black-Scholes-Merton framework.

BINARY STRIKE PRICES

As a first step, we assume that the strike price K is a random variable with two possible outcomes: K_1 and K_2. Under the risk neutral measure, K_1 will occur with probability q_K, and K_2 with probability $(1-q_K)$:

$$K = \begin{cases} K_1, & q_K \\ K_2, & 1 - q_K \end{cases}$$

We will price this option with Monte Carlo simulation and compare the result with c_1 and c_2, the prices of simple call options with strikes K_1 and K_2. We will use the expression 'stochastic option' for the option with the stochastic strike price. (We cannot use the 'binary option' term since it has a widespread and different meaning in the literature.) The price of the stochastic option will be denoted by c_K.

Monte Carlo simulation

Let us assume that the price of the underlying asset (S) follows geometric Brownian motion, with trend parameter equal to the risk-free interest rate (of course, we are talking about the dynamics under Q-measure):

$$dS = rSdt + \sigma SdW.$$

We simulated the cash flows of the option with binary K. In a sample of 10,000 realizations, we got the frequencies showed in Figure 2. The parameters used are: S_0=100, r=5%, σ=20%, T=3, K_1=60, K_2=75, q_K=50%. On the figure, we used the same realizations of S_T for all the three options.

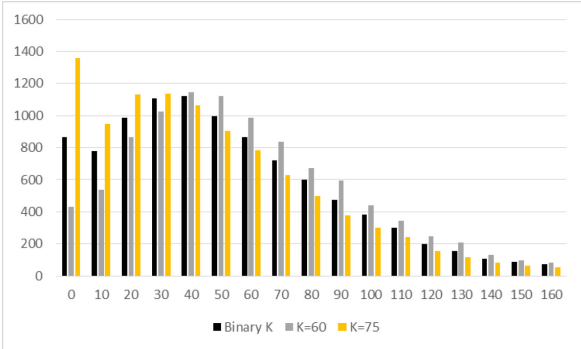

Figure 2: Payout frequencies

As for the price of the stochastic option, it will be obviously between the prices of the simple options with strike prices K_1 and K_2. It is also trivial that the higher the q_K parameter is, the closer the price of the stochastic option is to the price of the option with K_1. We plotted this relationship in Figure 3. Apart from q_K, the parameters are the same as before.

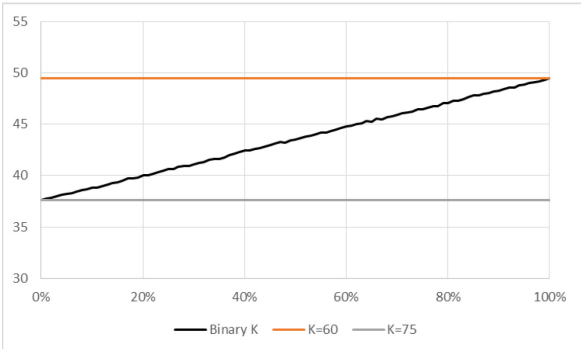

Figure 3: Prices as a function of q_K

The most interesting question is the relationship between the price of the stochastic option and the weighted average of the prices of the deterministic ones. Let us denote the latter one with c_{AVG}:

$$c_{AVG} \triangleq q_K c_1 + (1 - q_K) c_2 \tag{2}$$

Using Monte Carlo simulation again with 10,000 realizations, we observed that the price of the stochastic option is very close to c_{AVG}, actually it fluctuates around it. We repeated the Monte Carlo simulation 1,000 times and plotted the results on Figure 4, with the same parameters we used so far. On the figure, the constant c_{AVG} is calculated by the Black-Scholes formula.

