GRAPHING ZHUKOVSKI TRANSFORMATION IN DERIVE

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ABSTRACT
Zhukovski (noted also as Joukovsky) transformation is a conformal mapping and generates families of ortogonal lines. It is famous for its application in aerodynamics where it reduces the flow around a wing profile to the flow around a circle. The paper gives a systematic presentation of Zhukovski map and transformations associated to it, with an emphasis on their visualisation and the use in the simulation. Taking into account that all mathematical manipulations and graphs are made within DERIVE 5.05 from Texas Instruments, the paper shows the potential use of this computer algebra system in lecturing, modelling and simulating plane flows.

1. The Russian physicist Nikolai Zhukovski (1847-1921) worked on fluid mechanics in both theoretical and experimantal aspects, see e.g. Betyaev 2003, (Lazarev 1999). In particular, he showed that the investigation of the flow around an airfold (the tranversal section an aircraft wing or propeller) can be reduced to the flow around a circle. In particular, it lets to simulate the flow around wing profiles via the observation of easier case which are phenomena around circles. This reduction bases on the function, which is called after his name a Zhukovski map. It is defined by the formula

$$f(z) := (z + a/z)/2,$$

where a is a positive number (and, usually, a = 1). Zhukovski transformation f is a conformal mapping, i.e. it map from the plane (which can be regarded as $\mathbf{R}^2$ or **C**) to itself which preserves angles. That is, the angle between any two curves is the same as the angle between their images.

2. Using the Gauss representation $z = x + y \cdot i$, where both x and y are real and $i := \sqrt{-1}$, we have

$$f(x + y \cdot i) = u + v \cdot i,$$

where $u := x \cdot (s + 1)/(2s)$, $v := y \cdot (s - 1)/(2s)$

and $s := x^2 + y^2$.

Both the real part u and the imaginary part v of f satisfy Laplace equation of variables x and y (and, by the theory of analitic functions, it has to be), both they define lines in the real plane R^2. These lines form

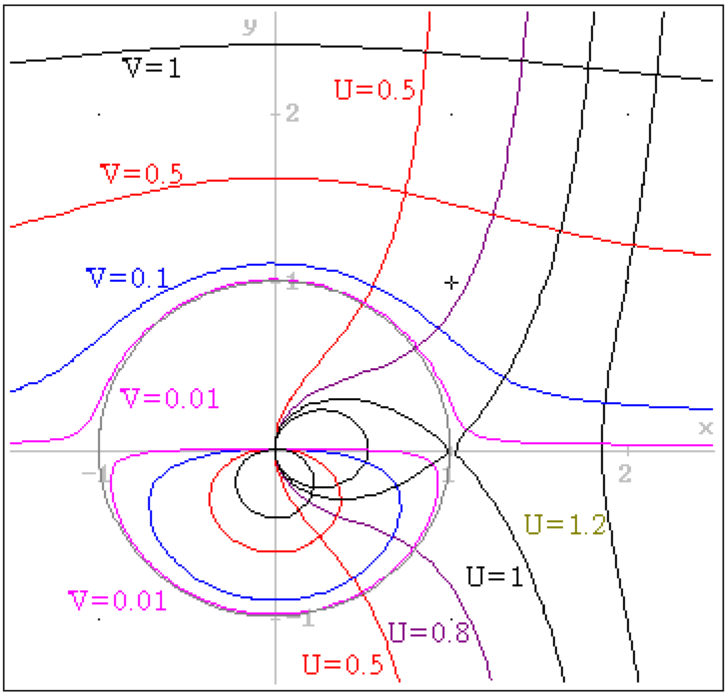

Figure 1: Lines governed by equations $u = U$ with $U = 0.5, 0.8, 1, 1.2$ and $v = V$ with $V = 0.01, 0.1, 0.5, 1$.Moreover, there is drawn the unit circle $x^2 + y^2 = 1$.

