# NEW EXTENSIONS OF THE CAYLEY-HAMILTON THEOREM WITH APPLICATIONS

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## ABSTRACT

New extension of the classical Cayley-Hamilton theorem for rectangular matrices, block matrices, discrete-time and continuous-time systems with delays, and singular systems are presented. Some applications of the extensions and illustrating examples are also given.

## INTRODUCTION

The classical Cayley-Hamilton theorem (Gantmacher 1974, Kaczorek 1988, Lancaster 1969) says that every square matrix satisfies its own characteristic equation. The Cayley-Hamilton theorem has been extended to rectangular matrices (Kaczorek 1995, Kaczorek 1988), block matrices (Kaczorek 1995), pairs of commuting matrices (Chang and Chan 1992, Lewis 1982, Lewis 1986, Kaczorek 1995), pairs of block matrices (Kaczorek 1998, Kaczorek 1988) and standard and singular two-dimensional linear (2-D) systems (Kaczorek 1992/1993, Kaczorek 1995, Smart and Barnett 1989, Theodoru 1989).

The Cayley-Hamilton theorem and its generalizations have been used in control systems, electrical circuits, systems with delays, singular systems, 2-D linear systems, etc., (Kaczorek 1992/1993, Busłowicz 1981, Busłowicz 1982, Kaczorek 1994, Lewis 1982, Mertizios and Christodoulous 1986).

In (Kaczorek 2005) the Cayley-Hamilton theorem has been extended to n-dimensional (n-D) real polynomial matrices. An extension of the Cayley-Hamilton theorem for discrete-time linear systems with delay has been given in (Busłowicz and Kaczorek 2004).

In this paper an overview of new generalizations of the Cayley-Hamilton theorem with some applications is presented. The classical Cayley-Hamilton theorem will be extended for rectangular matrices, block matrices, discrete-time and continuous-time systems with delays and singular systems. The extensions of the Cayley-Hamilton theorem will illustrated by examples.

## CAYLEY-HAMILTON THEOREM FOR SQUARE AND RECTANGULAR MATRICES

Let $C^{n\times m}$ be the set of complex ($n\times m$) matrices.

**Theorem 1.** (Cayley-Hamilton theorem). *Let* $A\in C^{n\times m}$ *and*

$$p(s)=\det\left[I_n s-A\right]=\sum_{i=0}^{n} a_i s^i \qquad (a_n=1) \tag{1}$$

*be the characteristic polynomial of* $A$, *where* $I_n$ *is the* ($n\times n$) *identity matrix.*
*Then*

$$p(A)=\sum_{i=0}^{n} a_i A^i = 0_n \tag{2}$$

*where* $0_n$ *is the* ($n\times n$) *matrix.*

The classical Cayley-Hamilton theorem can be extended to rectangular matrices as follows (Kaczorek 1988).

**Theorem 2.** (Cayley-Hamilton theorem for rectangular matrices). *Let*

$$A=\begin{bmatrix}A_1 & A_2\end{bmatrix}\in C^{m\times n},\ A_1\in C^{m\times m},\ A_2\in C^{m\times(n-m)},\ (n>m) \tag{3}$$

*and*

$$p_{A_1}=\det\left[I_m s-A_1\right]=\sum_{i=0}^{m} a_i s^i \quad (a_m=1) \tag{4}$$

*be the characteristic polynomial of* $A_1$
*Then*

$$\sum_{i=0}^{m} a_{m-i}\begin{bmatrix}A_1^{n-i} & A_1^{n-i-1}A_2\end{bmatrix}=0_{mn} \tag{5}$$

*where* $0_{mn}$ *is the* ($m\times n$) *zero matrix.*

**Theorem 3.** *Let* $A=\begin{bmatrix}A_1\\A_2\end{bmatrix}\in C^{m\times n}$, $m>n$ *and let the characteristic polynomial of* $A_1$ *have the form (4). Then*

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$$\sum_{i=0}^{n} a_{n-i}\begin{bmatrix} A_1^{m-i} \\ A_2A_1^{m-i-1}\end{bmatrix} = 0_{mn} \qquad (6)$$

## CAYLEY-HAMILTON THEOREM FOR BLOCK MATRIX

The classical Cayley-Hamilton theorem can be also extended for block matrices (Kaczorek 1998, Kaczorek 1988).

