INTERPOLATION FOR NON-REGULARLY LOCATED WELLS OF HYDROGEOLOGICAL MODELS

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KEYWORDS
Hydrogeological models, interpolation.

ABSTRACT

Locations of production and monitoring wells do not coincide with nodes of hydrogeological model (HM) and these locations may be considered as non – regular points that should be attached to HM by interpolation. The paper is devoted to this type of interpolation that improves accuracy of HM. New results are reported that are especially important for regional HM where approximation grids are coarse.

INTRODUCTION

The vector φ of the piezometric head is the numerical solution of a boundary field problem which is approximated in nodes of a HM grid of by the formula:

$$A\varphi = b\ ,\ A = A_{xy} + A_z - G\ ,\ b = \beta_\psi - G\psi \qquad (1)$$

where the matrices A_{xy}, A_z, G represent, correspondingly, current transmittivity a_{xy} of aquifers (these links are arranged in *xy*-planes), the vertical ties a_z originated by aquitards (if the semi-3D scheme is used) , the elements g_{xy} , g_z connecting nodes of the grid with the boundary conditions ψ; the vector β accounts for boundary flows. They also include the vector β_w of groundwater discharge/recharge from wells.

The φ and ψ-distribution of (1) must reproduce values of the head measured at monitoring wells. As a rule, locations of production and monitoring wells do not coincide with nodes of the HM grid. These wells should be attached to the grid by interpolation. The roughest interpolation method moves them to the one nearest node. This method not only worsens the accuracy of φ (due to shifting positions of production wells), but also deteriorates the role of monitored head values as calibration targets. These effects may be considerable for regional HM where the plane approximation step h is large (500 m – 4000 m).

This paper is devoted to interpolation for wells of the HM grid. The reported results represent development of methods described in (Lace et al., 1995). Interpolations for non – regular points are conditionally named as the forth and back ones if they are used for forming HM and for transferring obtained results to these points, respectively.

INTERPOLATION FOR PRODUCTION WELLS

Forth interpolation for production wells is considered by using the scheme of Figure 1 for an elementary $h\times h$ block of a uniform grid. Within the block, a single flow source 0 is sited. Its flow β_0 should be interpolated among nodes n = 5,6,7,8, as follows:

$$\beta_0 = \sum_{n=5}^{8}\beta_0^n\ ,\ \beta_0^n = c_{0n}\cdot\beta_0\ ,\ \sum_{n=5}^{8}c_{0n} = 1 \qquad (2)$$

where the position of the source within the block depends on the local coordinates h_{0i}, i = 1,2,3,4 (i - projections of 0 on edges), c_{0n} – the interpolation coefficients transferring β_0 to the nodes n = 5, 6, 7, 8.

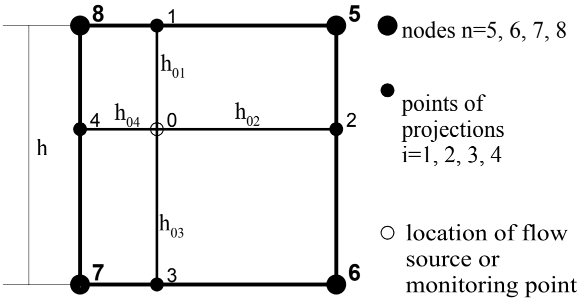

Figure 1: An elementary $h\times h$ block with a flow source or a monitoring point

The following structure of c_{0n} results from two interpolation stages ($0 \rightarrow i$; $i \rightarrow$ n):

$$c_{05} = c_{01}\cdot c_{15} + c_{02}\cdot c_{25}\ ,\quad c_{06} = c_{02}\cdot c_{26} + c_{03}\cdot c_{36}$$
$$c_{07} = c_{03}\cdot c_{37} + c_{04}\cdot c_{47}\ ,\quad c_{08} = c_{04}\cdot c_{48} + c_{01}\cdot c_{18}\ . \qquad (3)$$

