REVISION OF MATHEMATICAL APPROACH TO ELECTRICAL CIRCUIT MODELLING

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ABSTRACT

Discussion of two possible approaches to the same problem: linear electrical circuits modelling is used to define the mathematical background which students need to master before they start with the circuit analysis. Classical, analytical approach is compared to numerical one, and the better choice is proposed. Consequences to the teaching, for the moment, are unpredictable but intriguing.

An RLC circuit is a frequent example in the most of textbooks (Carter and Richardson 1972, Desoer and Kuh 1969, Nilsson and Riedel 1996) about electrical engineering theory fundamentals. Although the circuit is as simple as possible, it is of great educational value. Mathematics needed for the circuit analysis is a core of mathematic modelling used in all other linear electrical circuits analysis. The difference is in the number of components and equations complexity, but the principles are the same.

INTRODUCTION

It is customary to analyse the series RLC circuit (Fig. 1) with the initial circuit current i(0) and electrical charge q(0). The circuit's history is of no importance; the future is important! The R, L and C values are known, voltages and current are oriented and marked as shown in circuits diagram. Due to the lack of independent voltage or current sources, the circuit response on the initial conditions is the free one (zero-input response). The circuit energy is dissipated gradually on the resistance R in the form of heat, so it is to be expected that current and charge, sooner or later, are reduced to zero.

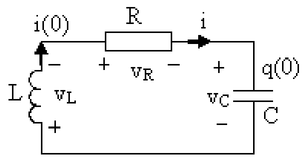

Figure 1: Series RLC Circuit Diagram

CIRCUIT ANALYSIS

For the clockwise referent loop direction, Kirchhoff's voltage law equation is:

$$v_L + v_R + v_C = 0 \tag{1}$$

Voltages are calculated from the current and charge:

$$v_L = L\frac{di}{dt} \tag{2a}$$

$$v_R = R \cdot i \tag{2b}$$

$$v_C = \frac{q}{C} \tag{2c}$$

$$i = \frac{dq}{dt} \tag{2d}$$

By inserting 2a, 2b and 2c into 1, differentiating and dividing by L, one could get:

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \tag{3}$$

Equation (3) is the second order homogenous differential equation with constant coefficients. Equation solution is started by the **assumption**, that it is in the following form:

$$i = Ae^{-rt} \tag{4}$$

The constants A and r are to be calculated from the circuit initial values and elements values. After (4) is inserted in (3), the so called characteristic equation is:

$$r^2 + \frac{R}{L}r + \frac{1}{LC} = 0 \tag{5}$$

Two new values: the damping factor δ and free oscillations angular frequency ω_0 are defined as follows:

$$\delta = \frac{R}{2L} \tag{6a}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{6b}$$

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With (6a) and (6b), (5) may be transformed to:

$$r^2 + 2\delta \cdot r + \omega_0^2 = 0 \qquad (7)$$

By **well known procedure**, two solutions are:

$$r_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2} \qquad (8)$$

Generally, characteristic equation (5) has two solutions. It is **expected** that the circuit current is:

$$i = Ae^{-r_1 t} + Be^{-r_2 t} \qquad (9)$$

Constants A and B are to be calculated from the initial condition:

$$\mathrm{A} + \mathrm{B} = \mathrm{i}(0) \qquad (10)$$

$$r_1 \mathrm{A} + r_2 \mathrm{B} = -\frac{\mathrm{i}(0)\mathrm{R}}{\mathrm{L}} - \frac{q(0)}{LC} \qquad (11)$$

With **a little bit longer procedure**, one could get a general solution for the circuit current i(t):

$$i(t) = i(0) \cdot e^{-\delta t}\left[ch\sqrt{\delta^2 - \omega_0^2}\,t - \frac{\delta}{\sqrt{\delta^2 - \omega_0^2}} sh\sqrt{\delta^2 - \omega_0^2}\,t\right] - \frac{q(0)\omega_0^2}{\sqrt{\delta^2 - \omega_0^2}} e^{-\delta t} sh\sqrt{\delta^2 - \omega_0^2}\,t \qquad (12)$$

By making use of:

$$q(t) = \int_0^t i(t)dt + q(0) \qquad (13)$$

it is obtained:

$$q(t) = \frac{i(0)e^{-\delta t} sh\sqrt{\delta^2 - \omega_0^2}\,t}{\sqrt{\delta^2 - \omega_0^2}} + q(0) \cdot e^{-\delta t}\left[\frac{\delta}{\sqrt{\delta^2 - \omega_0^2}} sh\sqrt{\delta^2 - \omega_0^2}\,t + ch\sqrt{\delta^2 - \omega_0^2}\,t\right] \qquad (14)$$

From (12, 14) and (2) it is possible to calculate voltages v_L, v_R and v_C. Then, it is possible to determine zero and extreme values, as well as powers and energies.

