NON-STATIONARY CHARACTERISTICS OF THE $M_K/G_K/1$ QUEUEING SYSTEM WITH THE GENERALIZED FOREGROUND-BACKGROUND PROCESSOR SHARING DISCIPLINE

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ABSTRACT

A single server queueing system with several independent Poisson input flows of customers, infinite buffers for each type of customers, and generalized foreground-background processor sharing discipline is studied. Non-stationary joint distribution of the number of customers of all types served until the moment t and being in the system at time t taking into account their lengths (for the customers being in the system at the moment t their served lengths are considered) is derived in terms of a triple transform (the Laplace transform on time and the generating functions of the number of customers).

1 PROBLEM STATEMENT

Let us consider a single server queueing system with K types of customers.
The several input flows of customers entering the system form independent Poisson flows with intensities $\lambda_k,\ k = \overline{1, K}$. We denote by $\lambda = \sum_{k=1}^{K} \lambda_k$ the total intensity of the Poisson flow which is the superposition of all flows entering the system.
The total number of customers of each type standing in the system is unlimited, i.e. we consider the system with infinite buffers for the customers of all types.
The service time (to be called length in what follows) of each customer of type k is distributed according to the distribution function $B_k(x)$ with mean value $\overline{b}_k = \int_0^{\infty} [1 - B_k(x)]\, dx \le \infty$. We consider a generalized foreground-background processor sharing discipline which is defined for this system in the following way. A k-type customer with served length x has a priority index $r_k(x)$; here for each k the condition $r_k(0) = 0$ is satisfied. We also suppose that the function $r_k(x)$ is continuous and strictly increasing.

Let $r_k^{(-1)}(y)$ be the inverse function for $r_k(x)$. We assume that $r_k^{(-1)}(y)$ has a bounded derivative $\gamma_k(y) = dr_k^{(-1)}(y)/dy$. Then at every time moment a customer of minimal priority index value is under service. In the case when l customers have the minimal value y of the priority index, they are being served simultaneously and if the ith customer from these l customers is of type k, his service speed is equal to $\gamma_{k_i}(y) \Big/ \sum_{j=1}^{l} \gamma_{k_j}(y)$.

We call $r_k(x)$ the priority index of a customer of type k and length x who is already served and left the system or of a similar customer who is just arriving into the system, i.e. it is his priority index at the moment of the service completion. Using Kendall's classification we codify such a system as $M_K/G_K/1/\infty/FBPS$.

For the future is convenient to introduce the following notations. We mark the vectors with components numbered with indices $k,\ k = \overline{1, K}$, with an arrow above, for example, $\vec{n} = (n_1, \ldots, n_K)$. We use the notations $|\vec{m}| = \sum_{i=1}^{K} m_i$ and $\vec{z}^{\vec{m}} = \prod_{k=1}^{K} z_k^{m_k}$. At last, we denote by $\vec{u} = (u_1, \ldots, u_K)$ (with or without indices) a set of arbitrary (measured) functions $u_k = u_k(t),\ 0 \le t < \infty,\ 0 \le u_k(t) \le 1,\ k = \overline{1, K}$, and by $\mathbf{1}$ a set of functions $u_k(t)$, which are identically equal to one.

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Let us note that the stationary characteristics of the $M_K/G_K/1/\infty$ queueing system under the generalized FBPS discipline has been obtained in (Pechinkin and Tatashev 1981), and the system $MAP_K/G_K/1/\infty$ — in (D'Apice and Pechinkin 2004). The work (D'Apice and Pechinkin 2004) contains also a bibliography about the investigation of queueing systems under FBPS discipline.

2 AUXILIARY SYSTEM

Let us introduce an y-system which is also a $M_K/G_K/1/\infty/FBPS$ queueing system, but in the y-system the length of a customer of type k has the distribution function

$$B_k^{(y)}(x) = \begin{cases} B_k(x), & x \le r_k^{(-1)}(y); \\ 1, & x > r_k^{(-1)}(y), \end{cases} \quad k = \overline{1,K}.$$

In other words, all customers with priority index larger than y (i.e. with lengths larger than $r_k^{(-1)}(y)$) arriving to the initial system have priority index in the y-system equal to y (i.e. the length $r_k^{(-1)}(y)$). It is obvious that an ∞-system coincides with the initial one.

Let us denote by $\rho^{(y)} = \sum_{k=1}^K \lambda_k \int_0^\infty \left[1 - B_k^{(y)}(x)\right] dx$ the traffic intensity of an y-system and $\rho = \rho^{(\infty)} = \sum_{k=1}^K \lambda_k \bar{b}_k$ — the traffic intensity of the initial one.