The coefficients c_{0i} and c_{in} represent the stages $0 \rightarrow i$, $i\rightarrow$ n, respectively; c_{0i} are obtained, by applying the inverse distance method (IDM):

$$c_{0i} = \frac{a_{0i}}{a_{00}}\ ,\quad a_{00} = \sum_{i=1}^{4}a_{0i}\ ,\quad a_{0i} = \frac{\sigma_i}{\left(h_{0i}/h + \varepsilon\right)^\nu}\ . \qquad (4)$$

For the original version of (4), ν=1.0 was used. Due to reasons explained later, ν=1.4 provides better results. In (4), the constant $\varepsilon \sim 10^{-5}$ averts the division by zero if h_{0i} = 0. Similar measures are needed for all interpolation

Proceedings 19th European Conference on Modelling and Simulation
Yuri Merkuryev, Richard Zobel, Eugène Kerckhoffs © ECMS, 2005
ISBN 1-84233-112-4 (Set) / ISBN 1-84233-113-2 (CD)

formulas to be considered further. To simplify their description, the necessary ε-protection is not displayed. The current transmittivity σ_i at the point i depends both on its position on the edge and on the transmittivities σ_n at nodes ending the edge:

$$\sigma_1 = \left(\frac{h_{02}}{h\sigma_8} + \frac{h_{04}}{h\sigma_5} \right)^{-1}, \sigma_2 = \left(\frac{h_{03}}{h\sigma_5} + \frac{h_{01}}{h\sigma_6} \right)^{-1},$$

$$\sigma_3 = \left(\frac{h_{04}}{h\sigma_6} + \frac{h_{02}}{h\sigma_7} \right)^{-1}, \ \sigma_4 = \left(\frac{h_{01}}{h\sigma_7} + \frac{h_{03}}{h\sigma_8} \right)^{-1}. \quad (5)$$

If $h_{0i} \rightarrow 0.5\ h$, $\sigma_i \rightarrow a_{xy}$ of A_{xy}. The first version of (4) applied $\sigma_i = a_{xy}$ (Lace et al., 1995).
The coefficient c_{in} depends only on the position of the point i on the edge:

$$c_{15} = c_{36} = \frac{h_{04}}{h}, \quad c_{18} = c_{37} = \frac{h_{02}}{h},$$

$$c_{25} = c_{48} = \frac{h_{03}}{h}, \quad c_{26} = c_{47} = \frac{h_{01}}{h}. \quad (6)$$

By introducing normalized distances $\rho_{0i}=h_{0i}/h$ and the local normalized coordinates ξ and η, with the node n=7 as the origin:

$$\xi = \frac{h_{04}}{h} = \rho_{04}, \ 1-\xi = \frac{h_{02}}{h} = \rho_{02},$$

$$\eta = \frac{h_{03}}{h} = \rho_{03}, \ 1-\eta = \frac{h_{01}}{h} = \rho_{01} \quad (7)$$

and accountimg for (6), the expression (3) takes the form:

$$c_{05} = c_{01} \cdot \xi + c_{02} \cdot \eta, \quad c_{07} = c_{03} \cdot (1-\xi) + c_{04} \cdot (1-\eta),$$

$$c_{06} = c_{03} \cdot \xi + c_{02} \cdot (1-\eta), \ c_{08} = c_{01} \cdot (1-\xi) + c_{04} \cdot \eta. \ (8)$$

If in (4) ν=1.0 and σ = const, the system (8) becomes much simpler:

$$c_{05} = \xi \cdot \eta, \ c_{06} = \xi \cdot (1-\eta),$$

$$c_{07} = (1-\xi) \cdot (1-\eta), \ c_{08} = (1-\xi) \cdot \eta \quad (9)$$

The system of (9) represents the set of rectangular hyperbolas projected on the normalized block 1×1. As an example, contours of c_{07} = const of (9) are shown in Figure 2. These contours have the following features:

- forth interpolation of β_0 is linear on any line parallel to the edges of the block;
- c_{07} = 0 if $\xi = \eta = 1$ (edges 8 - 5 and 5 - 6); therefore, the influence region for the node n = 7 represents the 2 × 2 area containing four elementary blocks surrounding the node;
- if ξ or η = 0 (edges 6-7 and 7-8) then β_0 gets distributed between two nodes ending the edges.