In science, it is customary that starting from general case, one get certain special cases. Equations (12) and (14) are generators of next examples.

Special Cases

In order to draw graphs of the circuit current i(t) and capacitor charge q(t), equations are to be slightly modified. Instead of real values of R, L and C, and initial values i(0) and q(0) qualitative waveforms depend on their ratio. Normalized time t_n and relative damping factor k are defined as:

$$\omega_0 t = t_n \qquad \frac{\delta}{\omega_0} = k \qquad (15)$$

Value K is:

$$K = \frac{q(0)\omega_0}{i(0)} \qquad (16)$$

There are four possible characteristic cases, dependent on the k value: k>1, k=1, k<1 and k=0. Two of them are presented.

Aperiodic Case, k>1

If k>1 then $\delta > \omega_0$ and $R > 2\sqrt{\frac{L}{C}}$ so (12) and (14) are:

$$i_n = \frac{i(t_n)}{i(0)} = e^{-kt_n}\left[ch\sqrt{k^2 - 1}\,t_n - \frac{k + K}{\sqrt{k^2 - 1}} sh\sqrt{k^2 - 1}\,t_n\right] \qquad (17)$$

$$q_n = \frac{q(t_n)}{q(0)} = e^{-kt_n}\left[(k + \frac{1}{K})\frac{sh\sqrt{k^2 - 1}\,t_n}{\sqrt{k^2 - 1}} + ch\sqrt{k^2 - 1}\,t_n\right] \qquad (18)$$

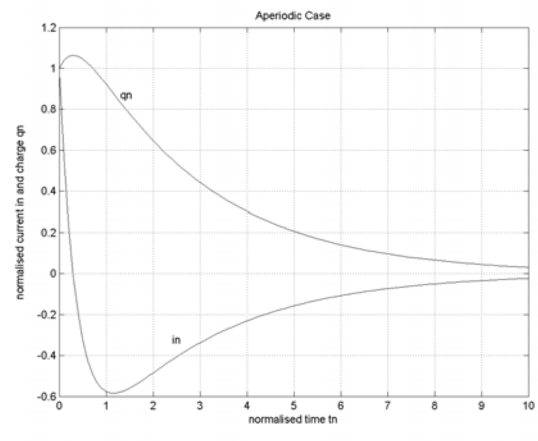


Figure 2: Aperiodic Case, K=2; k=1,5

Graphs in figure 2 are performed by MatLab 6. From Fig. 2 it is evident that the charge q(t) gains maximum value at the moment when the current is zero. The same conclusion comes from the energy conservation principle: the circuit energy is oscillating between the only two energy conservative elements: L and C.

Damped Oscillations, k<1

If k<1 then $\delta<\omega_0$ then $R < 2\sqrt{\frac{L}{C}}$

Instead of hyperbolic functions in (17) and (18), due to negative values under the square roots, one can gain trigonometric functions in the **well known manner**:

$$i_n = e^{-kt_n}\left[\cos\sqrt{1-k^2}\,t_n - \frac{k+K}{\sqrt{1-k^2}}\sin\sqrt{1-k^2}\,t_n\right] \tag{19}$$

$$q_n = e^{-kt_n}\left[(k+\frac{1}{K})\frac{\sin\sqrt{1-k^2}\,t_n}{\sqrt{1-k^2}} + \cos\sqrt{1-k^2}\,t_n\right] \tag{20}$$

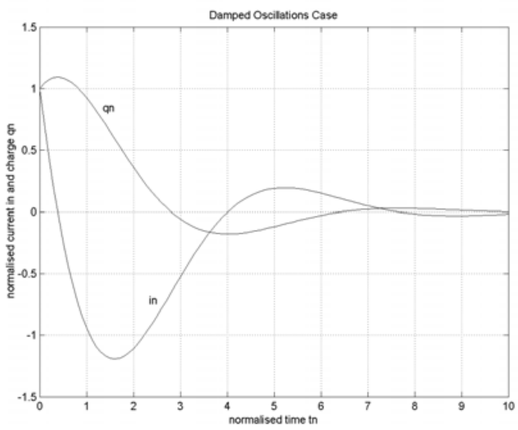


Figure 3: Damped Oscillations, K=2; k=0,5

Generally, starting from the given values of R, L, C, i(0) and q(0), δ, ω_0, k and K are to be calculated. One of the four possible cases is determined by the k-value. Adequate equations are picked up, K and k values inserted and normalized values calculated. Real values i(t) and q(t) are gained by i(0) and q(0) multiplying, respectively. Real time oscillations are calculated by scaling with the ω_0 value.