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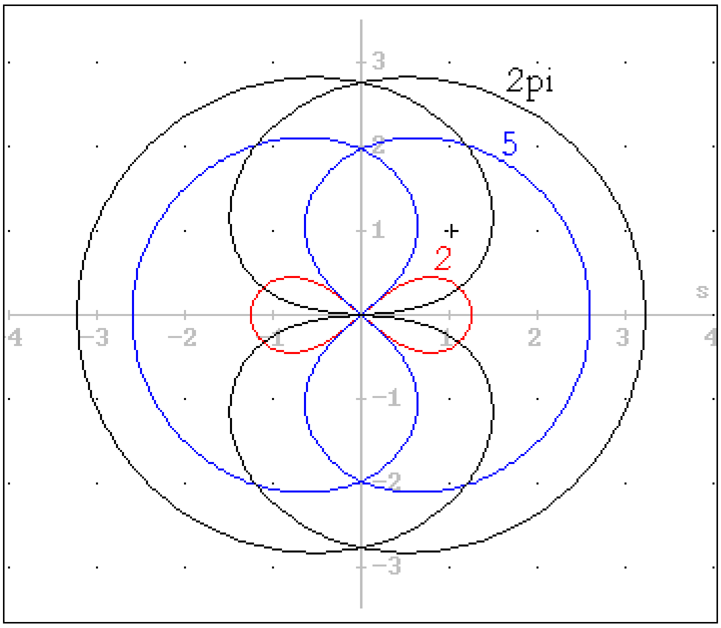

Figure 2: Lines in the polar coordinate system Osφ governed by equations s = (r + a/r)/2·cos(θ), φ = (r – a/r)/2·sin(θ), for a = 1, r = 2, 5, 2π, and θ running from 0 to 2π

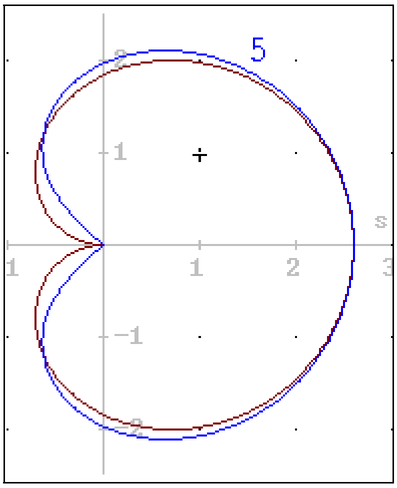

Figure 3: In the polar coordinate system Osφ: the line governed by the parametric representation s = 13/2·cos(θ), φ = 12/2·sin(θ), where θ runs from –π/2 to π/2, and the arc of Grandi rose s = 13/5·cos(φ/2) traced when φ runs from 0 to π

orthogonal families, see also (Puel 1999). Some lines governed by equations u = U and v = V are shown in Fig.1. In fluid mechanics these lines are called that of equivelocity lines and streamlines, respectively. The last ones are also shown in Fig.10.

3. Obviously, the trigonometric representation z = r·exp(i·θ), where both r, θ are real (and interpretated as the distance to the origin O and the angle, respectively), gives the same lines, in the polar coordinate system Orθ they are governed by the equations u = U and v = V, where u and v are real and imaginary parts of f(r·exp(i·θ)), i.e.

$$u = (r + a/r)/2 \cdot \cos(\theta),$$

$$v = (r - a/r)/2 \cdot \sin(\theta).$$

4. Let's use above relations to define the function

$$g(r,\theta) := [(r + a/r)/2 \cdot \cos(\theta),\ (r - a/r)/2 \cdot \sin(\theta)]$$

depending on two real parameters r and θ.
For any fixed r = R in the rectangular coordinate system Ouv these equations determine ellipses. Shapes traced in the polar coordinate system Osφ are also finite and closed graphs, see Fig.2.
For example, for a = 1 and R = 5 we have s = 13/5·cos(θ), φ =12/5·sin(θ), see Fig.3, and its shape is close a petal of Grandi rose (see Xah Lee, "A Visual Dictionary of Famous Plane Curves", www.xahlee.org/SpecialPlaneCurves_dir/Rose_dir/rose.html) covered by the equation s = 13/5·cos(φ). Setting r = a = 1 we get the interval <–1, 1> on the real axis.

Fixing the parameter θ, in the rectangular coordinate system Ouv we get the infinite curves which describe the stream lines of the fluid passing along the vertical axis Ov and hampered by an infinitesimaly thin segment laying on the horizontal axis from the point (a, 0) to the right. For a = 1 and some values of θ see these lines in Fig.4. They include θ = 0° and θ = 90°, and then we have the obstacle and the vertical axis, respectively.
The parameter θ fixed in the polar coordinate system Osφ, we get rather complicated curves. An arc of one of them is traced in Fig.5.