However, the contours of Figure 2 are not circular with respect to the node n = 7, at its vicinity This drawback can be corrected if ν = 1.4 is used for c_{0i} of (4). The improved contours are shown in Figure 3:

- their shape is still close to rectangular hyperbolas if $k_{07} < 0.35$;
- interpolation is linear on edges of the elementary block.

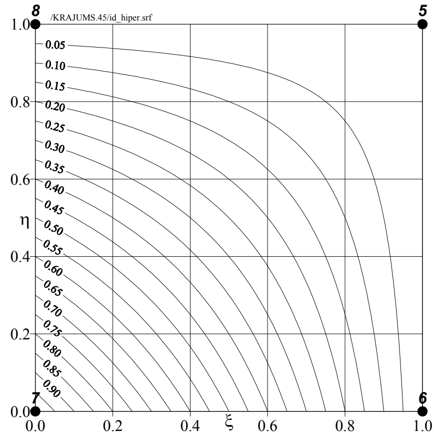

Figure 2. Contours of c_{07} as rectangular hyperbolas obtained by (9), σ = const

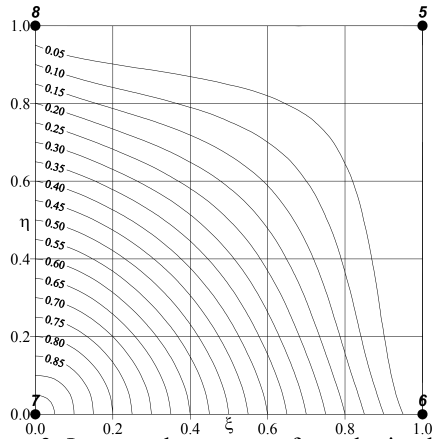

Figure 3. Improved contours of c_{07} obtained by (8) if ν=1.4, σ = const

Not any forth interpolation method possesses useful features of (8). For example, the classic IDM gives:

$$c_{07} = \left(\rho_{07} \cdot \sum_{n=5}^{8} \frac{1}{\rho_{0n}} \right)^{-1} \quad (10)$$

where $\rho_{0n} = r_{0n} / h$. In Figure 4, the contours c_{07} of (10) are shown. Their drawbacks are obvious:

- interpolation is nonlinear in any direction;
- no borderline c_{07} = 0 exists; c_{07} = 0 only at nodes n = 5,6,8; it is not possible to set justly the area of influence of the node n = 7;
- if the source β_0 is located on an edge of the elementary block then not only the endpoints

of the edge, but at least four neighboring nodes should be accounted for.

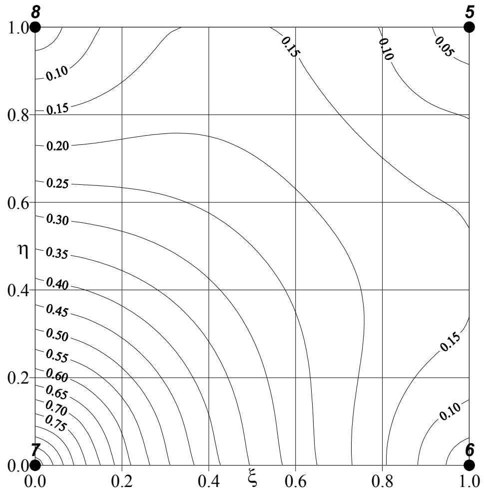

Figure 4. Contours of c_{07} obtained by the inverse distance method (10), σ = const

For a node n, the summary flow β_n resulting from interpolation of irregular β_j, which are located within the $2h \times 2h$ area of influence, is given by the formula:

$$\beta_n = \sum_{j=1}^{J} \beta_j^n \; , \qquad (11)$$

where β_j^n are the partial flows β_j of (2); J is the number of sources accounted for.

