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Proceedings of the 13th International Conference on Analytical and Stochastic Modelling Techniques and Applications

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Preface

This is the 13th Analytical and Stochastic Modelling Techniques and Applications (ASMTA06) Conference. The conference has become one of the most important conferences in the field in Europe. The programme of ASMTA06 comprises 23 high quality papers organised into 6 sessions. Every paper was reviewed by at least two (in most cases by three) reviewers. The reviewers were truly wonderful this year, too, and in most of the cases the reviews provided valuable comments that contributed to increasing the quality of the final versions of the papers. Therefore, I would like to express my deep gratitude and appreciation to members of the International Programme Committee and all the reviewers for the excellent work.

Few weeks ago, I had the pleasure to be present in Erlangen to pay gratitude to Gunter Bolch, who started the ASMTA conferences 12 years ago, in his retirement party on 30th March 2006. It was impressive to see the number of guests who all had something to tell about Gunter and especially the number of his students who became full professors such as Professor Ian Akyildiz from Georgia Tech and Professor Hermann De Meer from University of Passau.

Gunter has been very active researcher and great author of books and papers for more than 30 years. He has co-authored a number of books. It is enough, however, to mention his book “Queueing Networks and Markov Chains” published by John Wiley & Sons. This white book has been one of the most successful books in the field and the second edition has just been published.

I met Gunter in 1997 when he hosted me as Alexander von Humboldt Fellow. Since then we have been continuously collaborating including publishing our joint book “Practical Performance Modelling” by Kluwer in 2001. I am honoured to be able to carry his legacy with ASMTA conferences.

Therefore, I would like to dedicate this year’s conference to its founder Dr. Gunter Bolch on the occasion of his retirement and would say on behalf of all the participants of ASMTA06 (many who know Gunter personally):

Thank you very much Gunter!
We wish you and your lovely wife, Monika, all the bests and hope that you will enjoy many years of good health.
We also wish that you will continue to be with us at ASMTA conferences whenever you can!

Khalid Al-Begain
ASMTA 2006 Conference Chair

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SCIENTIFIC PROGRAMME

SESSION 1

QUEUEING SYSTEMS I

ANALYTIC COMPUTATION OF END-TO-END DELAYS IN QUEUEING NETWORKS WITH BATCH MARKOVIAN ARRIVAL PROCESSES AND PHASE-TYPE SERVICE TIMES

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Queueing systems, network models, stochastic models, performance modeling

ABSTRACT

This paper poposes an analytic approach to compute end-to-end delays in queueing networks composed of BMAP/PH/1 queues. The approximations for the first two moments of the end-to-end delays are based on corresponding moments for the delays at each node in the network. In a traffic-based decomposition of queueing networks, different techniques for the departure process approximation, including the ETAQA truncation, and traffic descriptions, like phase-type distributions and Markovian arrival processes (MAPs), have been implemented and are compared with simulation results. Extensions to networks with two classes of customers are also discussed.

1 INTRODUCTION

Many different kinds of queues and queueing networks have been studied to aid in the planning and optimization of computer and communication systems. Especially with respect to quality of service (QoS) in IP networks, end-to-end delays and their variability are crucial performance characteristics. However, end-to-end delay moments are hard to compute analytically even for simple queueing networks.

In contrast, various implementations of the principle of traffic-based decomposition have been proposed to obtain approximate performance measures for quite general queueing networks (see [Heindl 2001] for an overview). In this queue-by-queue decomposition, queues are analyzed in isolation and their output approximations serve as input traffic streams to downstream queues leading to a successive or iterative algorithmic procedure. Most such procedures focus on performance measures for single nodes, like queue length moments. Whitt [Whitt 1983] discusses a general approach to obtain estimates for the first two moments of the end-to-end delays. Ferng and Chang [Ferng and Chang 2001] also apply this approach in their decomposition using two-state Markov-modulated Poisson processes (MMPP(2)) as traffic descriptors. By employing more general correlated traffic models, like Markovian arrival processes (MAPs), with dedicated and recently developed decomposition techniques, this paper engineers state-of-the-art research results on queue analysis, output approximations and intermediate traffic characterizations into a decomposition framework to approximately compute the first two moments of end-to-end delays in queueing networks.

BMAP/PH/1 queues, where MAP arrivals may occur in batches and are processed with phase-type (PH) service times, represent the main building block in the considered queueing networks with Markovian routing. For the computation of the waiting time moments, we apply Lucantoni's methods for BMAP/G/1 queues [Lucantoni 1993, Lucantoni 1991], while the departure process characteristics are determined via ETAQA truncations developed for BMAP/MAP/1 queues [Zhang et al. 2005]. From these output characteristics (interdeparture moments and correlation coefficients), we construct various descriptors for the interqueue traffic to assess their efficiency and effectiveness in traffic-based decomposition. These traffic descriptors range from simple Poisson process and minimal acyclic PH distributions (fitted to the first three moments, [Bobbio et al. 2005]) to correlated MAPs of order 2 [Heindl 2004] and finally general correlated matrix-exponential processes resulting directly from the ETAQA truncation. We will also demonstrate in this paper how our approach extends to queueing networks with finite buffers at nodes and two traffic classes.

Section 2 discusses the interfaces of isolated queueing systems, which serve to embed them in the traffic-based decompostion framework described in Section 3. The examples in Section 4 illustrate how well the first two moments of end-to-end delays are approximated in queueing networks (as validated by simulation results).

2 ISOLATED QUEUEING SYSTEMS

In a queue-by-queue decomposition of queueing networks, the arrival process to an isolated system has to be determined first, before performance measures are computed and the departure process is approximated. For BMAP/PH/1 queues, we introduce the possible arrival processes, indicate how the first two moments of the waiting time are derived (as needed for network end-to-end delays) and discuss a flexible departure process truncation. Corresponding techniques for other queues in our framework are only broached at the end of this section.

2.1 The BMAP/PH/1 Queue

Necessarily, only the most important properties of BMAPs and the analysis of BMAP/PH/1 queues can be briefly recalled here and the reader will be referred to the relevant literature upon which our implementation is based.

2.1.1 *Batch Markovian Arrival Processes (BMAP)*

A BMAP is an ergodic Continuous-Time Markov Chain (CTMC) with finite state space $\{1, 2, \ldots, m_{\text{BMAP}}\}$, where the sojourn time in state i is exponentially distributed with parameter λ_i. At the end of a sojourn time in state i, a batch of size k $(k \geq 1)$ may occur with probability $p_{i,j}^{(k)}$, while the CTMC passes to state j $(1 \leq i, j \leq m_{\text{BMAP}})$. With probability $p_{i,j}^{(0)}$, there will be a transition to state j $(i \neq j)$ without an arrival and we thus require that

$$\sum_{\substack{j=1 \\ j \neq i}}^{m_{\text{MAP}}} p_{i,j}^{(0)} + \sum_{k=1}^{\infty} \sum_{j=1}^{m_{\text{MAP}}} p_{i,j}^{(k)} = 1 \quad , \quad \forall\, 1 \leq i \leq m_{\text{MAP}} \quad .$$

BMAPs can be represented by a sequence of matrices $\mathbf{D}_k$ $(k = 0, 1, \ldots)$ according to $(\mathbf{D}_k)_{i,j} = \lambda_i p_{i,j}^{(k)}$ for $k = 0, 1, \ldots$ and $(\mathbf{D}_0)_{i,i} = -\lambda_i$. This definition implies that $\mathbf{Q}_{\text{BMAP}} = \sum_{k=0}^{\infty} \mathbf{D}_k$ is the infinitesimal and irreducible generator of the Markov process. All matrices are of order $m_{\text{BMAP}} \times m_{\text{BMAP}}$, where arrivals of batches of size k are governed by the nonnegative rate matrices $\mathbf{D}_k$ $(k \geq 1)$ and $\mathbf{D}_0$ is a matrix with negative diagonal elements and nonnegative off-diagonal elements. Let $\boldsymbol{\pi}_{\text{BMAP}}$ be the stationary probability vector of the CTMC with $\boldsymbol{\pi}_{\text{BMAP}} \mathbf{Q}_{\text{BMAP}} = \mathbf{0}$, $\boldsymbol{\pi}_{\text{BMAP}} \mathbf{e} = 1$, where $\mathbf{0}$ and $\mathbf{e}$ denote the row/column vectors of zeros and ones of the appropriate dimension. Then, the fundamental arrival rate of the BMAP can be computed as

$$\lambda_{\text{BMAP}} = \boldsymbol{\pi}_{\text{BMAP}} \sum_{k=1}^{\infty} k \mathbf{D}_k \mathbf{e} \quad .$$

When arrivals occur singly (i.e., not in batches), the BMAP reduces to a Markovian Arrival Process (MAP, where $\mathbf{D}_k = \mathbf{0}$ for $k \geq 2$). Therefore, phase-type (PH) renewal processes (no batches, no correlation) and Poisson processes ($m_{\text{MAP}} = 1$, no batches) are also special cases of BMAPs. Usually, in our decomposition framework, BMAPs proper serve as external arrival processes to the network, while inter-queue traffic is characterized via MAPs.

2.1.2 *Moments Of The Waiting Time*

Let the arrival process and the service distribution of a BMAP/PH/1 queue be given by a BMAP with matrices $\mathbf{D}_k^{(A)}$ $(k \geq 0)$ and a PH service time $\langle \boldsymbol{\alpha}, \mathbf{T} \rangle$ with moments $E[S^i] = i! \boldsymbol{\alpha}(-\mathbf{T})^{-i}\mathbf{e}$, where $\mathbf{T}$ is the transition rate matrix and $\boldsymbol{\alpha}$ is the initial probability vector. Then, the definitions in [Lucantoni 1993, Lucantoni 1991](which we specialize from a BMAP/G/1 queue to a BMAP/PH/1 queue) can be used to compute the virtual waiting time distribution of the queue (without the service time) and thus the first and second moments of the waiting time W_k of the first customer in the batch of size k. Additionally, the probability that an arriving customer belongs to a batch of size k is

$$P_b^{(k)} = \frac{\boldsymbol{\pi}_{\text{BMAP}} k \mathbf{D}_k^{(A)} \mathbf{e}}{\lambda_{\text{BMAP}}} \quad .$$

As a consequence, the first two moments of the actual waiting time W of an arbitrary customer can be computed as

$$\begin{aligned} E[W] &= \sum_{k=1}^{\infty} P_b^{(k)} \left[E[W_k] + E[S] \left(\frac{k-1}{2} \right) \right] \quad , \\ E[W^2] &= \sum_{k=1}^{\infty} P_b^{(k)} \left[E[W_k^2] + E[S^2] \left(\frac{k-1}{2} \right) \right. \\ &+ \left. (k-1)\, E[W_k] E[S] + \frac{E[S]^2 (k-1)!}{3(k-3)!} \right] . \end{aligned}$$

From these moments (see [Söhnlein 2005] for a derivation), the squared coefficient of variation (SCV) can be expressed as $c_{\text{w}}^2 = \frac{E[W^2]}{E[W]^2} - 1$. The moments for the single queue flow into the computation of the end-to-end delay moments in a network (see Section 3.3).