Consider a busy period (BP) in an y-system and assume that this BP is opened by the arrival to the system of a customer of type k.
Let:
$\tau^{(y)}$ be the length of the BP;
$\nu_l^{(y)}$ be the number of customers of type l served on the BP and with priority index less than y (i.e. with lengths less than $r_l^{(-1)}(y)$);
$\xi_{l1}^{(y)}, \ldots, \xi_{l\nu_l^{(y)}}^{(y)}$ be their lengths;
$\mu_l^{(y)}$ be the number of customers of type l served on the BP and with priority index y (i.e. with lengths $r_l^{(-1)}(y)$).
We denote by $\vec{\mu}^{(y)}$ a vector with coordinates $\mu_l^{(y)}$.
Let us introduce the notations

$$W_k^{(y)}(t, \vec{u}, \vec{m}) = \mathsf{E}\chi\{\tau^{(y)} < t\}\chi\{\vec{\mu}^{(y)} = \vec{m}\} \prod_{l=1}^K \prod_{n=1}^{\nu_l^{(y)}} u_l(\xi_{ln}^{(y)}),$$

where $\chi(A)$ is the indicator function of the event A (here and in the sequel we use the convention: $\prod_{l=1}^0 \ldots = 1$). The function $W_k^{(y)}(t, \vec{u}, \vec{m})$ represents the generating function (GF) (of the number of served customers with priority index less than y) of the joint distribution of the following characteristics of the y-system:
the length of the BP (argument t),
the number of customers served on this BP and with priority index less than y taking into account their types and lengths (argument $\vec{u}$),
the number of customers served on this BP with priority index y taking into account their types (argument $\vec{m}$).

Let us set

$$\widetilde{W}_k^{(y)}(s, \vec{u}, \vec{z}) = \int_0^\infty \sum_{|m|=0}^\infty \vec{z}^{\vec{m}} e^{-st} W_l^{(y)}(dt, \vec{u}, \vec{m}), \quad 0 \le z_l \le 1.$$

The function $\widetilde{W}_k^{(y)}(s, \vec{u}, \vec{z})$ is a triple transform of the above characteristics: the Laplace-Stieltjes transform (LST) of the length of the BP of an y-system (argument s), the GF of the number of customers served on the BP with priority index less than y taking into account their types and lengths (argument $\vec{u}$), the GF of the number of customers served on the BP with priority index y taking into account their types (argument $\vec{z}$).

Let us denote by $\widetilde{W}_k^{(y)}(s, \vec{u}, \vec{z} \,|\, x)$ the function analogous to $\widetilde{W}_k^{(y)}(s, \vec{u}, \vec{z})$, but under the condition that the BP is opened by a customer of type k with priority index x, without taking into consideration the number of customers serviced on the BP .
For conservative service disciplines (the generalized FBPS discipline considered here belongs to this class of disciplines) the length of the BP and the numbers of customers of several types and lengths served on the BP are invariant characteristics respect to service discipline. This is why for calculation of $\widetilde{W}_k^{(y)}(s, \vec{u}, \vec{z} \,|\, x)$ and $\widetilde{W}_k^{(y)}(s, \vec{u}, \vec{z})$ we can use each conservative discipline. Then using the inverse service discipline (discipline LCFS, see, for instance, (Bocharov et al. 2004)), we obtain the equation

$$\left(\widetilde{W}_k^{(y)}(s, \vec{u}, \vec{z} \,|\, x)\right)'_x = \gamma_k(x) \left[\sum_{l=1}^K \lambda_l \widetilde{W}_l^{(y)}(s, \vec{u}, \vec{z}) - \lambda - s \right] \widetilde{W}_k^{(y)}(s, \vec{u}, \vec{z} \,|\, x)$$

with the initial condition

$$\widetilde{W}_k^{(y)}(s, \vec{u}, \vec{z} \,|\, 0) = 1$$

(it follows from the fact that the BP of an y-system, opened by a customer with priority index 0, i.e. with length 0, is equal to zero and any customer will not arrive on this BP) and the relation

$$\widetilde{W}_k^{(y)}(s,\vec{u},\vec{z})$$

$$= \int_0^{r_k^{(-1)}(y)} u_k(x)\widetilde{W}_k^{(y)}(s,\vec{u},\vec{z}\,|\,r_k(x))\,dB_k(x)$$