**Theorem 4** *(Cayley-Hamilton theorem for block matrices). Let*

$$A = \begin{bmatrix} A_{11} & \dots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \dots & A_{mm}\end{bmatrix} \in C^{mn\times mn} \qquad (7)$$

*where $A_{ij}\in C^{n\times n}$ are commutative, i.e.*

$$A_{ij}A_{kl} = A_{kl}A_{ij} \quad \text{for all } i,j,k,l = 1,2,\dots,m \qquad (8)$$

*Let*

$$\begin{aligned} P(S) &= \det\left[I_m \otimes S - A\right] = \\ &= S^m + D_1S^{m-1} + \dots + D_{m-1}S + D_m \end{aligned} \qquad (9)$$

*be the matrix characteristic polynomial of A, where $S\in C^{n\times n}$ is the matrix (block) eigenvalue of A, ⊗ denotes the Kronecker product of matrices (Kaczorek 1988). Then*

$$P(A) = \sum_{i=0}^{m}\left(I_m \otimes D_{m-i}\right)A^i = 0 \quad \left(D_0 = I_n\right) \qquad (10)$$

The matrix (9) is obtained by developing the determinant of the matrix $[I_n\otimes S-A]$ considering its commuting blocks as scalar entries (Kaczorek 1988).

**Theorem 5** *(Cayley-Hamilton theorem for rectangular block matrices). Let $\overline{A} = [A\ A_2]\in C^{mn\times(mn+p)}$ and let the matrix characteristic polynomial of A have the form (14). Then*

$$\sum_{i=0}^{m}\left(I_m \otimes D_{m-i}\right)\begin{bmatrix} A^{i+1} & A^iA_2\end{bmatrix} = 0 \quad \left(D_0 = I_n\right) \qquad (11)$$

The dual theorem has the form

**Theorem 6.** *Let*

$$\overline{A} = \begin{bmatrix} A \\ A_2\end{bmatrix} \in C^{(mn+p)\times mn},\ A\in C^{mn\times mn},\ A_2 \in C^{p\times mn}$$

*and let the matrix characteristic polynomial of A have the form (14). Then*

$$\sum_{i=0}^{m}\begin{bmatrix} A \\ A_2\end{bmatrix}\left(I_m \otimes D_{m-i}\right)A^i = 0 \quad \left(D_0 = I_n\right) \qquad (12)$$

## CAYLEY-HAMILTON THEOREM FOR SYSTEMS WITH DELAYS

### Discrete time-systems

Consider the discrete-time linear system with *h* delays described by the equation

$$x_{i+1} = A_0x_i + A_1x_{i-1} + \dots + A_hx_{i-h} + Bu_i \qquad (13)$$

where $x_i\in C^n$, $u_i\in C^m$ are the state and input vectors, $A_k\in C^{n\times n}$, $k=0,1,\dots,h$ and $B\in C^{n\times m}$.
The characteristic polynomial of (13) has the form

$$\begin{aligned} p(z) &= \det\begin{bmatrix} I_nz - A_0 & -A_1 & \dots & -A_{h-1} & -A_h \\ -I_n & I_nz & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -I_n & I_nz\end{bmatrix} = \\ &= \det\left[I_nz^{h+1} - A_0z^h - A_1z^{h-1} - \dots - A_h\right] = \\ &= z^N + a_{N-1}z^{N-1} + \dots + a_1z + a_0, \qquad N = n(h+1) \end{aligned} \qquad (14)$$

Let

$$\Phi_{i+1} = A_0\Phi_i + A_1\Phi_{i-1} + \dots + A_h\Phi_{i-h} \qquad (15)$$

and

$$\Phi_0 = I_n \quad \text{and} \quad \Phi_i = 0 \quad \text{for} \quad i<0 \qquad (16)$$

Knowing the matrices $A_k$, $k=0,1,\dots,h$ and using (15), (16) we may compute the matrices $\Phi_i$ for $i=1,2,\dots$ .