2.1.3 *Departure Process Characteristics Via The ETAQA Truncation*

In order to compute departure process characteristics, we pursue an ETAQA-based approach proposed in [Zhang et al. 2005]. Here, we exploit that a BMAP/PH/1 queue is an M/G/1-type Markov process with an infinitesimal generator $\mathbf{Q}_\infty$ of the form

$$\mathbf{Q}_\infty = \begin{bmatrix} \widehat{\mathbf{L}} & \mathbf{F}^{(1)} & \mathbf{F}^{(2)} & \mathbf{F}^{(3)} & \mathbf{F}^{(4)} & \cdots \\ \mathbf{B} & \mathbf{L} & \mathbf{F}^{(1)} & \mathbf{F}^{(2)} & \mathbf{F}^{(3)} & \ddots \\ \mathbf{0} & \mathbf{B} & \mathbf{L} & \mathbf{F}^{(1)} & \mathbf{F}^{(2)} & \ddots \\ \mathbf{0} & \mathbf{0} & \mathbf{B} & \mathbf{L} & \mathbf{F}^{(1)} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} . \tag{1}$$

The state space S of such a CTMC is partitioned into levels $S^{(j)} = \{s_1^{(j)}, \ldots, s_m^{(j)}\}$, for $j \geq 0$ and $m \geq 1$, where $S^{(0)}$ represents the state configuration when the queue is empty and the sets $S^{(j)}$, for $j \geq 1$, account for the state of the system when the queue is nonempty (with j customers). We use the letters "L", "F" and "B" according to whether they describe "local", "forward" and "backward" transition rates.
For BMAP/PH/1 queues, the block matrices in (1) can be defined as in [Zhang et al. 2005], where $\mathbf{D}_0^{(S)} = \mathbf{T}$ and $\mathbf{D}_1^{(S)} = (-\mathbf{T})\mathbf{e}\boldsymbol{\alpha}$ describe the PH service process as a MAP. The departure process of such a queue may be approximated via ETAQA [Riska and Smirni 2002], which provides a finite representation, from which moments of the marginal distributions and a set of coefficients of correlation can actually be computed exactly for the departure process. The number of block levels in this truncation (denoted by index $n\,;\ n > 1$) may be chosen flexibly, such that the overall order of the truncated representation is $(n+1)m_S m_A$, where m_A and m_S are the orders of the arrival and service process. This representation characterizes a so-called matrix-exponential (ME) process, which is an algebraic generalization of a MAP. An nth-level truncation preserves at least the marginal distribution (and thus all its moments) and $n-2$ coefficients of correlation ($n > 1$), which can be obtained as for MAPs by standard formulas (see e.g., [Zhang et al. 2005]).

2.2 Other Queues

For our investigations of the waiting times and end-to-end delays, we use several other models of single queues, like MAP/PH/1/K queues and MAP/PH/1 queues with two traffic classes. The above description contains the MAP/PH/1 queue, for which we employ more efficient specialized formulas for the waiting time moments [Heindl 2001] and the output characterization [Heindl et al. 2004]. If we define $\mathbf{F}^{(i)} = \mathbf{0}\ \ \forall i \neq 1$ in (1), and truncate this QBD-representation at a certatin level K, while the block element (K,K) is replaced by $\mathbf{L}+\mathbf{F}$, we get the generator $\mathbf{Q}_K$ for a MAP/PH/1/K queue. The computation of the actual waiting time moments of this kind of queue was also given in [Heindl 2001].

Contrary to the ETAQA truncation for queues with infinite capacity, we obtain the output process here directly by separating the generator $\mathbf{Q}_K$ into the two matrices $\mathbf{D}_{0,K}^{(D)}/\mathbf{D}_{1,K}^{(D)}$, where $\mathbf{D}_{1,K}^{(D)}$ governs all rates that cause a departure from the queue and $\mathbf{D}_{0,K}^{(D)}$ governs all other rates. Due to the resulting dimension of this representation of $(K+1)m_A m_S$, this approach is only feasible for small capacities K and better techniques should be implemented. A similar and straightforward approach can be applied to obtain the loss process for the MAP/PH/1/K queue.
In [Heindl and Gross 2006], an approximation for MAP/MAP/1 queues with two traffic classes and its ETAQA truncation was introduced. By using this model, the formulas in [Lucantoni 1993, Lucantoni 1991] can be adapted to approximate the actual waiting time moments for two classes (which we denote here as **t**agged and **b**ackground). Then a modified common service process has to be introduced, which can be done as follows: Given two MAP arrival processes with rates $\lambda^{(t)}$, $\lambda^{(b)}$ and two phase type service time distributions $\langle \boldsymbol{\alpha}^{(t)}, \mathbf{T}^{(t)} \rangle$, $\langle \boldsymbol{\alpha}^{(b)}, \mathbf{T}^{(b)} \rangle$. Then, we redefine the transition rate matrix $\mathbf{T}$ for the service process as

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}^{(t)} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^{(b)} \end{bmatrix} \quad ,$$

and the initial probability vector $\boldsymbol{\alpha} = \left[\boldsymbol{\alpha}_w^{(t)} \mid \boldsymbol{\alpha}_w^{(b)} \right]$ is weighted with

$$\boldsymbol{\alpha}_w^{(t)} = \left[\frac{\lambda^{(t)}}{\lambda^{(t)} + \lambda^{(b)}} \right] \boldsymbol{\alpha}^{(t)} \ ,\ \boldsymbol{\alpha}_w^{(b)} = \left[\frac{\lambda^{(b)}}{\lambda^{(b)} + \lambda^{(t)}} \right] \boldsymbol{\alpha}^{(b)}.$$

Additionally, the moments of the waiting times are adjusted as

$$\begin{aligned} E[S^i] &= \left[\frac{\lambda^{(t)}}{\lambda^{(t)} + \lambda^{(b)}} \right] i!\boldsymbol{\alpha}^{(t)}(-\mathbf{T}^{(t)})^{-i}\mathbf{e} \\ &+ \left[\frac{\lambda^{(b)}}{\lambda^{(b)} + \lambda^{(t)}} \right] i!\boldsymbol{\alpha}^{(b)}(-\mathbf{T}^{(b)})^{-i}\mathbf{e} \quad . \end{aligned}$$

With these service approximations, which are of course exact for identical service times for the two classes, the moments of the actual waiting times W_k^t, W_k^b of the first (and in this case only) customer in the batch of size 1, can be computed directly for both traffic classes (**t**agged and **b**ackground). More details for all mentioned isolated queues can be found in [Söhnlein 2005].

3 TRAFFIC-BASED DECOMPOSITION

Traffic-based decomposition is an approximate alternative to exact analytical solutions of queueing networks on the one hand (which are often not applicable due to state space constraints) and to time-consuming simulations on the other hand. The basic principle of decomposition is a successive (or iterative) analysis of isolated queues in a network, while the output process for each queue is approximated and may serve as an input process for subsequent queues – possibly after splitting and merging with other traffic descriptors. Network performance measures, like moments of end-to-end delays, are then derived from node performance measures, as already outlined in [Whitt 1983]

3.1 Employed Traffic Descriptors

The ETAQA truncation, as described in section 2.1.3, is one possibility to approximate the output process for queues with infinite capacity. Depending on the chosen index n, this may lead to high-order representations and therefore to long computation times for networks with many nodes. Alternatively, one can compute the moments and a set of correlation coefficients to yield a more tractable characterization of the output processes. In this paper, we use inverse characterizations for MAP(2)s with given three moments and a correlation parameter as described in [Heindl 2004] and a moment matching procedure for minimal acyclic phase type distributions as introduced in [Bobbio et al. 2005] in order to obtain low-order representations of the output processes. The necessary (moment and correlation) parameters can be computed from the ETAQA output model as described in [Zhang et al. 2005] or by similar methods for MAP/PH/1 queues with one or two traffic classes as described in [Heindl et al. 2004] and [Heindl and Gross 2006], respectively. Let us denote the reduced moment parameters by $r_i = \frac{E[X^i]}{i!}, i = 1, 2, 3$ and

$$h_2 = \frac{r_2 - r_1^2}{r_1^2} \quad , \quad h_3 = \frac{r_3 r_1 - r_2^2}{r_1^4} \quad . \tag{2}$$

For MAP(2)s, the correlation parameter γ might be computed from

$$\gamma = \text{corr}[X_0, X_1] \frac{2h_2 + 1}{h_2} \quad , \tag{3}$$

where corr$[X_0, X_1]$ is the first coefficient of correlation of the (wide-sense stationary) sequence of interevent times X_k. The parameters r_1, h_2, h_3 and γ have to be determined for the construction of a MAP(2), while for the representation as a PH process of minimal order n (in the following denoted as PH(n)), γ will be ignored (i.e., we set $\gamma = 0$). Even more drastically, by just taking the first moment into account, one may roughly approximate the departure process as a Poisson process.
Although these characterizations may be more efficient, the low-order representations usually impair the quality of the approximated waiting times of downstream queues and eventually of end-to-end delays. Also, mapping the output characteristics to a MAP(2) is not always feasible, when the moments or the correlation parameter are out of bounds for this restrictive representation (see [Heindl 2004]).