5. Let's take the quotient of quantities forming the parametric representation g(r,θ). Denoting it by q we have $q = (r^2 + a)/(r^2 - a) \cdot \cot(\theta)$. With θ fixed it defines the function of r. A graph of this function in the polar coordinate system Oqr (so here r is must be treated as the angle, not as the radius) for θ = 30° is shown at Fig.6. One can compare it to the graph plotted in Fig.7 besides, in J.Wassenaar's "Mathematical curves" (http://www.2dcurves.com) the graphs similar to that last are called atom spirals (first produced by Annie van Maldeghem in 2002).
Both relations, $q = (r^2 + a)/(r^2 - a) \cdot \cot(\theta)$
and $q = (r + a)/(r - a) \cdot \cot(\theta)$,
in the rectangular coordinate system Orq (so r marks the hoizontal axis, and q is measured along the vertical axis) describe well-known hiperbola and an other simple curve, see Fig.8.

6. A flow around a circular cylinder is a fundamental fluid dynamics problem of practical importance. In general case there isn't known its

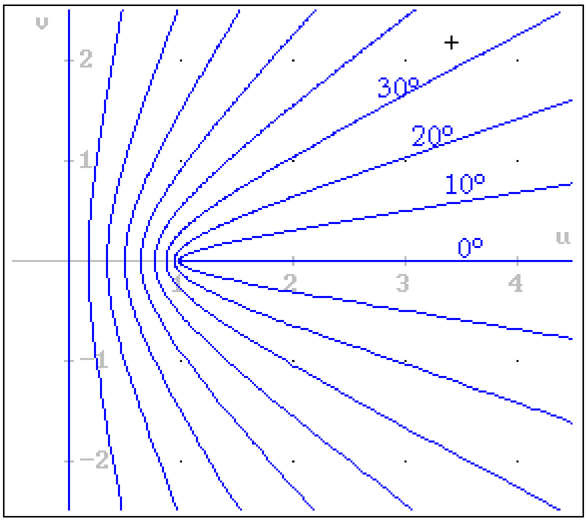

Figure 4: Lines in the rectangular coordinate system Ouv governed by equations $u = (r + a/r)/2 \cdot \cos(\theta)$, $v = (r - a/r)/2 \cdot \sin(\theta)$, for $a = 1$, $\theta = \cdot 0°, 10°, 20°, \ldots, 80°, 90°$, and r running the interval $(-\infty, +\infty)$

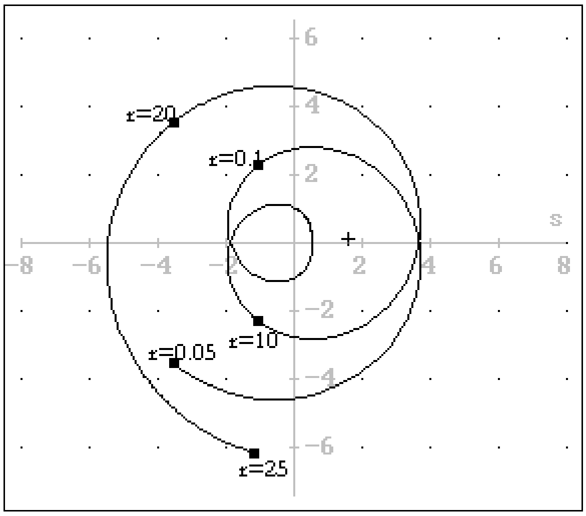

Figure 5: The graph in the polar coordinate system Osφ governed by the parametric respresentation $s = (r + a/r)/2 \cdot \cos(\theta)$, $\varphi = (r - a/r)/2 \cdot \sin(\theta)$, for $a = 1$, $\theta = 30°$, and r running the interval (0.05, 25)

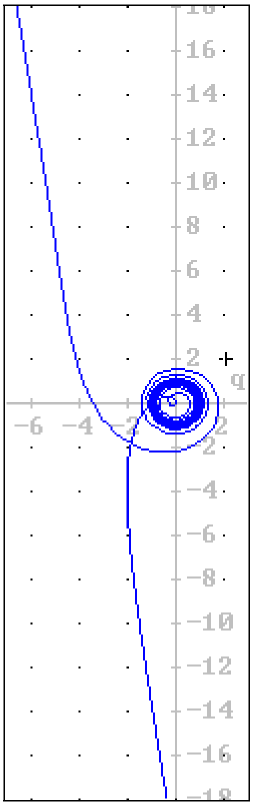

Figure 6: $q = \sqrt{3} \cdot (r+1)/(r-1)$ in the polar coordinates (q,r)