$$+ z_k\big[1 - B_k\big(r_k^{(-1)}(y)\big)\big]\widetilde{W}_k^{(y)}(s,\vec{u},\vec{z}\,|\,y),$$

which lead to the following expressions for $\widetilde{W}_k^{(y)}(s,\vec{u},\vec{z})$ and $\widetilde{W}_k^{(y)}(s,\vec{u},\vec{z}\,|\,x)$:

$$\widetilde{W}_k^{(y)}(s,\vec{u},\vec{z}\,|\,x) = e^{r_k^{(-1)}(x)\left[\sum_{l=1}^K \lambda_l \widetilde{W}_l^{(y)}(s,\vec{u},\vec{z})-\lambda-s\right]},$$

$$\widetilde{W}_k^{(y)}(s,\vec{u},\vec{z})$$

$$= \int_0^{r_k^{(-1)}(y)} u_k(x)\, e^{x\left[\sum_{l=1}^K \lambda_l \widetilde{W}_l^{(y)}(s,\vec{u},\vec{z})-\lambda-s\right]} dB(x)$$

$$+ z_k\big[1 - B_k\big(r_k^{(-1)}(y)\big)\big]$$

$$\times\, e^{r_k^{(-1)}(y)\left[\sum_{l=1}^K \lambda_l \widetilde{W}_l^{(y)}(s,\vec{u},\vec{z})-\lambda-s\right]}, \quad k=\overline{1,K}. \quad (1)$$

The system of functional equations (1) can be solved numerically using the methods provided in (Bocharov et al. 2004).

Further, let us set $\widetilde{W}_k(s,\vec{u}) = \widetilde{W}_k^{(\infty)}(s,\vec{u},\vec{z})$. The functions $\widetilde{W}_k(s,\vec{u})$ satisfy the system of equations

$$\widetilde{W}_k(s,\vec{u})$$

$$= \int_0^{\infty} u_k(x)\, e^{x\left[\sum_{l=1}^K \lambda_l \widetilde{W}_l(s,\vec{u})-\lambda-s\right]} dB(x), \quad k=\overline{1,K}.$$

These functions represent the LST of the lengths of the BP of the initial system and the GF of the number of customers served on this BP (taking into account their types and lengths) for several types of customers opening the BP.

Let us note that if the traffic intensity of an y-system is $\rho^{(y)} > 1$ (the traffic intensity of the initial system $\rho > 1$), then the equalities $\widetilde{W}_k^{(y)}(0,\mathbf{1},\vec{1}\,|\,x) < 1$ and $\widetilde{W}_k^{(y)}(0,\mathbf{1},\vec{1}) < 1$ (equality $\widetilde{W}(0,\mathbf{1}) < 1$) hold. This corresponds to the fact that the BP of the corresponding system will continue infinitely with a non-zero probability.

Now we turn back to the BP of an y-system opened by a customer of type k. Consider an arbitrary time moment t after the beginning of the BP.
Let
$\tau^{(y)}$ be the length of the BP;
$\nu_{1l}^{(y)}$ be the number of served customers of type l until the moment t;
$\xi_{1l1}^{(x)},\dots,\xi_{1l\nu_{1l}^{(y)}}^{(y)}$ be their lengths;
$\nu_{2l}^{(y)}$ be the number of customers of type l being in the system at the moment t;
$\xi_{2l1}^{(y)},\dots,\xi_{2l\nu_{2l}^{(y)}}^{(y)}$ be their served lengths.
Let us set

$$Q_{kij}^{(y)}(t,\vec{u}_1,\vec{u}_2) = \mathsf{E}\chi\{\tau^{(y)} > t\}$$

$$\times \prod_{l=1}^K \left(\prod_{k_1=1}^{\nu_{1l}^{(y)}} u_{1l}(\xi_{1lk_1}^{(y)})\right)\left(\prod_{k_2=1}^{\nu_{2l}^{(y)}} u_{2l}(\xi_{2lk_2}^{(y)})\right).$$

The function $Q_k^{(y)}(t,\vec{u}_1,\vec{u}_2)$ is the GF of the number of customers of all types (taking into account their lengths, for being in the system — served), served until the moment t and being in y-system at the moment t, multiplied by the indicator of the event that the first BP started at the moment 0 does not finish until the moment t (as earlier, the index k corresponds to the type of the customer that opened the BP).