Forth interpolation of β_w improves accuracy of HM. Unfortunately, this advantage can be exploited only then if back interpolation is available for irregularly located production and monitoring wells.

RESORATION OF HEADS FOR PRODUCTION WELLS

In comparison with forth interpolation of β_w, back interpolation for non-regular points is more complex. The value of φ_0 for the source of β_0 interpolated into the neighboring nodes must be restored. The following assumption is used:

$$\varphi_0 = \sum_{n=5}^{8} \varphi_n \cdot c_{0n} + \tau_0 \cdot \beta_0 = \varphi_0^n + s_0 \qquad (12)$$

where φ_n are the computed heads at four nodes of Figure 1; c_{0n} are the improved coefficients of (8); τ_0 is the local hydraulic resistance for the source; φ_0^n , s_0 are the head and local depression, caused by the grid solution and the source, respectively. The value of τ_0 should be predicted for any location of β_0 within the elementary block. To simplify this task, it is assumed that $\sigma_n = 1$, temporarily.

For any node of the grid, $\tau_0 = 0$. The maximum of τ_0 is expected at the centre of the block where $c_{05} = c_{06} = c_{07} = c_{08} = 0.25$. The other characteristic locus is the middle of an edge where β_0 is distributed in equal parts between the two nodes ending the edge. These two special values of τ_0 were obtained experimentally, as described below.

The elementary block was conditionally placed at the central part of a homogenous grid ($\sigma = 1.0$) containing 100×100 nodes; on the borderline of the grid, the condition $\psi = 0$ was specified. A single movable unity source of $\beta_0 = 1.0$ was applied as the flow condition to be positioned and interpolated within the elementary block. Then the grid solution of (1) can be interpreted as the resistances τ_{int} at nodes with respect to the nullified borderline. The maximal possible value $\tau_m = 0.8874$ was obtained when the source was located exactly at the node. This value was practically constant for all nodes of the grid, but the ones located nearby the borderline. If the source was sited at the centre of the elementary block then the minimal value $\tau_{int} = 0.6842$ appeared at four nodes where the interpolated partial flows $\beta_0^n = 0.25$ were applied. The value $\tau_{int} = 0.7634$ was obtained for two nodes if the source was in the middle of an edge. The local resistance τ_0 to be found is $\tau_0 = \tau_m - \tau_{int}$. Results of the experiment are summarized in Table 1.

Table 1. Computed resistances for various positions of the unity source

Nr	Position of source	Resistance at node τ_{int}	Local resistance τ_0	Equivalent radius r_s of source
1	node	0.8874	0	0.1972 h
2	edge	0.7634	0.1240	0.4299 h
3	centre	0.6842	0.2032	0.7071 h

The following analytic formula for the resistance τ between two coaxial cylinders (R and r are, correspondingly, radii of the outer and inner cylinders)

$$\tau = \frac{1}{2\pi\sigma} \ln \frac{R}{r} \qquad (13)$$

is applied to find the equivalent radius r_s of the interpolated source. If $\tau = \tau_{int}$, $\sigma = 1$ then $r_s = R/\exp(2\pi\tau_{int})$ where $R = 52.059\ h$ approximated the borderline of the grid area containing 100×100 nodes.

The values of τ_0 from Table 1 are exactly repeated by (13) if $R = r_s$, $r = 0.1972\ h$. There the ratio R/r does not include h. Therefore, τ_0 depends only on the position of the source within the block $h \times h$.