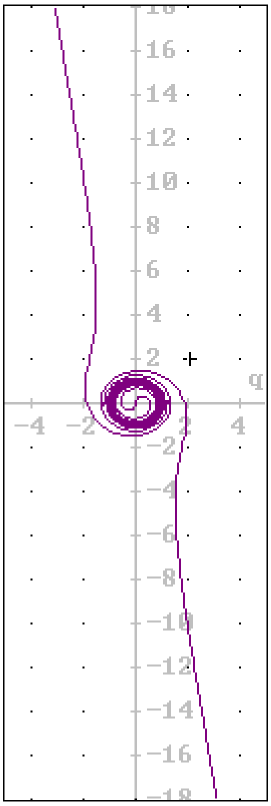

Figure 7: $q = \sqrt{3} \cdot (r^2+1)/(r^2-1)$ in the polar coordinates (q,r)

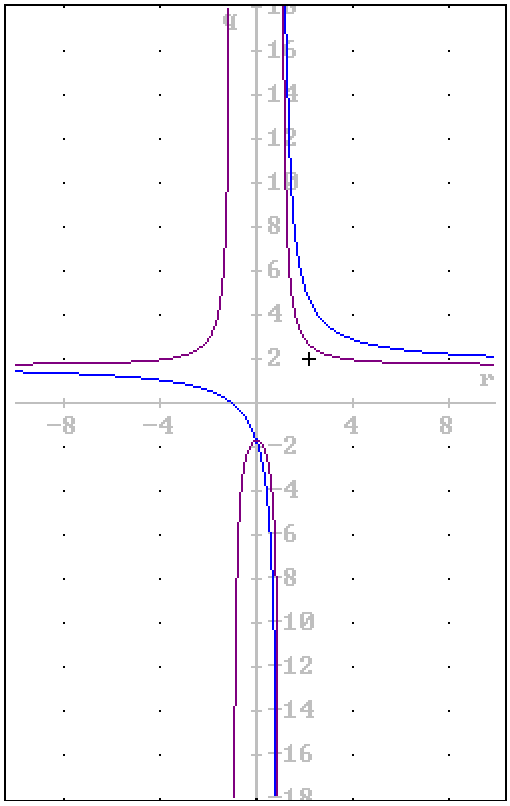

Figure 8: Hiperbola $q = \sqrt{3} \cdot (r+1)/(r-1)$ and the line $q = \sqrt{3} \cdot (r^2+1)/(r^2-1)$ in the rectangular coordinates (r,q)

analytical solution, the experiments reveal its character essentially depends on the Reynolds number: at its low values, below circa 80, the flow filed is symmetric, and the flow begins to separate (and the unsteady phenomenon, the vortex shedding, takes place) as the Reynolds number increases. In case of an uniform flow given simply as the map $z \to \nu \cdot z$, where ν stands for the velocity of the flow of the fluid, by the circle theorem (a.k.a. Milne-Thompson Theorem) we obtain that the complex potential of the flow past a disk of radius $\sqrt{a}$ is f(z), and its real and imaginary parts are called

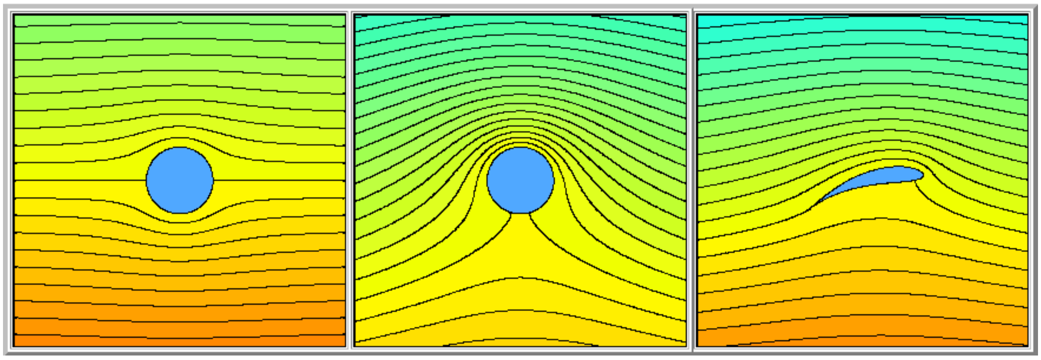

Figure 9: The streamlines of the uniform flow passing along the horizontal axis, from right to left, around a circle with zero circulation and with non-zero (two first captures)and around the airfoil (the right capture), where the attack angle is taken into account by the positioning of the airfoil (M. Colombini, "Moti irrotazionali piani di fluidi ideali", http://www.diam.unige.it/~irro/ lecture _e.html).