3.2 Splitting And Merging

If there are two or more arrival processes for a single queue, these processes have to be merged via superposition. For n BMAPs, this operation is defined using Kronecker notation in order to obtain the matrices $\mathbf{D}_i^{(A)}$ of the merged BMAPs:

$$\mathbf{D}_i^{(A)} \quad = \quad \mathbf{D}_i^{(1)} \oplus \mathbf{D}_i^{(2)} \oplus \mathbf{D}_i^{(3)} \oplus \cdots \oplus \mathbf{D}_i^{(n)} \quad .$$

In this definition, $\mathbf{D}_i^{(j)}$ represents the matrix of the batches of size i in the jth BMAP.
The splitting of departure processes (and of the loss process of MAP/PH/1/K queues), for which a MAP notation with $\mathbf{D}_0^{(D)}$ and $\mathbf{D}_1^{(D)}$ is sufficient, is performed as follows:

$$\begin{aligned} \mathbf{D}_0^{(k)} &= \mathbf{Q}^{(D)} - \mathbf{D}_1^{(k)} \quad , \\ \mathbf{D}_1^{(k)} &= q_k \cdot \mathbf{D}_1^{(D)} \quad . \end{aligned}$$

The departure process $\mathbf{D}_0^{(D)}, \mathbf{D}_1^{(D)}$, where $\mathbf{D}_0^{(D)} + \mathbf{D}_1^{(D)} = \mathbf{Q}^{(D)}$, will be routed to queue k with probability $0 \leq q_k \leq 1$.

3.3 End-To-End Delays

Using traffic-based decomposition, we are able to compute the actual waiting times at every node in a network consisting of queues with (batch) Markovian arrival processes and PH services. To compute end-to-end delays for different paths, we introduce the random variable T_i, which will either describe the waiting time in a queue or a service time at a node in a given network. Denoting a specific end-to-end delay by the random variable T, we compute its mean as

$$E[T] = \sum_{1 \leq i \leq m} E[T_i] \quad ,$$

where $m = 2p$ and p is the number of nodes on the path. In accordance to [Whitt 1983], the second moment of the end-to-end delay can be approximated as

$$E[T^2] = \sum_{1 \leq i \leq m} E[T_i^2] + 2 \sum_{\substack{1 \leq i,j \leq m \\ i \neq j}} E[T_i] \cdot E[T_j] \quad , \tag{4}$$

where we ignore possible covariances between the random variables T_i (e.g., between waiting and service times at the same node). Finally, the SCV (as a measure for the "jitter") can be defined as

$$c_{\mathrm{T}}^2 = \frac{E[T^2]}{E[T]^2} - 1 \quad .$$

The computation of end-to-end delay moments can be done accordingly for two separate traffic classes as introduced above.

4 EXAMPLES

In this section, we apply our approach to compute end-to-end delay moments to two example networks. To assess the quality of the approximations, we provide selected simulation results. All analytical computations were done with Maple, while the software tool AnyLogic was used to run the simulations (with a specified relative error of 5% and a confidence level of 95%).

4.1 BMAP/M/1 → ·/PH/1 Tandem Queue

In this example, we study the impact of increasing load in a BMAP/M/1 → ./PH/1 dual tandem queue on the waiting times, the end-to-end delays and the jitter.

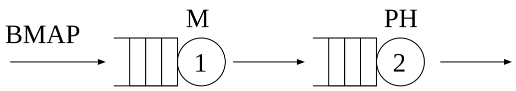

Figure 1: BMAP/M/1 → ./PH/1 dual tandem queue

The arrival process for queue 1 is a two-dimensional BMAP with maximal batch size 3. To obtain a BMAP with strong correlations, we transform a MAP(2) with a large correlation parameter γ (and an SCV of 2), described by $\mathbf{Q} = \mathbf{D}^{(A)}_{0,old} + \mathbf{D}^{(A)}_{1,old}$, which was constructed with parameters $r_1 = 2$, $h_2 = 0.5$, $h_3 = 0.25$, $\gamma = 0.9$. Applying the following formulas (for $K = 3$ and $q = 0.5$) results in the desired BMAP with finite batch sizes $k = 1, 2, \ldots, K$ and the same mean arrival rate as the MAP(2):

$$\begin{aligned}
\mathbf{D}^{(A)}_k &= \frac{(1-q)^2}{1+Kq^{K+1}-(K+1)q^K} q^{k-1}\mathbf{D}^{(A)}_{1,old} \quad , \\
\mathbf{D}^{(A)}_0 &= \mathbf{Q} - \sum_{k=1}^{K} \mathbf{D}^{(A)}_k \\
&= \mathbf{Q} - \frac{(1-q)(1-q^K)}{1+Kq^{K+1}-(K+1)q^K}\mathbf{D}^{(A)}_{1,old} \quad .
\end{aligned}$$

The high correlation in the initial MAP(2) leads to strong (though transformed) correlation in the resulting BMAP. The service process at this node is an exponential distribution with mean rate equal to 1.0σ, where σ is a scaling coefficient to adjust the load. The second queue has an Erlang-2 service process with mean service rate σ, which in MAP notation is given as

$$\mathbf{D}^{(S_2)}_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix} \cdot \sigma \quad , \quad \mathbf{D}^{(S_2)}_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \cdot \sigma \quad .$$

Figures 2 and 3 show how the increasing load affects the mean waiting times (MWT), their squared coefficients of variation (SCV) and the end-to-end delays. All necessary values for the graphs were computed analytically with an ETAQA-10 output approximation. As expected, the MWTs at both servers increase with the load (Figure 2), while the SCVs decrease (Figure 3). The

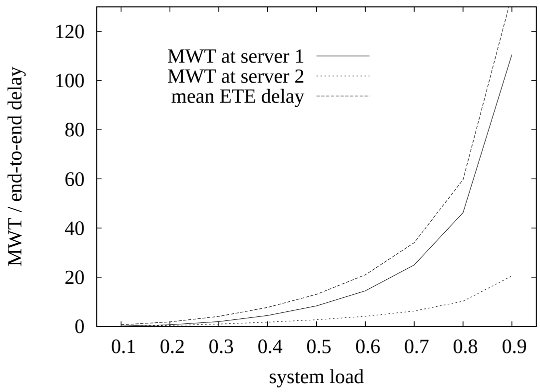

Figure 2: MWTs at both nodes of the tandem queue

mean end-to-end delays (also plotted in Figure 2) show a similar behavior as the per-node MWTs, but the SCV (also see Figure 3) reaches its peak under 60% utilization and decreases afterwards. We explicitly chose a strongly correlated arrival process in order to investigate the quality of ETAQA truncations (with different values for level n) and other departure process approximations. Table 4.1 includes the obtained results of the simulations (with confidence intervals), ETAQA-(10, 5, 2) truncations, MAP(2), PH(2) and Poisson (Exp.) output approximations for loads $\rho = 0.3$ and 0.8. S-2 and T stand for server 2 and end-to-end delay, respectively. The waiting times at the first queue are independent of the output approximation. Their mean and SCV are 1.9520 and 2.4416 for $\rho = 0.3$ and 46.2510 and 1.3863 for $\rho = 0.8$. Especially, the deviation of the simulated value 1.3105 ± 0.0316 for the SCV in high load reveals the difficulty of accurate simulations in the presence of correlations and high loads.
The maximal relative errors of the end-to-end delays with ETAQA-10 approximation are 1% and 4.9% (for loads 0.3 and 0.8), while the variation coefficients deviate with 18% and 5.3% (for loads 0.3 and 0.8). Interestingly for the SCV, the higher-order representations are not generally better than the lower-order representations

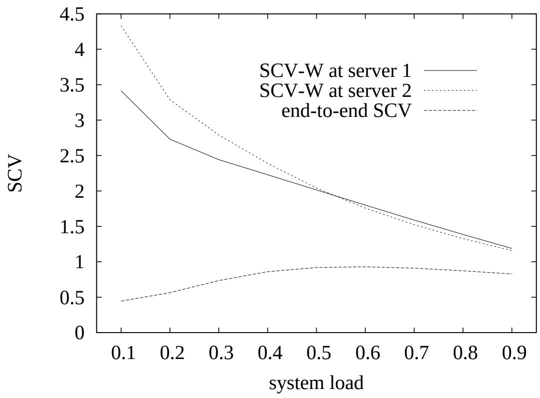

Figure 3: SCVs of waiting times at both nodes

Table 1: BMAP(2)/M/1 $\longrightarrow$ ./PH(2)/1 with different approximations

ρ	Approx.	S-2 MWT	S-2 SCV	T mean	T SCV
0.3	Simulation	0.94548	2.79664	4.12501	0.89792
		$\pm$0.0406	$\pm$0.0380	$\pm$0.0952	$\pm$0.0540
	ETAQA-10	0.93151	2.78998	4.08350	0.73548
	ETAQA-5	0.89481	2.70384	4.04680	0.73325
	ETAQA-2	0.79233	2.74297	3.94432	0.74338
	MAP(2)	0.53862	2.51340	3.69061	0.77621
	PH(2)	0.53291	2.51903	3.68490	0.77760
	Exp.	0.19286	5.14815	3.34485	0.89692
0.8	Simulation	13.73189	1.58330	62.77302	0.92153
		$\pm$0.2760	$\pm$0.0457	$\pm$0.9496	$\pm$0.0237
	ETAQA-10	10.22330	1.32749	59.67433	0.87284
	ETAQA-5	8.97571	1.31763	58.42674	0.90097
	ETAQA-2	7.87165	1.35060	57.32268	0.92917
	MAP(2)	6.55790	1.32916	56.00893	0.96481
	PH(2)	6.46699	1.33136	55.91802	0.96748
	Exp.	4.80000	1.44444	54.25103	1.02024

(see S-2 SCV with load 0.8 or T SCV with load 0.3). However, the mean end-to-end delays (and MWTs) consistently improve with refined output descriptors, from which we conclude that ignoring the covariances in (4) induces a rather high approximation error in the SCV computation.