It follows from Table 1 that any source located exactly in a node has the equivalent radius $r_s = 0.1972\ h$. It may be assumed that a non – regularly located source also has $r_s = 0.1972$ h. As a rule, $r_s > r_w$ where r_w is the real radius of the well. Due to this reason, the summary resistance τ_{0w} of the source is, as follows:

$$\tau_{0w} = \tau_0 + \tau_w \, , \; \tau_w = \frac{1}{2\pi\sigma_0} \ln\left(\frac{0.1972}{\rho_w}\right) ,$$

$$\rho_w = \frac{r_w}{h} \, , \; \rho_w \le 0.1972 \, ,$$

$$\sigma_0 = \sum_{i=1}^{4} \frac{\sigma_i}{\rho_{0i}} \Big/ \sum_{i=1}^{4} \frac{1}{\rho_{0i}} \qquad (14)$$

where τ_w is obtained by (13) if R = 0.1972; $r = \rho_w$; the current σ_0 is IDM interpolation on σ_i of (5).

The surface τ_0 is the main element enabling to restore heads at production wells by using (12). The initial version of the empiric formula for computing of τ_0, within the normalized block, was as follows:

$$\tau_0 = \left(0.3444 + 0.4960 a_{0i} a_i^{-1}\right) \Big/ \sum_{i=1}^{4} a_{0i} \, , \; a_{0i} = \frac{\sigma_i}{\rho_{0i}} \qquad (15)$$

where a_i were given by the expressions:

$$a_1 = \frac{\sigma_1}{\xi(1-\xi)} \, , \; a_2 = \frac{\sigma_2}{\eta(1-\eta)} \, ,$$

$$a_3 = \frac{\sigma_3}{\xi(1-\xi)} \, , \; a_4 = \frac{\sigma_4}{\eta(1-\eta)} \; . \qquad (16)$$

The formula (15) confirms the experimental values from Table 1. In Figure 5, the contours τ_0 of (15) are shown on the quarter of the normalized elementary grid block if σ =1. Contours of (15) have two disadvantages:

- in the vicinity of nodes, the contours are not circular towards the nodes as their origins;
- on edges, as borders between neighboring blocks, the values of τ_0 may not coincide when $\sigma_n \ne$ const.

These drawbacks are eliminated in the following improved formula:

$$\tau_0 = \left(0.3444 + 0.5697 a_{0i} c_{0i} a_i^{-1.1}\right) \Big/ \sum_{i=1}^{4} a_{0i} c_{0i} \, ,$$

$$c_{0i} = \sigma_i \cdot \rho_{0i}^{-1.05} \Big/ \sum_{i=1}^{4} \sigma_i \cdot \rho_{0i}^{-1.05} \; . \qquad (17)$$

Due to introduction of c_{0i} , values of τ_0 for neighboring blocks coincide on edges bordering them. The elements a_{0i} , a_i are common for (15) and (17). In Figure 6, the contours τ_0 of (17) are shown.

However, the formula (12) cannot give full value of s_0 if other nearby located flow sources are present. To account for this situation, the surface s_{0j} of the local depression cone caused by β_0 is necessary. This task is solved in the next section devoted to computing heads at monitoring wells.