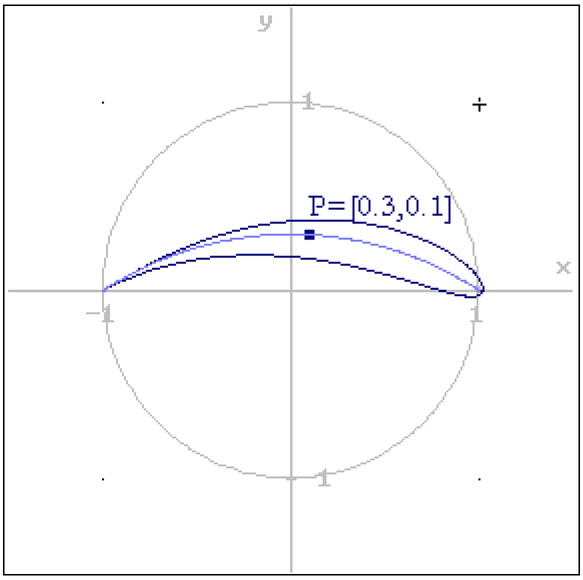

Figure 10: Zhukovski airfoil determined by the point P = (0.1, 0.3) and its central line which is Zhukovski profile determined by (0, 0.3)

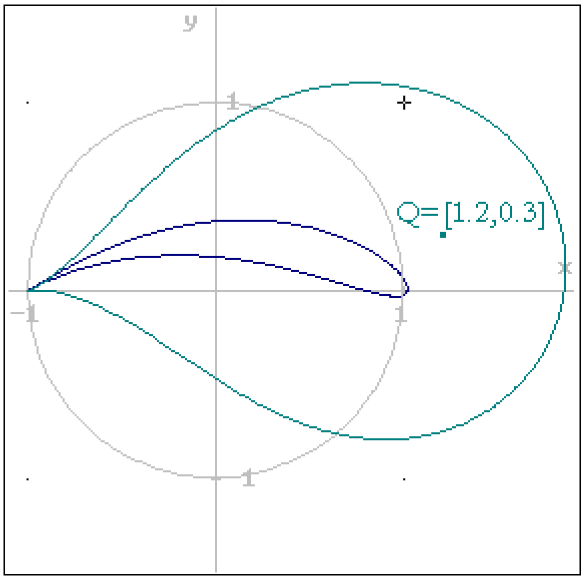

Figure 11: Zhukovski airfoil copied from Fig.10 and Zhukovski profile determined by Q = (1.2, 0.3)

velocity and stream function, respectively. To take into account the attack angle α and the circulation parameter γ we modify the Zhukovski function to have the map

$$z \to f(z \cdot \exp(\alpha \cdot i)) + \gamma \cdot \ln(z \cdot \exp(\alpha \cdot i))/(2\pi).$$

Analogical situation occurs in the aerodynamics, and it is even more difficult because there have tobe considered the shapes which are more complex than a cyclinder (see Fig.9). A typical shape is the airfoil (such as in Fig.10). The Zhukovsky function transform a circle (which, obviously, is a trace of a cylinder on a plane) into an airfoil, and its inverse, $w \to -w \pm \sqrt{(w^2 - a)}$, realises the back mapping.

7. The image of a circle under the Zhukovski function is called (its) Zhukovsky profile. Let a circle be centered at the point c_0 and passes through the point $(-a, 0)$. Then its Zhukovski profile reduces to the interval $<-a, a>$ if $c_0 = (0, 0)$, and to a symmetric arc if $c_0 = (0, y_0) \neq (0, 0)$.