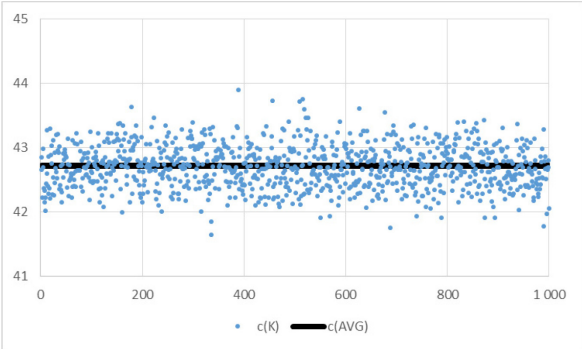

Figure 4: Monte Carlo prices

Binomial tree

To understand the nature of the stochastic option better, we analyse the problem in a one-period binomial model, with r=0. Let S_0 be 100 and the two possible values of S_1 are S_d=60 and S_u=140. In this case, the risk-neutral one-period probability (q) is 50%. As for the values of K_1 and K_2, it is worth distinguishing among six cases, according to their relation to S_d and S_u. Table 1 summarizes these cases and also contains the chosen values of the strike prices in each case.

Table 1: The six cases

Label	Case	K_1	K_2
A	$K_1<K_2<S_d<S_u$	10	30
B	$K_1<S_d<K_2<S_u$	10	100
C	$K_1<S_d<S_u<K_2$	10	170
D	$S_d<K_1<K_2<S_u$	80	120
E	$S_d<K_1<S_u<K_2$	80	170
F	$S_d<S_u<K_1<K_2$	170	200

It is easy to see that Case F is not really interesting. In this case, none of the examined options will be exercised, and hence their value is zero. This is true for the stochastic option as well.

Now we set q_K=40% and calculate c_1, c_2, and c_K with the well-known way: the price of the option is the risk-neutral expected value of its future cash flow. The results are summarized in Table 2, where we also showed c_{AVG}.

Table 2: Binomial prices

Case	K_1	K_2	c_1	c_2	c_K	c_{AVG}
A	10	30	90	70	**78**	**78**
B	10	100	90	20	**48**	**48**
C	10	170	90	0	**36**	**36**
D	80	120	30	10	**18**	**18**
E	80	170	30	0	**12**	**12**
F	170	200	0	0	**0**	**0**

We can observe that in this numerical example c_K=c_{AVG} in all the six cases. Now we show that this is not due to the special values we have chosen for the parameters.

For simplifying the analyses, we introduce the following notation:

$$c_{i,j} = \max(S_j - K_i, 0),$$

where i={1,2} and j={u,d}.

Thus, $c_{i,j}$ is the cash flow of the call option with strike price K_j, if the price of the underlying is S_j.

Under the risk-neutral measure, the cash flows of the options analysed are discrete random variables with the following possible values and probabilities. In the indices, T refers to the fact that we are talking about the cash flows to be paid at the maturity of the option.

$$c_{1,T} = \begin{cases} c_{1,u} & q \\ c_{1,d} & 1-q \end{cases}$$

$$c_{2,T} = \begin{cases} c_{2,u} & q \\ c_{2,d} & 1-q \end{cases}$$

According to the risk-neutral valuation, the prices of the deterministic options with our notations are:

$$c_1 = qc_{1,u} + (1-q)c_{1,d} \tag{3}$$

$$c_2 = qc_{2,u} + (1-q)c_{2,d} \tag{4}$$

Using the definition of (2) we end up with the following equation:

$$c_{AVG} = q_K\left[qc_{1,u} + (1-q)c_{1,d}\right] + \\ +(1-q_K)\left[qc_{2,u} + (1-q)c_{2,d}\right] \tag{5}$$

Now let us determine the distribution of the stochastic option's cash flow.

$$c_{K,T} = \begin{cases} c_{1,u} & q_K q \\ c_{2,u} & (1-q_K)q \\ c_{1,d} & q_K(1-q) \\ c_{2,d} & (1-q_K)(1-q) \end{cases}$$

From this, the price of the stochastic option is the following.

$$c_K = q_K q c_{1,u} + (1-q_K)qc_{2,u} + q_K(1-q)c_{1,d} \\ + (1-q_K)(1-q)c_{2,d} \tag{6}$$

Comparing (5) and (6) shows us that c_{AVG}=c_K.

Relaxing the assumption that r=0 would not change our results since both (5) and (6) would be multiplied by the same discount factor.

VARYING STRIKE PRICE

In this section, we analyse an American put option with a time-varying strike price. As a first step, we compare the simple European and American put options in the binomial framework (Table 3, bold numbers indicate the early exercise).