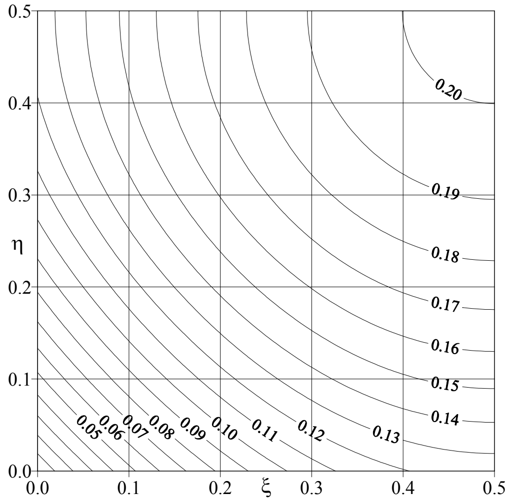

Figure 5. Contours of τ_0 if (15) is used, σ=1.0

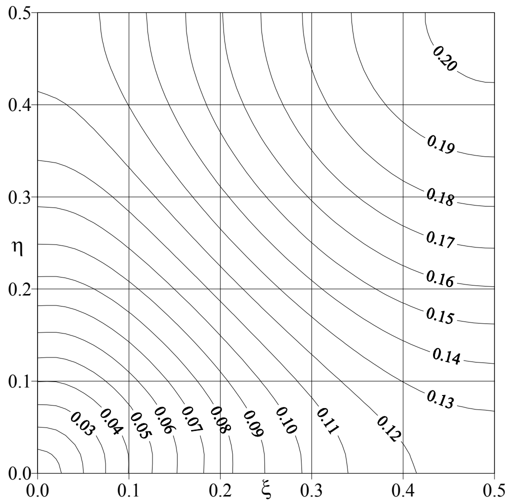

Figure 6. Improved contours of τ_0 if (17) is used, σ=1.0

COMPUTING OF HEADS FOR MONITORING WELLS

The task of computing φ_0 for a monitoring well of Figure 1 is more universal than the one devoted to restoring of the local maximum s_0 at the production well. It is assumed that the expression should be used:

$$\varphi_0 = \sum_{n=5}^{8} \varphi_n \cdot c_{0n} + s_{0j} \, , \; s_{0j} = \tau_{0j} \cdot \beta_j \, , \qquad (18)$$

where τ_{0j} is the transfer resistance of the source β_j towards the monitoring point 0. If the distance $\rho_{0j} \to 0$ then $\tau_{0j} \to \tau_0$ and (18) $\to$ (12).

The value of τ_{0j} must be zero at any node. The surface of τ_{0j} must be flat on the level τ_j (given by (17)) within a circle with the centre β_j and the radius ρ = 0.1972. Such a surface may be approximated by the modified IDM, as follows:

$$\tau_{0j} = \frac{\tau_j \rho_{0j}^{-3}}{\rho_{0j}^{-3} + \sum_{p=1}^{P} \rho_{0p}^{-1.5}}, \quad \rho_{0j} \le 2.0 \qquad (19)$$

where ρ_{0j} and ρ_{0p} are the normalized distances between the monitoring point 0 and the source β_j and the nearby nodes *p = 1, 2,…,.P,* correspondingly; these distances should not exceed 2.0. The expression (19) is empiric. It was calibrated for the elementary block by accounting for the two characteristic source positions when σ_n =1.0 : 1) the center; 2) the middle of an edge. The results for the centre are shown in Figure 7 and Figure 8. In Figure 7, the contours of τ_{0j} are exposed. The top $\rho_{0j} \le 0.1972$ of the surface τ_{0j} is not ideally flat on $\tau_{0j} = 0.2032$ and this fact causes errors. In (19), the powers -3.0 and -1.5 are chosen to minimize the error Δ_{0j} , along the diagonal of the normalized block 1 × 1 (Figure 8):

$$\Delta_{0j} = \frac{1}{2\pi} \ln \frac{1}{\sqrt{2}\rho_{0j}} - \tau_{0j}, \text{ if } \frac{1}{\sqrt{2}} \ge \rho_{0j} \ge 0.1972,$$
$$\Delta_{0j} = 0.2032 - \tau_{0j}, \text{ if } 0.1972 > \rho_{0j} > 0. \qquad (20)$$

where the analytic standard of (20) is represented by (13) if $1/\sqrt{2} \ge \rho_{0j} \ge 0.1972$, R = $1/\sqrt{2}$.