8. A really wide applications of Zhukovski function f is in aerodynamics, where it transforms a circle into the curve called Zhukovski profile. If $a = 1$, this circle contains the point $(-1, 0)$ of the Cartesian plane. Thus in the complex plane Oz it has the equation

$$z = s_0 + r_0 \cdot \exp(i \cdot \theta),$$

where $s_0 = x_0 + y_0 \cdot I$,

$$r_0 = \sqrt{\{(x_0 + 1)^2 + y_0^2\}}.$$

Now the formulas $x = \mathrm{Re}(f(s_0 + r_0 \cdot \exp(i \cdot \theta)),$

$$y = \mathrm{Im}(f(s_0 + r_0 \cdot \exp(i \cdot \theta))$$

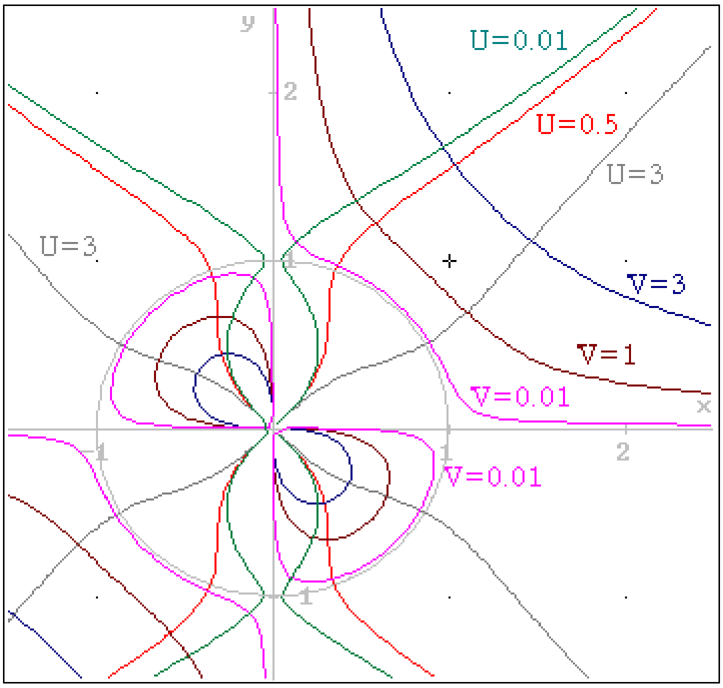

Fig.12. Lines governed by equations
$u = U$ ($U = 0.01, 0.5, 1, 3$)
and $v = V$ ($V = 0.01, 1, 3$),
where u and v are real and imaginary parts of the expression $f^2(x+y\cdot i)$.
Identically as in Fig.1, it is also drawn the unit circle $x^2 + y^2 = 1$.

cover the Zhukovski profile with respect to the point s_0. For example for $s_0 = (1+3\cdot i)/10$ they are

$$x = (\sqrt{130}\cos(\theta)+1)\cdot(m+1/20)$$

$$y = (-\sqrt{130}\sin(\theta)+3)\cdot(m+3/20)$$

where $m:=\sqrt{10}/\{4\cdot(\sqrt{13}\cos(\theta)+3\cdot\sqrt{13}\sin(\theta)+7\sqrt{10})\}$, see Fig.10.

9. It's clear, the Zhukovski transformation may be involved in other operations. An example is given in (Nersessian and Ter-Antoyan 1997) where Zhukovski function is used to parametrize the oscillator trajectories. It leads to the square of Zhukovski function, i.e. the z complex plane is mapped by the function

$$f^2 : z \to (z + 1/z)^2 = z^2 + 2 + z^{-2}.$$

Some of curves determined by this transformation are reproduced in Fig.12, and it is worth to compare them to that in Fig.1.
Another example dealing with Zhukovski and Karman-Trefftz mappings is (Cześnik and Prosnak 2002).
A wide spectrum of web-pages with Zhukovski airfoils is offered by J.H.Matthews in "Joukovski transformation and airfoils" (http://math.fullerton. edu /mathews/c2003/Joukowski Trans Bib.html). In R.Ferréol's "Coubre de Joukovski, profile d'aile d'avion" (http://www.mathcurve.com/courbes2d/ joukowski/joukowski.shtml) there is presented the animation showing that the Zhukovski airfold is the locus of middle points of appropriate segments having ends on two circles.