**Theorem 7.** *The matrices $\Phi_i$ for $i=1,2,\dots$ defined by (15) and (16) satisfy the equation*

$$\sum_{i=0}^{N} a_{i+k}\Phi_{i+k} = 0 \quad \text{for} \quad k = 0,1,\dots \quad \left(a_N = 1\right) \qquad (17)$$

*where $a_i$, $i=0,1,\dots,N-1$ are the coefficients of the characteristic polynomial (14).*

**Proof.** From definition of the inverse matrix we have

$$\begin{aligned} &\left[I_nz^{h+1} - A_0z^h - A_1z^{h-1} - \dots - A_h\right]_{ad} = \\ &= \left[I_nz^{h+1} - A_0z^h - A_1z^{h-1} - \dots - A_h\right]^{-1} \times \\ &\det\left[I_nz^{h+1} - A_0z^h - A_1z^{h-1} - \dots - A_h\right] \end{aligned} \qquad (18)$$

From (15) and (16) it follows that

$$\begin{aligned}&\left[I_n z^{h+1} - A_0 z^h - A_1 z^{h-1} - \ldots - A_h\right]^{-1} = \\ &= I_n z^{-1} + \Phi_1 z^{-2} + \Phi_2 z^{-3} + \ldots\end{aligned} \tag{19}$$

From (18), (19) and (14) we obtain

$$\begin{aligned}&\left[I_n z^{h+1} - A_0 z^h - A_1 z^{h-1} - \ldots - A_h\right]_{ad} = \\ &= \left[I_n z^{-1} + \Phi_1 z^{-2} + \Phi_2 z^{-3} + \ldots\right] \\ &\left(z^N + a_{N-1} z^{N-1} + \ldots + a_1 z + a_0\right)\end{aligned} \tag{20}$$

Comparing the coefficient at the same power of $z^{-(k+1)}$ of the equality (20) we obtain the equation (17).

■

**Example 1.** Let $h=1$ and

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \tag{21}$$

The characteristic polynomial (14) in this case has the form

$$\begin{aligned}p(z) &= \det\left[I_2 z^2 - A_0 z - A_1\right] = \begin{vmatrix} z^2 - z & -1 \\ -1 & z^2 - z - 2 \end{vmatrix} = \\ &= z^4 - 2z^3 - z^2 + 2z - 1\end{aligned} \tag{22}$$

Using (15) and (21) we obtain

$$\begin{aligned}&\Phi_1 = A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Phi_2 = A_0\Phi_1 + A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \\ &\Phi_3 = A_0\Phi_2 + A_1\Phi_1 = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \quad \Phi_4 = A_0\Phi_3 + A_1\Phi_2 = \begin{bmatrix} 2 & 5 \\ 5 & 12 \end{bmatrix}\end{aligned} \tag{23}$$

Hence by (17) for $k=0$ we obtain

$$\begin{aligned}&\Phi_4 - 2\Phi_3 - \Phi_2 + 2\Phi_1 - I_2 = \begin{bmatrix} 2 & 5 \\ 5 & 12 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} + \\ &- \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\end{aligned} \tag{24}$$

Therefore, the matrices (23) satisfy the equation (22).