NUMERICAL, COMPUTER BASED APPROACH

Starting from the same initial equation (1) we get:

$$v_L = -v_R - v_C \tag{21}$$

Then, equations (2) are modified in the difference equation form instead of differential form.

$$v_{Li} = L\frac{\Delta i_i}{\Delta t} \tag{22a}$$

$$v_{Ri} = R \cdot i_i \tag{22b}$$

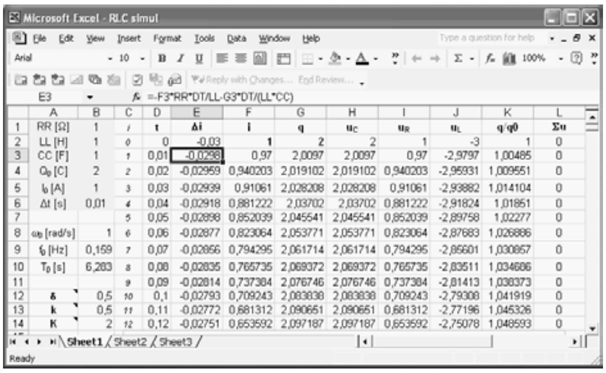
$$v_{Ci} = \frac{q_i}{C} \tag{22c}$$

$$i_i = i_{i-1} + \Delta i_{i-1} \tag{22d}$$

By inserting (22a), (22b) i (22c) into (21), one gets:

$$\Delta i_i = -\frac{R}{L}\cdot i_i \cdot \Delta t - \frac{1}{L\cdot C}q_i \cdot \Delta t \tag{23}$$

Equation (23) in spreadsheet MS Excel is programmed (Fig. 4.). In A and B columns there are circuit values and calculated values. In C column is time index, in column D is discrete time t_i, in E is discrete current difference Δi_i and in the F column, the current is calculated based on (22d) equation. Formula (23) written in Excel programming syntax is presented at fig. 6. too.


Microsoft Excel - RLC simul

E3 =-F3*RR*DT/LL-G3*DT/(LL*CC)

	A	B	C	D	E	F	G	H	I	J	K	L
1	RR [Ω]	1	i	t	Δi	i	q	u_C	u_R	u_L	q/q0	Σu
2	LL [H]	1	0	0	-0,03	1	2	2	1	-3	1	0
3	CC [F]	1	1	0,01	-0,0298	0,97	2,0097	2,0097	0,97	-2,9797	1,00485	0
4	Q_0 [C]	2	2	0,02	-0,02959	0,940203	2,019102	2,019102	0,940203	-2,95931	1,009551	0
5	I_0 [A]	1	3	0,03	-0,02939	0,91061	2,028208	2,028208	0,91061	-2,93882	1,014104	0
6	Δt [s]	0,01	4	0,04	-0,02918	0,881222	2,03702	2,03702	0,881222	-2,91824	1,01851	0
7			5	0,05	-0,02898	0,852039	2,045541	2,045541	0,852039	-2,89758	1,02277	0
8	ω_0 [rad/s]	1	6	0,06	-0,02877	0,823064	2,053771	2,053771	0,823064	-2,87683	1,026886	0
9	f_0 [Hz]	0,159	7	0,07	-0,02856	0,794295	2,061714	2,061714	0,794295	-2,85601	1,030857	0
10	T_0 [s]	6,283	8	0,08	-0,02835	0,765735	2,069372	2,069372	0,765735	-2,83511	1,034686	0
11			9	0,09	-0,02814	0,737384	2,076746	2,076746	0,737384	-2,81413	1,038373	0
12	δ	0,5	10	0,1	-0,02793	0,709243	2,083838	2,083838	0,709243	-2,79308	1,041919	0
13	k	0,5	11	0,11	-0,02772	0,681312	2,090651	2,090651	0,681312	-2,77196	1,045326	0
14	K	2	12	0,12	-0,02751	0,653592	2,097187	2,097187	0,653592	-2,75078	1,048593	0

Sheet1 / Sheet2 / Sheet3

Ready

Figure 4: MS Excel spreadsheet table

Instead of equation (13) for the time function of the charge q(t) time-discrete charge values are calculated by recursive addition:

$$q_i = i_i \cdot \Delta t + q_{i-1} \tag{24}$$

Therefore, problems with numerical integration are, for the moment, avoided. Calculated values are in the G-columns.