10. In this paper all figures except Fig.9 are plotted by DERIVE 5.04, a computer algebra system from Texas Instrumants, Inc.; for more information see producer's webpage as well as numerous papers, a.o. (Jankowski and Marlewski 2005), where there are widely discussed advances of the use of symbolic algebra in sciences and education. All mathematical transformations were also did within this system. This paper is completed with an Appendix, a collection of definitions to handle the subject in DERIVE. If this system is in use, in lessons in complex variable functions and fluid mechanics the functions listed in Appendix can help teachers and students to handle classical Zhukovski mapping as well as similar transformations, as Karman-Trefftz maps discussed in (Iollo and Zannetti 2000) and (Simakov et al. 2000).

11. COCLUSIONS Maple from Waterloo Maple Inc., Mathematica from Wolfram Research Inc., MatLab from MathWorks Inc. and other computer algebra systems offer additional packages or files to handle Zhukovski transformation. Till now in this family of computer programs for symbolic manipu-lations there is absent the system DERIVE, where a good quality and high achieve-ments meet moderate price and user-friendly control, so it conquers still more and more students all over the world, and its educational are well stated, see e.g. (Koepf 2000). Its research utility is also well acclaimed, see e.g. (Marlewski and Kołodziej 1997). Surely, DERIVE may be applied to investigate Zhukovski transformation, too. This paper proposes the way at which it can be done and, at the same time, wants to show potential areas to be covered by this approach.

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ADAM MARLEWSKI, born in Czempiń, Poland, studied numerical methods at Adam Mickieiwcz University in Poznań. In 1981 he got Ph.D. for the dissertation on the Bernstein polynomial approximation. Since 1973 he works for Poznań University of Technology, in 1992-94 he lectured in Universidade da Beira Interior in Covilhã, Portugal. He published books on numerical methods and DERIVE, he co- and authored papers in pure and applied mathematics, on computer algebra systems, induction machines, simulation in remote sensing and in civil engieneering. His webpage is http://www.math.put.poznan.pl/~amarlew.

APPENDIX

A collection of expression to handle Zhukovski map and its associates in DERIVE 5.04

ZhukovskiFunction(a,z):=(z+a/z)/2

ZhukovskiFunction(a,x+y·î) simplifies to a·x/(2·(x^2+y^2))+x/2+î·(y/2–a·y/(2·(x^2+y^2))).

u:=RE(ZhukovskiFunction(a,x+y·î)) and v:=IM(ZhukovskiFunction(a,x+y·î))

returns the real and imaginary parts of the above expression, respectively.

u=U and v=V plot graphs as that shown in Fig.1, if earlier it is set a:=1.

flowAroundCylinder(α,γ,z):=(z·EXP(α·î) + 1/z·EXP(-α·î))/2 + γ/(2·π)·LN(z·EXP(α·î))

where α and γ stand for the attack angle and the circulation parameter, respectively.

flowAroundCylinder(α,γ,x+y·î) simplifies to Gauss form of the above complex expression.

velAC(α,γ,x,y):=RE(flowAroundCylinder(α,γ,x+y·î)) returns the velocity.

strAC(α,γ,x,y):=IM(flowAroundCylinder(α,γ,x+y·î)) returns the stream.

The simplification of velAC(α,γ,x,y)=c is plotted as the line of the velocity equal to a constant c.

The simplification of strAC(α,γ,x,y)=c is plotted as the streamline corresponding to a constant c.

complexCircle(c,θ):=c_1+ î·c_2+√((c_1+1)^2+c_2^2)·(COS(θ)+î·SIN(θ))

defines a circle centered at a point c=(c_1,c_2) and passing through the point (–1,0).

Given c and a=1, the simplification p:=ZhukovskiFunction(complexCircle(c,θ))

returns the complex expression, its real and imaginary parts compose the vector

airfoil(c,θ):=[RE(p),IM(p)]

which parametrically describes a Zhukovski profile. For appropriate point c (c has to be a point within the unit circle, but not on the vertical axis) it returns the Zhukovski airfold. For example, airfold([0.3,0.1],θ) simplifies to [√10·(√130·COS(θ)+1)/(4·(√13·COS(θ)+3·√13·SIN(θ)+7·√10))+√130·COS(θ)/20+1/20,
-√10·(√130·SIN(θ)+3)/(4·(√13·COS(θ)+3·√13·SIN(θ)+7·√10))+√130·SIN(θ)/20+3/20],

and its graph, traced for θ running the interval <0,2π>, is shown in Fig.10.