**Continuous-time systems**

Consider the continuous-time linear system with $h$ delays described by the equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-d) + \ldots + A_h x(t-hd) + Bu(t) \tag{25}$$

where $x(t) \in R^n$, $u(t) \in R^m$ are the state and input vectors, $A_k \in R^{n \times n}$, $k=0,1,\ldots,h$, $B \in R^{n \times m}$ and $d$ is the delay.
The characteristic polynomial of (25) has the form

$$\begin{aligned}p(s,w) &= \det\left[I_n s - A_0 - A_1 w - \ldots - A_h w^h\right] = \\ &= s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0\end{aligned} \tag{26}$$

where $w=e^{-ds}$ and

$$\begin{aligned}&a_{n-1} = a_{n-1}(w) = a_{n-1,h} w^h + \ldots + a_{n-1,1} w + a_{n-1,0} \\ &a_{n-2} = a_{n-2}(w) = a_{n-2,2h} w^{2h} + \ldots + a_{n-2,1} w + a_{n-2,0} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &a_0 = a_0(w) = a_{0,nh} w^{nh} + \ldots + a_{0,1} w + a_{0,0}\end{aligned} \tag{27}$$

The coefficients $a_{kj}$, $k=0,1,\ldots,n-1$, $j=0,1,\ldots,n-1$ depend on the entries of matrices $A_0, A_1, \ldots, A_h$.
Let

$$\begin{aligned}&\left[I_n s - \left(A_0 + A_1 w + \ldots + A_h w^h\right)\right]^{-1} = \\ &= I_n s^{-1} + \Phi_1 s^{-2} + \Phi_2 s^{-3} + \ldots\end{aligned} \tag{28}$$

where

$$\begin{aligned}&\Phi_i = \Phi_i(w) = \left(A(w)\right)^i \quad \text{for } i = 1,2,\ldots \\ &\text{and} \quad A(w) = A_0 + A_1 w + \ldots + A_h w^h\end{aligned} \tag{29}$$

Using the well-known relation Adj $M=M^{-1}$det $M$ and (26), (28) we may write

$$\begin{aligned}\operatorname{Adj} A(w) = &\left[I_n s^{-1} + \Phi_1 s^{-2} + \Phi_2 s^{-3} + \ldots\right] \\ &\left(s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0\right)\end{aligned} \tag{30}$$

Note the adjoint matrix Adj $A(w)$ is a polynomial matrix in nonnegative powers of $s$. Thus the comparison of the coefficients at the same powers of $s^{-1}$ of the equality (30) yields

$$\begin{aligned}&\Phi_{n+k-1} + a_{n-1}\Phi_{n+k-2} + \ldots + a_1\Phi_k + a_0\Phi_{k-1} = 0 \\ &\text{for } k = 1,2,\ldots \quad \left(\Phi_0 = I_n\right)\end{aligned} \tag{31}$$

From (31) for $k=1$ we have the equality (the Cayley-Hamilton theorem)

$$\Phi_n + a_{n-1}\Phi_{n-1} + \ldots + a_1\Phi_1 + a_0 I_n = 0 \tag{32}$$

with coefficients $a_k$ depending on $w$.

From (29) we have

$$\begin{aligned}&\Phi_i = \left(A_0 + A_1 w + \ldots + A_h w^h\right)^i = \\ &= A_0^i + \left(A_0 A_1 A_0^{i-2} + A_1 A_0^{i-1} + \ldots + A_0^{i-2} A_1 A_0\right) w + \left(A_0 A_2 A_0^{i-2} + \right. \\ &+ A_1^2 A_0^{i-2} + A_2 A_0^{i-1} + A_1 A_0 A_1 A_0^{i-3} + A_0 A_1 A_0 A_1^{i-3} + A_1 A_0^2 A_1^{i-3} + \ldots \\ &\left. \ldots A_0^2 A_1^{i-2} A_1^2\right) w^2 + \ldots + A_h^i w^{hi}, \quad \text{for } i = 1,2,\ldots\end{aligned} \tag{33}$$