4.2 Capacity Planning

Here, we examine a capacity planning problem in a queueing network with three nodes. The first node is a

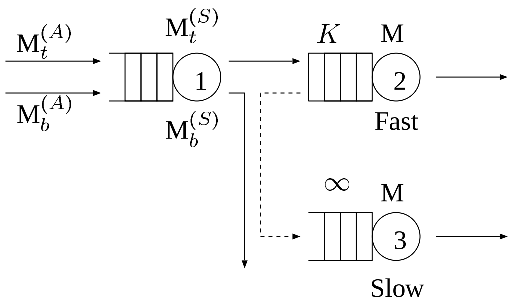

Figure 4: three-node-network

queue with two traffic classes (tagged and background), where both arrival processes are uncorrelated with exponentially distributed interarrival times of means $r_{1,t}^{(A)} = r_{1,b}^{(A)} = 1.0$, respectively. Arrivals are served in FIFO manner with exponentially distributed service times of means $r_{1,t}^{(S)} = r_{1,b}^{(S)} = 0.375$. The background stream is sorted out after this node, while the tagged stream is routed to queue 2, which has a (fast) server with mean service time $r_1 = 1.0$, but only a finite capacity K. If the queue is full, arriving packets will be re-routed to queue 3, which has a slower server ($r_1 = 1.8$) than queue 2, but an infinite capacity. The question is how to choose the capacity K of queue 2 to get minimal end-to-end delays (weighted for both possible paths). To this end, we analyze the behavior of the mean end-to-end delays and their SCVs under different capacities K. Figures 5 and 6 show the mean end-to-end delays and the SCVs for the two possible paths: server 1 $\longrightarrow$ server 2 or server 1 $\longrightarrow$ server 3. In Figure 5 we also plotted the mean end-to-end delays of both paths, weighted with the respective flow proportion. All analytic results were computed with an ETAQA-5 approximation (because the loss process of queue 2 depends on the capacity K). For the path server 1 $\longrightarrow$ server 3, the mean end-to-end delay and its SCV do not change significantly with a capacity greater than $K = 6$, while for path server 1 $\longrightarrow$ server 2, the end-to-end delay increases constantly with higher capacities and its SCV decreases. Regarding the plots for the separate paths, we are not able to make any statements about how to set the capacity K. But, considering the weighted mean end-to-end delay we see, that for capacity $K = 3$, the graph reaches its minimum. Therefore, the capacity should neither be set to a lower nor to a higher value. In Table 4.2, we provide selected

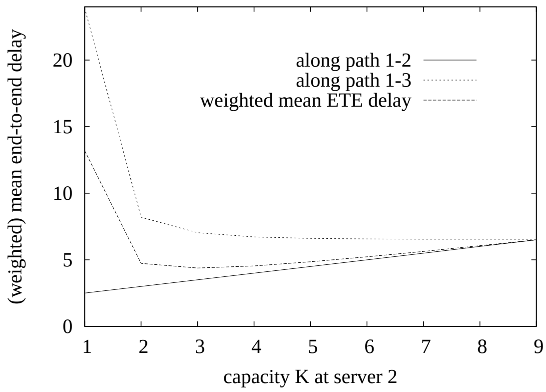

Figure 5: Mean end-to-end delays separately and weighted for both paths

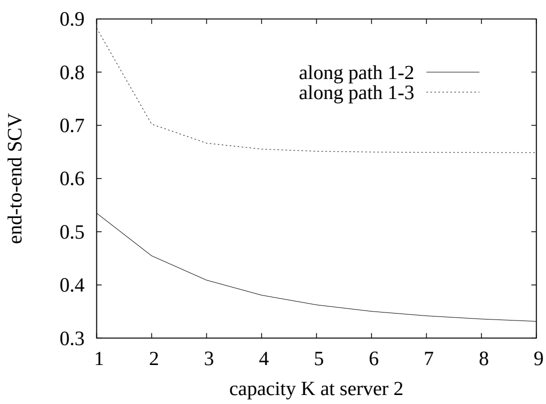

Figure 6: End-to-end SCVs along both paths

analytic/simulation results for capacities $K = 1$ and $K = 3$. S-2 and S-3 stand for server 2 and server 3 (mean/SCV for waiting time) and T 1-2 / T 1-3 / T W stand for path server 1 $\rightarrow$ server 2 / path server 1 $\rightarrow$ server 3 / weighted measure (mean/SCV for end-to-end delay). With capacity $K = 1$, the waiting time (without service time) at server 2 is 0. Regarding the first queue with one-dimensional arrival and service processes with equal rates, the tagged and background waiting times are equal, too, and their mean and SCV amount to 1.1251 and 1.6666, respectively. The maximal relative errors of

analysis

K	meas.	S-2	S-3	T 1-2	T 1-3	T W
1	MWT	0	20.6018	2.5001	23.9018	13.2009
	SCV	0	1.1747	0.5350	0.8825	
3	MWT	1.0000	3.7371	3.5001	7.0371	4.3843
	SCV	1.6667	1.9633	0.4090	0.6664	

simulation

K	meas.	S-2	S-3	T 1-2	T 1-3	T W
1	MWT	0	20.2577	2.5046	23.5606	13.0326
			$\pm$0.463	$\pm$0.007	$\pm$0.464	
	SCV	0	1.1010	0.5208	0.8181	
			$\pm$0.048	$\pm$0.004	$\pm$0.036	
3	MWT	1.0005	3.7342	3.4990	7.0367	4.3834
		$\pm$0.005	$\pm$0.084	$\pm$0.013	$\pm$0.091	
	SCV	1.6650	1.9216	0.4019	0.6475	
		$\pm$0.013	$\pm$0.080	$\pm$0.003	$\pm$0.022	

Table 2: Results for the three-node-network

the mean end-to-end delays are about 1.5%, while the corresponding SCVs deviate with at most 7.9%.

5 CONCLUSIONS

This paper gives an overview of the current state of our research to compute end-to-end delays in queueing networks by means of traffic-based decomposition. Various standard and novel techniques for per-node analysis are combined to obtain network performance characteristics. While mean end-to-end delays can be computed efficiently with a satisfactory accuracy, the approximations for end-to-end delay jitter (in terms of squared coefficient of variation) deserve further refinement.
The computation times (and the accuracy) in this decomposition approach heavily depend on (the order of) the employed traffic descriptors and may range from seconds to hours. Future research will focus on identifying the traffic descriptors needed in particular network situations in order to achieve a good trade-off between efficiency and accuracy.

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THE $SM/PH/N$ QUEUEING SYSTEM WITH BROADCASTING SERVICE DISCIPLINE

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KEYWORDS

Semi-Markovian Arrival Process, Phase Type Service Process, Broadcasting Service Discilpine, Performance Modelling

ABSTRACT

We consider a multi-server queueing model with an infinite buffer, the Semi-Markovian Arrival Process and phase-type service time distribution. Recently introduced customers admission discipline is under study. The customer, which sees several free servers upon arrival, is served simultaneously and independently by all these servers. Such situation occurs, e.g. in modeling wireless communication network with broadcasting. Stationary distribution of a number of customers in the system is calculated.

1 Introduction

Multi-server queueing systems model many real-life objects and processes and have got a lot of attention in literature since the pioneering works by the Danish mathematician and engineer A.K. Erlang in the early 1900th. He has investigated the $M/M/N/\infty$ and $M/M/N/N$ queueing model under the standard assumption that each customer admitted into the system is served by one server. Recently, see (Lee, Dudin, Klimenok, 2006a), (Lee, Dudin, Klimenok, 2006b), another customer admission discipline was introduced into consideration. According to this discipline, the customer gets a service from all servers that are idle at the customer arrival epoch. Such a discipline is realistic in modeling, e.g., the wireless communication network with broadcasting. If the system have many antennas, it makes sense to employ all of free antennas to transmit the arriving information unit. It creates some redundancy, but, as it was shown in (Lee, Dudin, Klimenok, 2006a), (Lee, Dudin, Klimenok, 2006b), it helps to decrease the average information delivering time if the system is not overloaded while the transmission time has high variation. The diversification of the ways of transmission helps to decrease the average time of delivering of a first copy of the broadcasted information.

Here we consider a very general $SM/PH/N$ type queueing model with broadcasting service discipline.

The multi-server model of the $MAP/PH/N$ type with the broadcasting service discipline was considered in (Lee, Dudin, Klimenok, 2006a). Here MAP means the Markovian Arrival Process, see, e.g., (Chakravarthy, 2001), (Lucantoni, 1991), and PH means phase-type service time distribution, see, e.g., (Neuts, 1981). In (Lee, Dudin, Klimenok, 2006b), the $SM/M/N$ type multi-server model with the broadcasting service discipline was considered. Thus, it is assumed in the paper (Lee, Dudin, Klimenok, 2006b) that the input flow is described in terms of the SM (Semi-Markovian) arrival process. It means that the successive inter-arrival times are defined by the sojourn times of some finite state Semi-Markovian stochastic process in its states. The SM arrival process is the maximally general among known in literature descriptors of the arrival process which still allows to get analytically tractable results for characteristics of the queueing model. The SM arrival process is, generally speaking, the correlated process and so it well suits for modeling the real life flows in modern telecommunication networks. As a particular case, the SM arrival process includes the set of all GI (General Independent) recurrent service processes, in which the inter-arrival times are independent identically arbitrary distributed random variables, and the MAP. So, the arrival process considered in (Lee, Dudin, Klimenok, 2006b) is, generally speaking, much more complicated process than one in (Lee, Dudin, Klimenok, 2006a). However, it is assumed in (Lee, Dudin, Klimenok, 2006b) that the service time distribution is exponential one while in (Lee, Dudin, Klimenok, 2006a) the more general PH type service time distribution is dealt with.

Operation of the $MAP/PH/N$ is described in terms of the multi-dimensional continuous time Markov which is the quasi-birth-and-death process with non-standard boundary behavior. To investigate the $SM/M/N$ system, one has to analyze first the two-dimensional discrete time Markov chain embedded at the epochs of the customers arrival into the system. The investigated in this case stochastic process also is not directly immersed into some well-known class of random processes and its investigation is not quite straightforward. Assumption that service time distribution in this model is exponential was imposed in (Lee, Dudin, Klimenok, 2006b) to allow derivation of more or less simple explicit formulas for the one-step transition probability matrices for the embedded two-dimensional Markov chain. In the present paper we essentially weaken this assumption and allow the PH type service time distribution. Thus, both models considered in (Lee, Dudin, Klimenok, 2006a) and (Lee, Dudin, Klimenok, 2006b) are the partial cases of the model investigated in this paper.

The rest of the paper is organized as follows. In section 2, the model is described. In section 3, the stationary distribution of the embedded at the customer arrival epochs Markov chain is computed. In section 4, an arbitrary time distribution of the number of customers in the system is analyzed. Section 5 contains some concluding remarks.