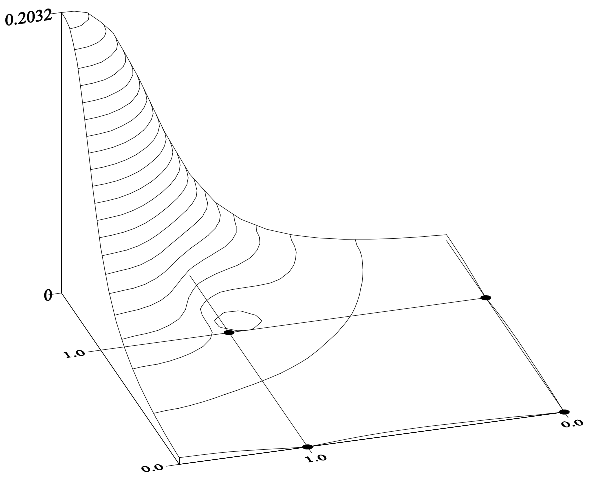

Figure 7. The source at the center of the normalized block. Contours of τ_{0j} on the quarter of the region 3×3, σ = 1.0; the isoline step is 0.01

It follows from Figure 8 that the graph of Δ_{0j} has two maximal values 0.014 and - 0.012 when ρ_{0j} = 0.2 and 0.35, respectively. Therefore, the relative error 100 Δ_{0j} /0.2032 given by (20) does not exceed 7%.
If ρ_{0j} < 0.1972, the following analytic correction is necessary:

$$\left(\tau_{0j}\right)_w = \tau_{0j} + \left(\tau_{0j}\right)_{0.2h},$$

$$\left(\tau_{0j}\right)_{0.2h} = \frac{1}{2\pi\sigma_j} \ln\left(\frac{0.1972}{\rho_{0j}}\right) \qquad (21)$$

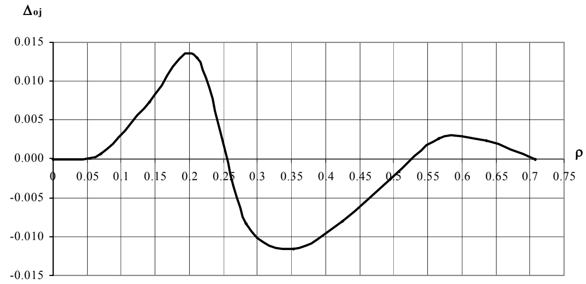

Figure 8. Source at the center of the block 1 × 1. The graph Δ_{0j} along the diagonal if $1/\sqrt{2} \ge \rho \ge 0$, σ=1.0

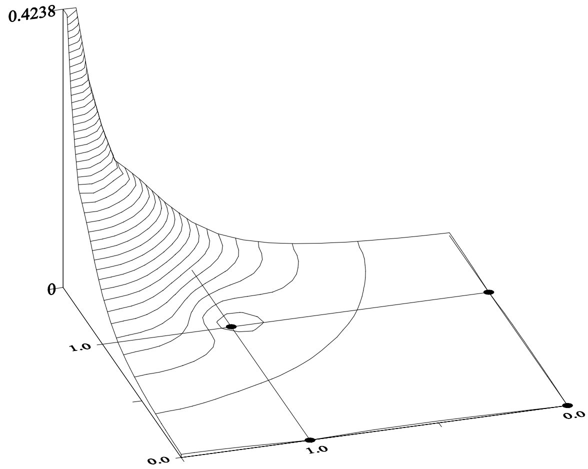

Figure 9. The source at the center of the normalized block. Contours of $\tau_{0j}+(\tau_{0j})_{0.2h}$ on the quarter of the region 3×3, σ=1.0; the isoline step is 0.01

There $(\tau_{0j})_{0.2h}$ represents the analytic complement provided by (13) if $R = 0.1972h$. In Figure 9, contours of $(\tau_{0j})_{0.2h}$ are shown if the minimal ρ_{0j} = 0.25 × 0.1972. Due to the error Δ_{0j} caused by the slantwise top of τ_{0j} , the junction of the surfaces τ_{0j} and $(\tau_{0j})_{0.2h}$ is not smooth. In Figure 10, Figure 11 the results are presented when β_j is at the middle of the edge. Contours of τ_{0j} are shown in Figure 10. For the source β_j , $\tau_{0j} = \tau_j = 0.124$. The error Δ_{0j} of τ_{0j} is evaluated on the edge where β_j is positioned:

$$\Delta_{0j} = \frac{0.837}{2\pi} \cdot \ln \frac{1}{2\rho_{0j}} - \tau_{0j}, \quad \text{if } 0.5 \ge \rho_{0j} \ge 0.1972,$$
$$\Delta_{0j} = 0.124 - \tau_{0j}, \text{ if } 0.1972 > \rho_{0j} > 0. \qquad (22)$$