Substitution of (33) and (27) into (32) yields

$$
\begin{aligned}
&A_0^n + \left(A_0 A_1 A_0^{n-2} + A_1 A_0^{n-1} + \ldots + A_0^{n-2} A_1 A_0\right) w + \\
&+\left(A_0 A_2 A_0^{n-2} + A_1^2 A_0^{n-2} + \ldots + A_0^2 A_1^{n-2} A_1^2\right) w^2 + \\
&+\ldots + A_h^n w^{nh} + \left(a_{n-1,h} w^h + \ldots + a_{n-1,1} w + a_{n-1,0}\right) \\
&\Big[A_0^{n-1} + \left(A_0 A_1 A_0^{n-3} + A_1 A_0^{n-2} + \ldots + A_0^{n-3} A_1 A_0\right) w + \\
&+\left(A_0 A_2 A_0^{n-3} + A_1^2 A_0^{n-3} + \ldots + A_0^2 A_1^{n-3} A_1^2\right) w^2 + \ldots \\
&\ldots + A_h^{n-1} w^{(n-1)h} + \ldots + \left(a_{1,n(h-1)} w^{n(h-1)} + \ldots + a_{11} w + a_{10}\right) \\
&\left[A_0 + A_1 w + \ldots + A_h w^h\right] + \left(a_{0,nh} w^{nh} + \ldots + a_{01} w + a_{00}\right) I_n = 0
\end{aligned}
\tag{34}
$$

From (34) we have the following $(nh+1)$ equations

$$
\begin{aligned}
&A_0^n + a_{n-1,0} A_0^{n-1} + \ldots + a_{10} A_0 + a_{00} I_n = 0 \\
&A_0 A_1 A_0^{n-2} + A_1 A_0^{n-1} + \ldots + A_0^{n-2} A_1 A_0 + \\
&+a_{n-1,0}\left(A_0 A_1 A_0^{n-3} + A_1 A_0^{n-2} + \ldots + A_0^{n-3} A_1 A_0\right) + \\
&+a_{n-1,1} A_0^{n-1} + \ldots + a_{10} A_1 + a_{11} A_0 + a_{01} I_n = 0 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&A_h^n + a_{0,nh} I_n = 0
\end{aligned}
\tag{35}
$$

Therefore, the following theorem has been proved

**Theorem 8.** *Matrices $A_k \in R^{n\times n}$, $k=0,1,\ldots,h$ of the continuous-time linear system with h delays (25) satisfy the ($nh+1$) algebraic matrix equations (35).*

Note that the first equation of (35) express the Cayley-Hamilton theorem for the system (25) without delay (*h=0*).

**Example 2.** Consider the system with

$$
A_0 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{36}
$$

In this case the characteristic polynomial (26) has the form

$$
\begin{aligned}
p(s,w) &= \det\left[I_2 s - A_0 - A_1 w - A\, w^2\right] = \\
&= \begin{vmatrix} s-1-w & -2-w^2 \\ -1-w^2 & s-w \end{vmatrix} = \\
&= s^2 - (2w+1)s - \left(w^4 + 2w^2 - w + 2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&a_1 = a_1(w) = a_{11} w + a_{10} = -2w - 1, \\
&a_0 = a_0(w) = a_{04} w^4 + a_{03} w^3 + a_{02} w^2 + a_{01} w + a_{00} = \\
&\qquad = -w^4 - 2w^2 + w - 2
\end{aligned}
$$

Taking into account that *n=h=2* from (35) we obtain the following equations

$$
\begin{aligned}
&A_0^2 + a_{10} A_0 + a_{00} I_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^2 - \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&A_0 A_1 + A_1 A_0 + a_{11} A_0 + a_{10} A_1 + a_{01} I_2 = \\
&= 2\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&A_1^2 + A_0 A_2 + A_2 A_0 + a_{10} A_2 + a_{11} A_1 + a_{02} I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \\
&+\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&\qquad A_1 A_2 + A_2 A_1 + a_{11} A_2 + a_{03} I_2 = \\
&\qquad = 2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&\qquad A_2^2 + a_{04} I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