2 Mathematical Model

We consider an N-server queueing system. The flow of customers arriving into the system is of SM type. It means that the arrivals are directed by some Semi-Markovian process. Let us denote this process by $\nu_t, t \geq 0$. The process $\nu_t, t \geq 0$, has a finite state space $\{1, \ldots, K\}$. Behavior of the process is described by the Semi-Markovian kernel $A(t)$. This kernel is the square matrix of size K with entries $(A(t))_{k,k'}$, $k, k' = \overline{1, K}$. Function $(A(t))_{k,k'}$ has the meaning of the probability that sojourn time of the process in the state k will be no longer than t and after that the process jumps into the state k', not necessary different from the state k, $k, k' = \overline{1, K}$. The matrix $A(\infty)$ has a meaning of the one-step probability matrix of the Markov chain embedded at epochs of all jumps of the process $\nu_t, t \geq 0$. It is assumed that the embedded Markov chain is irreducible and sojourn times of the process $\nu_t, t \geq 0$, in its states are positive and finite. Then the stationary distribution of the embedded Markov chain exists. Denote by $\boldsymbol{\theta}$ the row vector of the stationary distribution of the embedded Markov chain. It is well-known that this vector is the unique solution to the following system of equations:

$$\boldsymbol{\theta} = \boldsymbol{\theta} A(\infty), \quad \boldsymbol{\theta}\mathbf{e} = 1. \tag{1}$$

Here and in the sequel $\mathbf{e}$ is the column vector consisting of 1's.

The customers in the SM arrival process arrive at the epochs of jumps of the Semi-Markovian process $\nu_t, t \geq 0$. The value λ defined by formula

$$\lambda^{-1} = \boldsymbol{\theta} \int_0^\infty t dA(t)\mathbf{e} \tag{2}$$

is called the average intensity or fundamental rate of the SM arrival process.

The servers of the system are assumed to be independent of each other and identical. The service process in each server is assumed to be of phase (PH) type. The PH type service process is described by a continuous time Markov process η_t, $t \geq 0$. The state of this process at the epoch of a service beginning is defined according to a probabilistic row-vector $\boldsymbol{\beta} = (\beta_1, ..., \beta_M)$. Further transitions of the process η_t, $t \geq 0$, are defined by a matrix S of dimension M. The diagonal entries $S_{k,k}, |S_{k,k}| < \infty$, $k = \overline{1, M}$, of the matrix are negative and $-S_{k,k}$ defines the parameter of the exponentially distributed sojourn time of the process in the state $k = \overline{1, M}$. The non-diagonal entries of the matrix S define the intensities of transitions of the process η_t, $t \geq 0$, in the state space $\{1, ..., M\}$. The value $-\sum_{k'=1}^{M} S_{k,k'}$ defines the intensity of the transition of the process η_t, $t \geq 0$, from state k into the absorbing state. The epoch of the transition of the process η_t, $t \geq 0$, into this absorbing state defines the service completion epoch. We set $S_0 = -S\mathbf{e}$. It is assumed that all the entries of the column-vector S_0 are non-negative and at least one of them is positive. The average service time b_1 is given by

$$b_1 = \boldsymbol{\beta}(-S)^{-1}\mathbf{e}. \tag{3}$$

The pair $(\boldsymbol{\beta}, S)$ is called as irreducible representation of the random having PH distribution. For more details on PH type distributions see the book (Neuts, 1981). Mention that the class of the PH type distributions includes such well known in literature distributions as Erlangian and Hyper-Markovian and suits for approximating a wide range of more general distributions arising in the practical models (Weibull, Lognormal, etc.)

If the arriving into the system customer meets several free servers upon arrival, all these servers start, independently of others, the service of this customer. Since this epoch, all multiple copies of this customer are considered as the independent customers serving in the system. These copies are not deleted from the system when some copy of the customer finishes the service or if new customers arrive into the system. If all the servers are busy upon arrival, the customer is placed into the buffer of an infinite capacity and then it will be picked up from the queue according to the $FIFO$ (First In - First Out) discipline.

3 Distribution of the number of customers in the system at arrival epochs

Let $i_t,\ i_t \geq 0$, be the number of customers in the system at epoch $t, t \geq 0$. Our aim is to study the stationary behavior of the process $i_t,\ i_t \geq 0$. However, this process is non-Markovian one and its direct investigation is not possible. So, we apply the method of embedded Markov chains for its investigation. To this end, we first consider the states of the process $i_t,\ i_t \geq 0$, only at the epochs $t_n - 0$ immediately before the nth customer arrival into the system, $n \geq 1$. Let $i_n = i_{t_n-0}, n \geq 1$, $\nu_n = \nu_{t_n-0}, n \geq 1$, $m_n^{(j)}$ be the state of the directing process of the service in the j-th busy server at the epoch $t_n - 0$, $m_n^{(j)} = \overline{1,M},\ j = \overline{1, \min\{i_n, N\}},\ n \geq 1$. We assume here that the busy servers are numerated in order of their occupying, i.e. the server, which begins the service, is appointed the maximal number among all busy servers; when several servers start the service of a customer simultaneously, they get sequential numbers in a random order; when some server finishes the service, the other busy servers are correspondingly enumerated.

It is easy to see that the multi-dimensional process

$$\zeta_n = (i_n, m_n^{(1)}, \ldots, m_n^{(\min\{i_n,N\})}, \nu_n),\ i_n \geq 0,$$

$$m_n^{(j)} = \overline{1,M},\ j = \overline{1, \min\{i_n, N\}}, \nu_n = \overline{1,K},\ n \geq 1,$$

is an irreducible discrete time Markov chain.

Denote the stationary probabilities of this process as

$$\pi(i, s_1, \ldots, s_{\min\{i,N\}}, k) = \qquad (4)$$

$$= \lim_{n\to\infty} P\{\, i_n = i,\ m_n^{(1)} = s_1, \ldots,$$

$$m_n^{(\min\{i_n,N\})} = s_{\min\{i,N\}}, \nu_n = k\},$$

$$i \geq 0,\ s_j = \overline{1,M},\ j = \overline{1,\min\{i,N\}},\ k = \overline{1,K}.$$

The problem of the establishing conditions for existence of limits (4) will be discussed a bit later.

Let us enumerate the states of the Markov chain $\zeta_n,\ n \geq 1$, in the lexicographic order and form the row-vectors $\boldsymbol{\pi}_i$ of the stationary-state probabilities $\pi(i, , s_1, \ldots, s_{\min\{i,N\}}, k)$, corresponding to the state i of the first component of the chain, $i \geq 0$.

Analogously, we form the matrices $P_{i,l}$ of one-step transition probabilities of the Markov chain $\zeta_n, n \geq 1$, as the matrices consisting of the conditional probabilities

$$P\{i_{n+1} = l, m_{n+1}^{(1)} = s_1^{'}, \ldots, m_{n+1}^{(\min\{l,N\})} = s^{'}_{\min\{l,N\}},$$

$$\nu_{n+1} = k' | i_n = i, m_n^{(1)} = s_1, \ldots, m_n^{(\min\{i,N\})} = s_{\min\{i,N\}},$$

$$\nu_n = k\},\ k, k' = \overline{1,K},\ s_j = \overline{1,M},\ j = \overline{1, \min\{i,N\}},$$

$$s_j^{'} = \overline{1,M},\ j = \overline{1,\min\{l,N\}},\ i, l \geq 0.$$

Lemma 1. *The non-zero matrices $P_{i,l},\ i,l \geq 0$, of one-step transition probabilities are calculated by*

$$P_{i,l} = \Omega_{i+1-l} = \int_0^\infty P(i+1-l, t) \otimes dA(t), \qquad (5)$$

$$N \leq l \leq i+1,$$

$$P_{i,l} = \int_0^\infty \int_0^t P(i-N, y)\mathcal{F}_1(e^{Q(t-y)})_{N,l} dy \otimes dA(t), \ (6)$$

$$0 \leq l < N,\ i \geq N,$$

$$P_{i,l} = \Gamma_i \mathcal{A}_l, i = \overline{0, N-1},\ \ l = \overline{0,N}, \qquad (7)$$

where

$$\Gamma_i = (I_{M^i} \otimes \boldsymbol{\beta}^{\otimes(N-i)}),\ i = \overline{0, N-1}, \qquad (8)$$

$$\mathcal{A}_l = \int_0^\infty (e^{Qt})_{N,l} \otimes dA(t),\ l = \overline{0,N}, \qquad (9)$$

the matrices $P(l,t)$ are defined as coefficients of expansion

$$\sum_{l=0}^\infty P(l,t) z^l = e^{(\mathcal{F}_0 + \mathcal{F}_1 z)t}, \qquad (10)$$

the matrices $\mathcal{F}_0,\ \mathcal{F}_1$ are defined by

$$\mathcal{F}_0 = S^{\oplus N},\ \mathcal{F}_1 = (S_0 \boldsymbol{\beta})^{\oplus N},$$

generator Q has the following form:

$$\begin{pmatrix} O & O & O & \ldots & O & O \\ S_0 & S & O & \ldots & O & O \\ O & S_0^{\oplus 2} & S^{\oplus 2} & \ldots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \ldots & S^{\oplus(N-1)} & O \\ O & O & O & \ldots & S_0^{\oplus N} & S^{\oplus N} \end{pmatrix}, \ (11)$$

$\otimes$ *is the sign of Kronecker's product, and* $\oplus$ *is the sign of Kronecker's sum of matrices,* $\beta^{\otimes l} \stackrel{def}{=} \underbrace{\beta \otimes \ldots \otimes \beta}_{l}$, $S^{\oplus l} \stackrel{def}{=} \underbrace{S \oplus \ldots \oplus S}_{l}$,

$S_0^{\oplus l} \stackrel{def}{=} \sum_{m=0}^{l-1} I_{M^m} \otimes S_0 \otimes I_{M^{l-m-1}}, \quad l \geq 1,$ $S^{\oplus 0} \stackrel{def}{=} 0$, I_L *denotes the identity matrix of size* L, $L \geq 1$.

Proof of the lemma consists of analysis of one-step transitions of the Markov chain $\zeta_n, \ n \geq 1$. It is straightforward if we take into account the probabilistic meaning of the involved matrices.