In terms of the Laplace transform (LT)

$$\widetilde{Q}_k^{(y)}(s,\vec{u}_1,\vec{u}_2) = \int_0^\infty e^{-st} Q_k^{(y)}(t,\vec{u}_1,\vec{u}_2)\,dt,$$

the function $Q_k^{(y)}(t,\vec{u}_1,\vec{u}_2)$ satisfies the equation

$$\big(\widetilde{Q}_k^{(y)}(s,\vec{u}_1,\vec{u}_2)\big)'_y = \left[\sum_{n=1}^K \gamma_n(y) u_{2n}\big(r_n^{(-1)}(y)\big)\right.$$

$$\left.\times\, \big(\widetilde{W}^{(y)}(s,\vec{u}_1,\vec{z})\big)'_{z_n}\Big|_{z_n=u_{2n}(r_n^{(-1)}(y))}\right]$$

$$\times \left[1 + \sum_{l=1}^K \lambda_l \widetilde{Q}_l^{(y)}(s,\vec{u}_1,\vec{u}_2)\right], \quad k=\overline{1,K}. \quad (2)$$

The system of equations (2) is a system of differential equations of the first order. The initial condition

$$\widetilde{Q}_k^{(0)}(s,\vec{u}_1,\vec{u}_2) = 0, \quad k=\overline{1,K},$$

for this system is obtained observing that the BP of a 0-system is equal to zero.
Unfortunately, unlike the system $M/G/1/\infty/FBPS$ with one type of customers, the equation (2) can not be solved in explicit form, and its solution can be find only by numerical methods.

The function $\widetilde{Q}_k(s,\vec{u}_1,\vec{u}_2) = \widetilde{Q}_k^{(\infty)}(s,\vec{u}_1,\vec{u}_2)$ gives in terms of a triple transform (the LT on t and the GF of the number of customers) the joint distribution of the numbers of customers served until the moment t and being in the initial system at this moment (taking into account their types and lengths) and the event that the first BP started at the moment 0 by service completion of a customer of type k has not been finished during the time t.

3 NON-STATIONARY CHARACTERISTICS

Now we turn back to the initial $M_K/G_K/1/\infty/FBPS$ queueing system. We assume that at the initial moment 0 the system is empty. We consider an arbitrary time moment $t > 0$ and let:
ν_{1k} be the number of customers of type k served until the moment t;
$\xi_{1k1}, \dots, \xi_{1k\nu_{1k}}$ be their lengths;
ν_{2k} be the number of customers of type k being in the system at the moment t;
$\xi_{2k1}, \dots, \xi_{2k\nu_{2k}}$ be their served lengths.
Let us introduce the notations

$$P_{ij}(t, \vec{u}_1, \vec{u}_2) = \mathsf{E} \prod_{k=1}^{K} \left(\prod_{n_{1k}=1}^{\nu_{1k}} u_{1k}(\xi_{1kn_{1k}}) \right) \times \left(\prod_{n_{2k}=1}^{\nu_{2k}} u_{2k}(\xi_{2kn_{2k}}) \right).$$

The function $P(t, \vec{u}_1, \vec{u}_2)$ is the GF of the numbers of served customers and being in the system at the moment t (taking into account their types and lengths, for being in the system — served).
Let us put

$$\widetilde{P}(s, \vec{u}_1, \vec{u}_2) = \int_0^\infty e^{-st} P(t, \vec{u}_1, \vec{u}_2)\, dt.$$

Now using the elements of renewal theory, we obtain

$$\widetilde{P}(s, \vec{u}_1, \vec{u}_2) = \left[s + \lambda - \sum_{k=1}^{K} \lambda_k \widetilde{W}_k(s, \vec{u}_1) \right]^{-1} \times \left[1 + \sum_{k=1}^{K} \lambda_k \widetilde{Q}_k(s, \vec{u}_1, \vec{u}_2) \right]. \tag{3}$$

The formula (3) defines in terms of a triple transform (the LT on time and the GF of the numbers of customers served until the moment t and being in the system at the moment t) the joint non-stationary distribution of the numbers of customers served until the moment t and being in the system at the moment t (taking into account their types and lengths, for being in the system — served).

Using the results above we can find other characteristics of the $M_K/G_K/1/\infty/FBPS$ queueing system, for example, the stationary distribution of the number of customers in the system (taking into account their types and lengths), non-stationary and stationary distributions of the numbers of customers of given served lengths and being in the system, stationary and non-stationary distributions of the remaining work, the moments of the numbers of customers of given served lengths and being in the system, for stationary and non-stationary cases.

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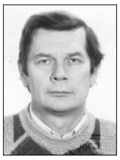

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