## CAYLEY-HAMILTON THEOREM FOR SINGULAR SYSTEMS

Consider the singular systems

$$
E\dot{x} = Ax + Bu \tag{37}
$$

where $x \in R^n$, $u \in R^m$, $E$, $A \in R^{n\times n}$, $B \in R^{n\times m}$.
It is assumed that det $E=0$,

$$
\det\left[Es - A\right] \neq 0 \quad \text{for some} \quad s \in C \tag{38}
$$

and

$$
EA = AE \tag{39}
$$

If the assumption (38) holds then it is easy to show (Kaczorek 1988) that the matrices

$$
\bar{E} = \left[Es - A\right]^{-1} E, \quad \bar{A} = \left[Es - A\right]^{-1} A
$$

satisfy the conditions

$$
\overline{E}\overline{A} = \overline{A}\overline{E}
$$

**Theorem 9.** *Let*

$$
\begin{aligned}
&\det\left[Es - A\right] = a_r s^r + a_{r-1} s^{r-1} + \ldots + a_1 s + a_0 \\
&\left(r = \operatorname{rank} E < n\right)
\end{aligned}
\tag{40}
$$

*Then*

$$
\sum_{i=0}^{r} a_i A^i E^{n-i} = 0 \tag{41}
$$

**Proof.** Let

$$\operatorname{Adj}\left[Es-A\right]=B_{n-1}s^{n-1}+\ldots+B_1s+B_0 \tag{42}$$

be the adjoint matrix of [*Es-A*].
From definition of the inverse matrix and (40), (42) we have

$$\begin{aligned}&\left[Es-A\right]\left[B_{n-1}s^{n-1}+\ldots+B_1s+B_0\right]=\\&=I_n\left(a_rs^r+a_{r-1}s^{r-1}+\ldots+a_1s+a_0\right)\end{aligned} \tag{43}$$

The comparison of the coefficients at the same powers of *s* of the equality (43) yields

$$\begin{bmatrix}E & 0 & 0 & \ldots & 0 & 0 & 0\\ -A & E & 0 & \ldots & 0 & 0 & 0\\ 0 & -A & E & \ldots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & -A & E & 0\\ 0 & 0 & 0 & \ldots & 0 & -A & E\\ 0 & 0 & 0 & \ldots & 0 & 0 & -A\end{bmatrix}\begin{bmatrix}B_{n-1}\\ B_{n-2}\\ B_{n-3}\\ \vdots\\ B_2\\ B_1\\ B_0\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ \vdots\\ a_rI_n\\ \vdots\\ a_1I_n\\ a_0I_n\end{bmatrix} \tag{44}$$

Premultiplying (44) by the row matrix

$$\begin{bmatrix}A^n & A^{n-1}E & A^{n-2}E^2 & \ldots & AE^{n-1} & E^n\end{bmatrix} \tag{45}$$

and using (39) we obtain the equation (41).

■

**Example 3.** Let

$$E=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ a & 0 & 0\end{bmatrix},\quad A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{46}$$

The pair (46) satisfies the condition (39).
In this case

$$\det\left[Es-A\right]=\begin{vmatrix}-1 & s & 0\\ 0 & -1 & s\\ as & 0 & -1\end{vmatrix}=as^3-1\quad\left(a_0=1,a_3=a\right) \tag{47}$$

Using (46) and (47) we obtain
for $a\neq 0$

$$\begin{aligned}&\sum_{i=0}^{3}a_iA^iE^{3-i}=-E^3+aA^3=\\&=-a\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}+a\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\end{aligned}$$

and for $a=0$

$$a_0E^3=-E^3=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}^3=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

## CONCLUDING REMARKS

New extensions of the classical Cayley-Hamilton theorem are presented with some applications and illustrating examples.

The Cayley-Hamilton theorem has been extended for rectangular matrices, block matrices, discrete-time and continuous-time systems with delays and singular systems.

It has been shown that rectangular matrices satisfy many different algebraic equations and the matrices $A_k$, *k=0,1,...,h* of the continuous-time systems (25) with *h* delays satisfy *nh+1* algebraic equations.
With slight modifications the presented extensions for systems with delays can be extended for rectangular matrices and block matrices.

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