It follows from the graph of Δ_{0j} provided by (22) that its maximum Δ_{0j} = 0.0145 when ρ_{0j} ~ 0.2 (Figure 11). It is caused by the slantwise top of the surface τ_{0j} if ρ_{0j} < 0.2. This maximum is practically the same as for the graph of Figure 8.

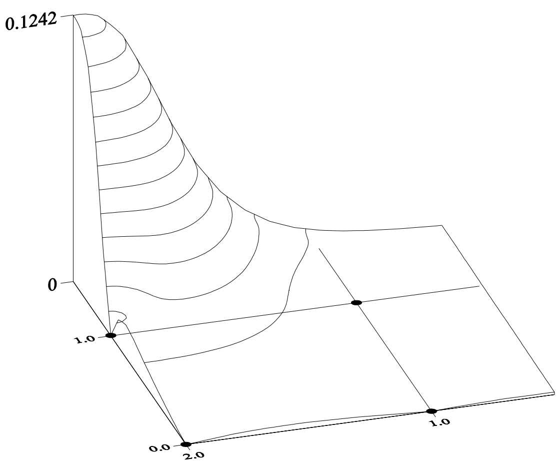

Figure 10. The source at the middle of the edge of the block 1 × 1. Contours of τ_{0j} on the quarter of the region 3 × 3, σ = 1.0; the isoline step is 0.01

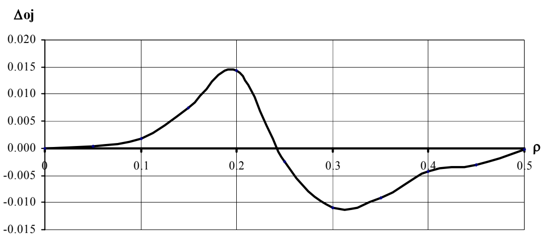

Figure 11. The source at the middle of the edge of the block 1 × 1. The graph of Δ_{0j} along the edge if $0.5 \ge \rho \ge 0$, σ=1.0

If the surface τ_{0j} of (19) is used as a tool for computing of s_{0j} at the monitoring well 0 then it is possible to account for the influence of various sources β_j, j = 1, 2, …, J. They are located within a circle of the radius ρ_{0max} = 2.0 with the centre 0 where superposition of s_{0j} is applied. The final formula is generalization of (18):

$$\varphi_0 = \sum_{n=5}^{8} \varphi_n \cdot c_{0n} + \sum_{j=1}^{J} s_{0j}\ ,\ s_{0j} = \tau_{0j} \cdot \beta_j\ . \qquad (23)$$

It is supposed that the nearest source is β_1, j = 1, ρ_{01} = min. If ρ_{01} < 0.1972 then the complement of (21) should be used for obtaining of $s_{01} = (\tau_{01})_w\ \beta_1$. When $\rho_{01} \to \rho_{w1}$ then $s_{01} \to s_1 = \tau_{1w}\ \beta_1$ of (14), as the maximum of the local depression caused by β_1.

If compared with the original version of back interpolation, the expression (23) is more universal. Formulation of its main components τ_j and τ_{0j} have been improved considerably.

CONCLUSIONS

1. Methods for forth and back interpolation have been developed for non-regularly located production and monitoring wells.
2. The methods improve accuracy of hydrological models, especially, if grid plane steps are large.
3. The improved interpolation methods have been implemented in software developed by the Environment Modelling centre of the Riga Technical University.

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