In B1 cell there is R value, in B2 is L, C is in B3. Initial charge value is in B4 cell and initial current in B5 is placed. Time-step value Δt is in B6 cell. Calculated free-oscillations angular frequency is in B8, frequency is B9 and period in B10. Values δ, k and K are in cells B12, B13 and B14, respectively.

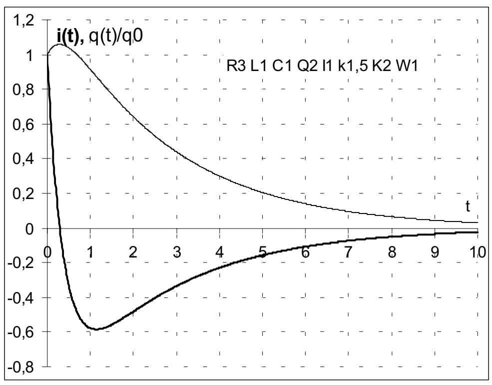


Figure 5: Aperiodic Case, K=2; k=1,5

Each of four characteristic cases are calculated and graphically presented, Figs. 5 and 6.

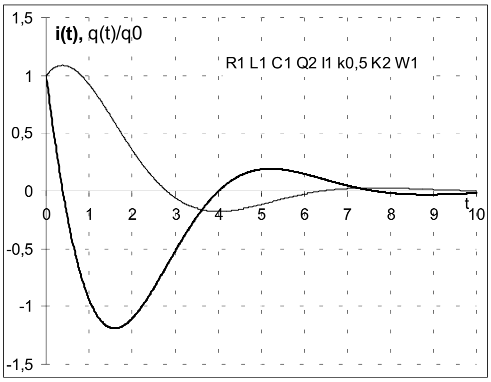


Figure 6: Damped Oscillations, K=2; k=0,5

CONCLUSION

Illustrated by a well known simple example from electrical networks, the intention of this paper is to mark mathematical knowledge and skill needed for teaching and learning of electrical circuits analysis. Together with fundamental laws of physics and electricity, the student has to implement trigonometric and hyperbolic functions, algebraic equations solving and their roots discussion, linear differential equations solving, and limits, derivates and integrals calculating. Higher semester students should be able to solve those problems by Laplace transformation, too.

Expected previous knowledge is bolded in the paper above:

- Waveform assumption, equation (4).
- Well known differential equations (4) to (7) solving procedure.
- Routine procedure from (10, 11) gives A and B...

Only a few students can understand processes dynamics directly from equations (12) or (14) even in the case of simple circuit such as series RLC. The saying "Picture is thousand words worth!" is confirmed here, i.e. it is obvious why visualization and graphs are so frequently used. So, if graphs are a suitable way to present electrical circuits processes, it is reasonable to pose a question:

Is it possible to make a bypass, and generate graphs by simpler, better or more accessible mathematical procedure? From the point of a student, of course!

Or:
Could it be, that computer based approach will reduce and simplify mathematical background the student needs to understand processes to be learned?

Today, there is broad spectrum of software tools for circuits analysis even without any higher mathematics knowledge. All one needs is to understand the problem, know how to insert data and how to interpret results obtained.

But, it is doubtful for the knowledge level if lecturer's theoretical explanation is illustrated and supported by simulation programs or virtual laboratories only. What is student's knowledge benefit if he or she has to type data in marked fields, press "ENTER" and look to or print data table or graph? And do it tents or hundreds of time!

The middle way seems to be more productive: Computer modelling in general purpose programming language, based on well known fundamentals of physics (charge and energy conservation, Ohm's and Kirchhoff's law) and mathematics which students have already mastered.

In the previous chapter such an approach is described. Outcomes of numerical and analytical models are quite satisfactory and close to each other. Mathematics used in numerical modelling vs. analytical is quite simple and MS Excel programming is a basic of 21-th century literacy!

Consequently, teaching subjects like Electrical Engineering Fundamentals does not have to wait for Mathematics to be completed. It is even more important now, when all European countries are faced with education system crisis and Bologna initiative as possible solution. Shortened first study level (three years) gives no time for a traditional sequence (Math I, Math II, Physics...). Each reasonable way to start with core teaching subjects without any introduction is to be considered.

The example presented is well known just for the reason to be widely recognized: RLC circuit and its mathematical modelling, comparison of the analytical and numerical one. It is shown that the starting point: physical laws of a serial RLC electrical circuit and final result to be discussed: current, voltage and charge diagrams are the same! But, in the case of numerical approach, the "path" is much shorter (just few equations) and simpler (multiplication and addition instead of differential equations solving). Instead of detailed mathematical analysis, numerical mathematics is to be preferred! If needed, deeper analysis based on higher mathematics may be postponed for the second, graduated study level.

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