The entries of the matrix $P(l,t)$ define probability of finishing the service of l customer during the time interval $[0,t)$ and corresponding transitions of the directing processes of the PH type service in all N servers if these servers were permanently busy during this interval. Then, the entries of the matrix Ω_l define probability of $l, \ l \geq 0$, customers departure from the system between the successive customer arrival epochs when all servers are permanently busy during this inter-arrival time.

Let $n_t, \ n_t = \overline{0,N}$, be the number of busy servers at epoch $t, \ t \geq 0$, and $m_t^{(l)}, \ m_t^{(l)} = \overline{1,M}$, be the states of the directing process of the service in the l-th, server, $l = \overline{1,n_t}$. The matrix Q is generator of the reducible continuous time Markov chain

$$\xi_t = \{n_t, m_t^{(1)}, \ldots, m_t^{(n_t)}\}, n \geq 1.$$

Note that the Markov chain $\xi_t, \ n \geq 1$, describes behavior of the service process in the queueing model under consideration when new customers do not enter the servers. This implies that the entries of the matrix $(e^{Qt})_{N,l}$ define probability that the number of busy servers will decrease since N to l and the directing processes of the PH type service in the servers make the corresponding transitions during the time interval $[0,t)$ conditional that new customers do not enter the service during this interval.

Taking into account these probabilistic considerations and the formula of total probability, we easy get formulae (5) - (11) be proven.

Lemma 2. *The matrix* $\mathcal{P} = (P_{i,l})_{i\geq 0, l\geq 0}$ *of one-step transition probabilities of the Markov chain* $\zeta_n, \ n \geq 1$, *has the following structure:*

$$\mathcal{P} = \tag{12}$$

$$= \begin{pmatrix} \Gamma_0\mathcal{A}_0 & \Gamma_0\mathcal{A}_1 & \ldots & \Gamma_0\mathcal{A}_{N-1} & \Gamma_0\mathcal{A}_N & O & O & \ldots \\ \Gamma_1\mathcal{A}_0 & \Gamma_1\mathcal{A}_1 & \ldots & \Gamma_1\mathcal{A}_{N-1} & \Gamma_1\mathcal{A}_N & O & O & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Gamma_{N-1}\mathcal{A}_0 & \Gamma_{N-1}\mathcal{A}_1 & \ldots & \Gamma_{N-1}\mathcal{A}_{N-1} & \Gamma_{N-1}\mathcal{A}_N & O & O & \ldots \\ P_{N,0} & P_{N,1} & \ldots & P_{N,N-1} & \Omega_1 & \Omega_0 & O & \ldots \\ P_{N+1,0} & P_{N+1,1} & \ldots & P_{N+1,N-1} & \Omega_2 & \Omega_1 & \Omega_0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The statement of the lemma directly stems from lemma 1.

Two essential distinctions of the matrix $\mathcal{P} = (P_{i,l})_{i\geq 0, l\geq 0}$ of one-step transition probabilities of its prototype in the case of the classic service discipline have to be mentioned. The first one is that the matrix $\mathcal{P}$ in the form (12) is not completely low hessenbergian as its prototype. This distinction makes investigation of the Markov chain more complicated. The second distinction consists of the fact that the first N block rows of the matrix $\mathcal{P}$ coincide up to the multiplier $\Gamma_i, i = \overline{0, N-1}$. It appears that this distinction simplifies investigation and, eventually, results, which will be derived below, have more nice analytic form comparing the system with the classic service discipline.

Theorem 1. *Stationary distribution* $\boldsymbol{\pi}_i, i \geq 0$, *of the Markov chain* $\zeta_n, \ n \geq 1$, *exists if and only if the inequality*

$$\rho = \frac{\lambda b_1}{N} < 1 \tag{13}$$

is fulfilled where λ *is the fundamental rate of the arrival process defined by formula (2) and* b_1 *is the mean service time defined by formula (3).*

Vectors $\boldsymbol{\pi}_i, i \geq 0$, *are computed as follows:*

$$\boldsymbol{\pi}_k = \boldsymbol{\delta}\mathcal{A}_k + \mathbf{c}\mathcal{B}_k, \ \ k = \overline{0, N-1}, \tag{14}$$

$$\boldsymbol{\pi}_i = \mathbf{c}\mathcal{R}^{i-N+1}, \ \ i \geq N, \tag{15}$$

where the matrices $\mathcal{A}_k, \ k = \overline{0,N}$, *are defined by formulae (9), the matrices* $\mathcal{B}_k$, *are computed by*

$$\mathcal{B}_k = \tag{16}$$

$$\mathcal{R} \int_0^\infty \int_0^t \sum_{m=0}^{\infty} \mathcal{R}^m \Big(P(m,y)\mathcal{F}_1(e^{Q(t-y)})_{N,k} dy \otimes dA(t) \Big),$$

the matrix $\mathcal{R}$ *is the minimal non-negative solution to the equation*

$$\mathcal{R} = \sum_{m=0}^{\infty} \mathcal{R}^m \Omega_m, \tag{17}$$

$\boldsymbol{\delta}$ *is the row vector computed by*

$$\boldsymbol{\delta} = \mathbf{c} \sum_{k=0}^{N-1} \mathcal{B}_k \Gamma_k (I - \sum_{l=0}^{N-1} \mathcal{A}_l \Gamma_l)^{-1}, \tag{18}$$

$\mathbf{c}$ *is the row vector computed as the unique solution to the system*

$$\mathbf{c}\big[\sum_{k=0}^{N-1}\mathcal{B}_k\Gamma_k(I-\sum_{l=0}^{N-1}\mathcal{A}_l\Gamma_l)^{-1}\mathcal{A}_N-\mathcal{R}+\mathcal{B}_N\big]=\mathbf{0}, \tag{19}$$

$$\mathbf{c}\big[\mathcal{R}(I-\mathcal{R})^{-1}+\sum_{k=0}^{N-1}\mathcal{B}_k\Gamma_k(I-\sum_{l=0}^{N-1}\mathcal{A}_l\Gamma_l)^{-1}\big]\mathbf{e}=1. \tag{20}$$

Proof. It is well-known that the vectors $\boldsymbol{\pi}_i, i\geq 0$, defining the stationary distribution of the Markov chain $\zeta_n,\ n\geq 1$, satisfy Chapman-Kolmogorov's equations (or equilibrium equations)

$$(\boldsymbol{\pi}_0,\boldsymbol{\pi}_1,\boldsymbol{\pi}_2,\dots)\mathcal{P}=(\boldsymbol{\pi}_0,\boldsymbol{\pi}_1,\boldsymbol{\pi}_2,\dots). \tag{21}$$

It can be seen from the structure (12) of the transition probability matrix $\mathcal{P}$ that, starting from the $(N+1)$st block row, the matrix becomes be low hessenbergian. So, results by M. Neuts (Neuts, 1981) concerning so called $GI/M/1$ type Markov chains (or Markov chains possessing skip-free to the right property) can be applied in some extent. In particular, because the limiting behavior of the Markov chain (and stability condition) does not depend on the transitions of the Markov chain in the boundary states, stability condition (13) directly follows from (Neuts, 1981).

To prove relations (14) - (20) we will solve, step-by-step, equilibrium equations (21) with the transition probability matrix generator $\mathcal{P}$ of form (12).

The kth, $k\geq N+2$, equation of the system (21) can be rewritten as

$$\boldsymbol{\pi}_k=\sum_{l=k-1}^{\infty}\boldsymbol{\pi}_l\Omega_{l+1-k},\quad k\geq N+1. \tag{22}$$

By the direct substitution, we can make sure that the probability vectors $\boldsymbol{\pi}_k$, which satisfy system (22), have the following form:

$$\boldsymbol{\pi}_k=\mathbf{c}\mathcal{R}^{k-N+1},\quad k\geq N, \tag{23}$$

where $\mathbf{c}$ is some constant vector and the matrix $\mathcal{R}$ is solution to equation (17). See M. Neuts' book (Neuts, 1981) for more reasonings and explanations.

The first N equations of the system (21) can be rewritten as

$$\boldsymbol{\pi}_k=\sum_{l=0}^{N-1}\boldsymbol{\pi}_l\Gamma_l\mathcal{A}_k+\sum_{m=N}^{\infty}\boldsymbol{\pi}_m P_{m,k},\quad k=\overline{0,N-1}, \tag{24}$$

where the matrices $\mathcal{A}_k,\ k=\overline{0,N-1}$, are defined by formula (9) and the matrices $P_{m,k}$ are defined by formula (6).

By substituting expressions (6) and (15) into the infinite sum in (24) and calculating this sum, we get relation

$$\boldsymbol{\pi}_k=\boldsymbol{\delta}\mathcal{A}_k+\mathbf{c}\mathcal{B}_k,\quad k=\overline{0,N-1}, \tag{25}$$

where

$$\boldsymbol{\delta}=\sum_{l=0}^{N-1}\boldsymbol{\pi}_l\Gamma_l, \tag{26}$$

and the matrices $\mathcal{B}_k$ are given by formula (16).

By multiplying equations (25) by the matrices Γ_k and summing up, we yield the following expression for the unknown vector $\boldsymbol{\delta}$:

$$\boldsymbol{\delta}=\mathbf{c}\sum_{k=0}^{N-1}\mathcal{B}_k\Gamma_k(I-\sum_{l=0}^{N-1}\mathcal{A}_l\Gamma_l)^{-1}. \tag{27}$$

Mention that the inverse matrix in (27) exists due to Hadamard's theorem because the matrix $\sum\limits_{l=0}^{N-1}\mathcal{A}_l\Gamma_l$ is sub-generator.

The $(N+1)$st equation of the system (21) can be rewritten as

$$\boldsymbol{\pi}_N=\sum_{l=0}^{N-1}\boldsymbol{\pi}_l\Gamma_l\mathcal{A}_N+\sum_{m=N}^{\infty}\boldsymbol{\pi}_m\Omega_{m+1-N}. \tag{28}$$

By substituting expressions (5) and (15) into the infinite sum in (28) and calculating this sum, we conclude that relation (25) holds good for $k=N$ as well. So, we have two different expressions (15) and (25) for the vector $\boldsymbol{\pi}_N$. This implies relation

$$\boldsymbol{\delta}\mathcal{A}_N+\mathbf{c}\mathcal{B}_N=\mathbf{c}\mathcal{R}. \tag{29}$$

Relations (27) and (29) yield the following homogeneous system of linear algebraic equations for the unknown vector $\mathbf{c}$:

$$\mathbf{c}\big[\sum_{k=0}^{N-1}\mathcal{B}_k\Gamma_k(I-\sum_{l=0}^{N-1}\mathcal{A}_l\Gamma_l)^{-1}\mathcal{A}_N-\mathcal{R}+\mathcal{B}_N\big]=\mathbf{0}.$$

This system coincides with (19). Equation (20) is obtained from normalization condition

$$\sum_{i=0}^{\infty}\boldsymbol{\pi}_i\mathbf{e}=1.$$

This completes the proof of the theorem.