Table 3: American put with fix K*

n	S0	K	IC	u	d	q	DF	dt	DFn
3	**200**	**210**	**1.25**	**2**	0.5	**0.50**	0.8	1	0.512

S(i,j)	0	1	2	3
0	200			
1	100	400		
2	50	200	800	
3	25	100	400	1 600

eu put	0	1	2	3
0	**33.0**			
1	64.8	17.6		
2	118	44	0	
3	**185**	**110**	**0**	**0**

Intrin	0	1	2	3
0	10			
1	**110**	0		
2	**160**	10	0	
3	185	110	0	0

am put	0	1	2	3
0	**51.0**			
1	**110**	18		
2	160	44	0	
3	185	**110**	0	0

PV EV	0	1	2	3
0	**51**			
1	82	**18**		
2	118	44	0	
3				

**Spot price development (S), European put value (eu put), intrinsic value of the put (Intrin), American put value (am put) and the present value of the expected value (PV EV). Time is changing vertically; the states are horizontally (j- number of up moves). IC = 1+ r, DF=1/IC (discount factor), DFn is the DF for n periods (years)*

The value of the right to sell at time T=n=3 for K=200, when the price of the underlying is S=200, is p=33 (European put), while the American put (right to sell the underlying *until* T) has a value of 51. The exercise takes place either in the state (1,0) when S=100 and we sell the underlying for K=210, or at maturity in the state (3,1). In the state (2,1) we could sell the underlying for K=210 with a gain of 10 (S=200), but it is better to hold it, since we can expect a payout of 55, and its discounted value is 44.

Now, let us determine the value of an American put when K is a deterministic function of time (i) and the underlying price (the state, j). Table 4 compares this option with the previous, simple American put. We can see that this rather strange put option has a higher value (**58.2** as compared to **51.0**), reflecting the fact that the strike price is increasing.

Table 4: American put with varying K*

n	S0	K	IC	u	d	q	DF	dt	DFn
3	**200**	**210**	**1.25**	**2**	0.5	**0.50**	0.8	1	0.512

S(i,j)	0	1	2	3
0	200			
1	100	400		
2	50	200	800	
3	25	100	400	1 600

am put	0	1	2	3
0	**51.0**			
1	110	18		
2	160	44	0	
3	185	110	0	0

K	0	1	2	3
0	210			
1	220	240		
2	230	250	270	
3	240	260	280	300

am p K	0	1	2	3
0	**58.2**			
1	120	25.6		
2	180	64	0	
3	215	160	0	0

**Spot price development (S), simple American put value (am put), exercise price as function of time and state (K), American put value with varying K(am p K);*

We can separate the effect of the increasing K due to *time* (Table 5, left side), and due to *price changes* of the underlying (Table 5, right side). The second one leads to a stochastic exercise price, but the source of randomness is the uncertainty in the underlying price itself (perfect correlation).

Table 5: Separated effects of the varying strike price

K	0	1	2	3
0	210			
1	220	220		
2	230	230	230	
3	240	240	240	240

57.0

K	0	1	2	3
0	210			
1	210	230		
2	210	230	250	
3	210	230	250	270

52.3

It is a question whether the 58.2 – 51 = **7.2** option premium increase can or cannot be divided along this separation of exercise increase. The early exercise feature of the American options can lead to several complications, this is why we do not have a closed form formula even for the simplest case.

CONCLUSION

Relaxing the assumption that the strike price is constant might lead to very interesting and novel research in the option pricing field. In this paper, we showed two possible ways how this generalization may be initiated. For the first sight, in case of traditional financial options, it can be rather strange to imagine that K is volatile. However, in the real world applications of the option pricing principal – the martingale approach –, sometimes it is the best way to treat *K* as a deterministic or stochastic function, changing over time and/or across states. For example, Kornai's famous *Soft Budget Constraint* phenomenon could be treated as an American put option with a not well-defined exercise price: the agent that gets into financial trouble does not know when it will be bailed out, and the extension of the bail-out is stochastic (might be zero as well).

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