Corollary 1. *Average number L of customers in the system at the customer arrival epochs is computed by*

$$L=\sum_{k=1}^{\infty}k\boldsymbol{\pi}_k\mathbf{e}=$$

$$=\Big[\boldsymbol{\delta}\sum_{k=1}^{N}k\mathcal{A}_k+\mathbf{c}\Big(\sum_{k=1}^{N}k\mathcal{B}_k+\mathcal{R}^2(N(I-\mathcal{R})+I)(I-\mathcal{R})^{-2}\Big)\Big]\mathbf{e}.$$

Average number N_s of servers, which process an arbitrary customer in the system, is computed by

$$N_s = \Big[\sum_{k=0}^{N-1}(N-k)\boldsymbol{\pi}_k + \sum_{k=N}^{\infty}\boldsymbol{\pi}_k\Big]\mathbf{e} =$$

$$= \Big[\sum_{k=0}^{N-1}(N-k)\boldsymbol{\pi}_k + \mathbf{c}\mathcal{R}(I-\mathcal{R})^{-1}\Big]\mathbf{e}.$$

4 Distribution of the number of customers in the system at arbitrary epoch

Having the stationary distribution of the embedded Markov chain $\zeta_n, n \geq 1$, been computed, now we can calculate the stationary distribution of the non-Markovian process $i_t,\ i_t \geq 0$, of the number of customers in the system at epoch $t, t \geq 0$.

Let us denote the stationary probabilities of the multi-dimensional process

$$\zeta_t = (i_t, m_t^{(1)}, \dots, m_t^{(\min\{i_t,N\})}, \nu_t),\ i_t \geq 0,$$

$$m_t^{(j)} = \overline{1,M},\ j = \overline{1,\min\{i_t,N\}}, \nu_t = \overline{1,K},$$

by

$$p(i, s_1, \dots, s_{\min\{i,N\}}, k) =$$

$$= \lim_{t\to\infty} P\{\, i_t = i,\ m_t^{(1)} = s_1, \dots,$$

$$m_t^{(\min\{i_t,N\})} = s_{\min\{i,N\}}, \nu_t = k\},$$

$$i \geq 0,\ s_j = \overline{1,M},\ j = \overline{1,\min\{i,N\}},\ k = \overline{1,K},$$

and form the row-vectors $\mathbf{p}_i$ of these stationary-state probabilities corresponding to the state i of the first component of the process $\zeta_t,\ t \geq 0$.

Theorem 2. *The stationary-state probability vectors $\mathbf{p}_i,\ i \geq 0$, are computed by*

$$\mathbf{p}_i = \lambda\mathbf{c}\mathcal{R}^{i-N}\sum_{m=0}^{\infty}\mathcal{R}^m\tilde{\Omega}_m,\ \ i \geq N, \tag{30}$$

$$\mathbf{p}_i = \lambda\Big[\sum_{j=\max\{0,i-1\}}^{N-1}\boldsymbol{\pi}_j\Gamma_j\tilde{\mathcal{A}}_i + \mathbf{c}\sum_{j=N}^{\infty}\mathcal{R}^{j-N+1}\tilde{P}_{j,i}\Big], \tag{31}$$
$$i = \overline{0, N-1},$$

where

$$\tilde{\Omega}_m = \int_0^{\infty} P(m,t)\otimes(A(\infty)-A(t))dt,\ m \geq 0,$$

$$\tilde{\mathcal{A}}_l = \int_0^{\infty}(e^{Qt})_{N,l}\otimes(A(\infty)-A(t))dt, l = \overline{0,N},$$

$$\tilde{P}_{i,l} = \int_0^{\infty}\int_0^{t} P(i-N,y)\mathcal{F}_1(e^{Q(t-y)})_{N,l}dy\otimes(A(\infty)-A(t))dt,$$
$$0 \leq l < N,\ i \geq N.$$

Proof exploits the theorem (6.12) in (Cinlar, 1975) and is straightforward. So, it is omitted.

5 Conclusion

We have analyzed the stationary distribution of the queue in the $SM/PH/N$ type queueing systems with infinite buffer when the customer arriving into the system is served simultaneously by all free servers. Analysis can be easy extended to the cases when errors, which break the service or just cause wrong result of the service, can occur during the service of a customer. As it was shown for the partial cases of the considered model in (Lee, Dudin, Klimenok, 2006a), (Lee, Dudin, Klimenok, 2006b), broadcasting service discipline can essentially improve such performance measures of the system as the mean sojourn time and probability of successful delivering of a customer.

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OPTIMAL MULTITHRESHOLD CONTROL FOR BMAP/SM/1 QUEUE WITH MAP-INPUT OF DISASTERS

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ABSTRACT
BMAP/SM/1 queueing system with controllable mode of operation is considered. Operational mode is changed according to a multithreshold strategy. System also has Markovian input of disasters. Disaster is a special kind of negative customer which interrupts the service and causes all the customers to leave the system unserved immediately whenever it arrives to the busy system. The stationary state distribution of embedded Markov chain and performance characteristics (mean queue length, mean interdeparture time, mean number of customers lost per time unit) are obtained under the fixed control strategy. The optimal control strategy is determined numerically.

INTRODUCTION
Queueing systems with dynamic control allow to redistribute the system resources during system operation. In particular, such systems have variable parameters of input flow, service and other processes characterizing the queueing system. The each group of parameters used simultaneously is referred to as operational mode. At the decision making moments the certain operational mode is selected for customer processing correspondingly to some control strategy (rule of mode selection). Queueing models with controllable mode of operation can be effectively applied for modelling the telecommunication network fragments. That is the resource of telecommunication network which hands a mixture of flows having different requirements to the response time; systems which combine transmission of several types of information having different requirements to the quality of service and having a possibility to distribute the bandwidth dynamically, etc.

There are two directions of the controllable queueing systems investigation. The papers of the first direction, e.g. (Rykov 1999) and (Artalejo and Economou 2004), deal with the problem of the optimal strategy existence and properties of such strategies. In the papers of the second direction including the present paper the class of control strategies is chosen in advance (e.g., threshold or hysteresis strategies) and the problem to find the optimal strategy in the given class is dealt with. Threshold strategies to control the queues with two operational modes were considered in (Nishimura and Jiang 1995) and (Nobel and Tijms 1999); queues with hysteresis strategies were analyzed in (Dudin 2002) and (Dudin and Nishimura 2000). Multithreshold control strategies were investigated in (Dudin 1998), (Dudin and Chakravarthy 2003) and (Kim et al. 2006) for the queues with several operational modes. The latter works include analysis of queues having the BMAP (Batch Markovian Arrival Process) input of customers. Nowadays BMAP is the most general customer input model which allows to analyze the queue with such input analytically, (Neuts 1989, Lucantoni 1991).

In this paper the analysis of controllable BMAP/SM/1 queue having n, $n \geq 2$, operation modes and input of disasters is presented. Control strategies are multithreshold that is mode of operation can be change at definite epochs (customer departure epochs) accordingly to the relation between queue length and prespecified integers called the thresholds. The system also has an additional input of disasters which cause all customers to leave the system unserved. Disaster is the special kind of negative arrival that interrupts the service and removes all the customers from the system whenever it enters the busy system. Theory of negative arrivals was originated by E. Gelenbe (Gelenbe 1989) and since is being developed significantly. Detailed review of achieved results is presented in reviews (Artalejo 2000) and (Bocharov and Vishnevskij 2003). BMAP/SM/1 queue with disasters was investigated in (Dudin and Nishimura 1999). Note that queueing models with disasters can be applied to computer networks with virus infections and migration processes with catastrophes.

The present paper generalizes the results of (Semenova 2004) obtained for the queue with two modes of operation to the case of several operation modes.

MODEL

A single-server queue having n, $n \geq 2$, modes of operation and input of disasters is under consideration.

The rth mode is described as follows. Customers arrive to the system accordingly to Batch Markovian Arrival Process (BMAP) which is governed by a stochastic process $\nu_t, t \geq 0$, with a state space $\{0, 1, ..., W\}$. Transitions of the process $\nu_t, t \geq 0$ and arrivals of customers are performed with a matrix generation function $D^{(r)}(z) = \sum_{m=0}^{\infty} D_m^{(r)} z^m, |z| < 1$. The matrix $D_0^{(r)}$ governs the transitions corresponding to no arrivals, and $D_m^{(r)}$ governs the transitions corresponding to arrivals of size m batch of customers, $m \geq 1$.

Denote by $\boldsymbol{\gamma}^{(r)}$ the stationary probability row vector of the Markov chain $\nu_t, t \geq 0$. It is given by the system of equations: $\boldsymbol{\gamma}^{(r)} D^{(r)}(1) = \mathbf{0}, \quad \boldsymbol{\gamma}^{(r)} \mathbf{1} = 1$. Here and bellow $\mathbf{0}$ is a null column-vector of appropriate size, $\mathbf{1}$ is a row-vector consisting of ones. The intensity $\lambda^{(r)}$ of BMAP is given by

$$\lambda^{(r)} = \boldsymbol{\gamma}^{(r)} \frac{dD^{(r)}(z)}{dz}\bigg|_{z=1} \mathbf{1}.$$

The variation coefficient c_{var} of intervals between successive group arrivals is given by the formula $c_{var}^2 = 2\lambda_g^{(r)} \boldsymbol{\gamma}^{(r)} (-D_0^{(r)})^{-1} \mathbf{1} - 1$, where $\lambda_g^{(r)} = \boldsymbol{\gamma}^{(r)} (-D_0^{(r)}) \mathbf{1}$ is the intensity of group arrivals. The correlation coefficient c_{cor} of intervals between customer group arrivals is given by $c_{cor} = (\lambda_g^{(r)} \boldsymbol{\gamma}^{(r)} (-D_0^{(r)})^{-1} (D^{(r)}(1) - D_0^{(r)})(-D_0^{(r)})^{-1} \mathbf{1})/c_{var}^2$.

More detailed description of BMAP and assumptions about the matrix function $D^{(r)}(z)$ can be found e.g. in (Lucantoni 1991).

The service process is of SM-type. It means that successful service times are the sojourn times of a semi-Markovian process $m_t, t \geq 0$. This process has a state space $\{1, ..., M\}$ and a semi-Markovian kernel $B^{(r)}(x) = \left(B_{m,m'}^{(r)}(x)\right)_{m,m'=\overline{1,M}}$ when system works under the rth mode. The function $B_{m,m'}^{(r)}(x)$ is the conditional distribution function of the sojourn time of the process $m_t, t \geq 0$ in a state m under the condition that the next state will be m', $m, m' = \overline{1, M}$.

For the system under consideration the service is interrupted at a disaster arrival epoch. It is assumed that the states of the service directing process $m_t, t \geq 0$ are changed at service completion epochs accordingly to the matrix $P^{(r)} = B^{(r)}(\infty)$ in the rth operational mode regardless of whether service is completed successfully or is cancelled by disaster appearance.

Denote by $b^{(r)}$ the mean service time which is not interrupted by a disaster arrival in the rth mode. It's value is given by the formula $b^{(r)} = \boldsymbol{\delta} \int_0^{\infty} t dB^{(r)}(t) \mathbf{1}$, where $\boldsymbol{\delta}$ is invariant row vector of the matrix $P^{(r)}$.

As (Lucantoni and Neuts 1994) and (Neuts 1989) we assume that the matrix $P^{(r)}$ is indecomposable, $B^{(r)}(+0) = 0$ and $b^{(r)} < \infty$.

Disaster arrival process is MAP (Markovian Arrival Process) which is the particular case of BMAP allowing the ordinary arrivals only. Disaster arrival to the busy system interrupts the service and causes all customers to leave the system instantaneously. If disaster arrives to the empty system it is ignored. We suppose that input of disasters is governed by the process $\eta_t, t \geq 0$, having a state space $\{0, 1, ..., N\}$ and a matrix generating function $F^{(r)}(z) = F_0^{(r)} + F_1^{(r)} z, |z| \leq 1$. Description of the rth operation mode is completed, $r = \overline{1, n}$.

The operation mode can be switched at the service completion epochs according to multithreshold strategy.

The aim of the system control is to minimize the cost criterion

$$C = a\Lambda L + \sum_{r=1}^{n} c_r \Phi_r + dR, \tag{1}$$

where L is the mean queue length at customer departure epoch; Λ^{-1} is the mean interdeparture time; Φ_r s the average fraction of time, when the r-th mode is in use, $r = \overline{1, n}$; R is the average number of customers lost per time unit due to disaster arrival; a, c_r, $r = \overline{1, n}$, and d are the cost coefficients.

Multithreshold control strategy is determined as follows. The nonnegative integers $(j_1, j_2, \ldots, j_{n-1})$, $-1 = j_0 < j_1 \leq j_2 \leq \cdots \leq j_{n-1} < j_n = \infty$, which are called the thresholds are fixed. If a queue length i at a given customer departure epoch satisfies the inequality $j_{r-1} + 1 \leq i \leq j_r$, the rth mode is selected for the next customer service, $r = \overline{1, N}$.

To obtain the performance characteristics involved in (1) under the fixed set of thresholds we use the method of embedded Markov chains and derive the stationary state distribution of the embedded chain describing the system's behavior at service completion epochs.

EMBEDDED MARKOV CHAIN

Let t_k be the kth epoch of customers departure from the system, $k \geq 1$. Note that it is a service completion epoch or a disaster arrival epoch at a busy period.

Introduce into consideration the following five-dimensional Markov chain:

$$\xi_k = \{i_k, u_k, \zeta_k\}, \quad k \geq 1,$$

where $\zeta_k = \{\nu_k, \eta_k, m_k\}$, ν_k is the state of arrival directing process ν_t, $t \geq 0$, at the epoch t_k, $\nu_k = \overline{0, W}$, η_k is the state of disaster directing process η_t, $t \geq 0$, at the epoch $t_k + 0$, $\eta_k = \overline{0, N}$; m_k is the state of the service directing process m_t, $t \geq 0$, at the epoch $t_k + 0$, $m_k = \overline{1, M}$. The component u_k possesses the following values:

— $u_k = 0$, if t_k is the epoch of successful service completion, in this case i_k means the queue length at the epoch $t_k + 0$, $i_k \geq 0$;

— $u_k = 1$, if t_k is a disaster arrival epoch, in this case i_k means the number of customers that leave the system at epoch t_k, $i_k \geq 1$, $k \geq 1$.

Let the states of process ζ_k, $k \geq 1$, be listed in lexicographic order of the components $\{\nu_k, \eta_k, m_k\}$ increase and be numbered from 1 to $H = (W + 1)(N + 1)M$. For convenience the state $\{\nu, \eta, m\}$ of the triple $\{\nu_k, \eta_k, m_k\}$ will be further replaced with it's serial number.

Introduce into consideration one-step transition probabilities

$$P\{(i, u, \zeta) \to (l, u', \zeta')\} = P\{i_{k+1} = l, u_{k+1} = u', \zeta_{k+1} = \zeta' | i_k = i, u_k = u, \zeta_k = \zeta\}, \quad i, l \geq 0,\ u, u' = \overline{0, 1},\ \zeta, \zeta' = \overline{1, H},$$

and form the matrices

$$P^{(u,u')}_{i,l} = \big(P\{(i, u, \zeta) \to (l, u', \zeta')\}\big)_{\zeta,\zeta'=\overline{1,H}}, \quad i, l \geq 0, u, u' = \overline{0, 1}.$$

Introduce also the matrices Ψ_k, $k \geq 1$, and $S^{(r)}_l$, $\Omega^{(r)}_l$, $l \geq 0$, $r = \overline{1, n}$, having the following probabilistic sense.

The entry $[\Omega^{(r)}_k]_{s,m}$ of the matrix $\Omega^{(r)}_k$ is the probability that k customers arrive but no disaster arrives to the system during the customer processing in the r-th mode, and the process ζ_l, $l \geq 1$, transits from the state s into the state m, $s, m = \overline{1, H}$, $k \geq 0$, $r = \overline{1, n}$.

The entry $[S^{(r)}_k]_{s,m}$ of the matrix $S^{(r)}_k$ is the probability that disaster appears during customer service in the r-th mode, k customers arrive to the queue and the process ζ_l, $l \geq 1$, transits from the state s into the state m during the uncompleted service time, $s, m = \overline{1, H}$, $k \geq 0$, $r = \overline{1, n}$.

The entry $[\Psi_k]_{s,m}$ of the matrix Ψ_k is the probability that system's busy period starts by the arrival of customers batch of size k and the process ζ_l, $l \geq 1$, transits from the state s into the state m during the idle period, $s, m = \overline{1, H}$, $k \geq 1$.

Analyzing the one-step transition probabilities of the Markov chain ξ_k, $k \geq 1$, the following statement can be proved.

Lemma 1. Nonzero matrices $P^{(u,u')}_{i,l}$, $i, l \geq 0$, $u, u' = \overline{0, 1}$, of one-step transition probabilities of the Markov chain ξ_k, $k \geq 1$, are given as

$$P^{(0,0)}_{0,l} = P^{(1,0)}_{i,l} = \sum_{k=1}^{l+1} \Psi_k \Omega^{(1)}_{l-k+1},\ i > 0,\ l \geq 0,$$
$$P^{(0,1)}_{0,l} = P^{(1,1)}_{i,l} = \sum_{k=1}^{l} \Psi_k S^{(1)}_{l-k},\ i, l \geq 1, \tag{2}$$
$$P^{(0,0)}_{i,l} = \Omega^{(r)}_{l-i+1}, \quad i > 0,\ l \geq i-1,\ j_{r-1} < i \leq j_r,$$
$$P^{(0,1)}_{i,l} = S^{(r)}_{l-i}, \quad i > 0,\ l \geq i,\ j_{r-1} < i \leq j_r, \quad r = \overline{1, n},$$

where the matrices Ψ_k, $k \geq 1$, и $S^{(r)}_l$, $\Omega^{(r)}_l$, $l \geq 0$, $r = \overline{1, n}$, can be determined from the following matrix expansions:

$$\Psi(z) = \sum_{k=1}^{\infty} \Psi_k z^k = -[(D^{(1)}_0 \oplus F^{(1)}(1))^{-1} \otimes I_M] \times \times ((D^{(1)}(z) - D^{(1)}_0) \otimes I_{(N+1)M}),$$
$$S^{(r)}(z) = \sum_{k=0}^{\infty} S^{(r)}_k z^k = \int_0^{\infty} e^{D^{(r)}(z)t} \otimes \big(e^{F^{(r)}_0 t} F^{(r)}_1\big) \otimes \otimes (P^{(r)} - B^{(r)}(t))\, dt,$$
$$\Omega^{(r)}(z) = \sum_{k=0}^{\infty} \Omega^{(r)}_k z^k = \int_0^{\infty} e^{D^{(r)}(z)t} \otimes e^{F^{(r)}_0 t} \otimes \otimes dB^{(r)}(t),\ r = \overline{1, n},$$

$\otimes$ and $\oplus$ are the symbols of the Kronecker product and the Kronecker sum, $I_\bullet$ denotes an identity matrix of corresponding size.

Markov chain $\xi_k, k \geq 1$ is ergodic under any parameters of input and disaster flows, service and mode switching processes if the matrices $F^{(r)}_1$, $r = \overline{1, N}$, are nonzero. If one of these matrices is zero matrix the embedded Markov chain is ergodic if and only if the inequality $\lambda^{(r)} b^{(r)} < 1$ holds (Lucantoni and Neuts 1994), where r is the number of mode having zero matrix $F^{(r)}_1$.

Introduce into consideration the stationary state probabilities

$$p(i, \zeta) = \lim_{l \to \infty} P\{i_l = i, u_l = 0, \zeta_l = \zeta\},\ i \geq 0,$$
$$k(i, \zeta) = \lim_{l \to \infty} P\{i_l = i, u_l = 1, \zeta_l = \zeta\},\ i \geq 1, \zeta = \overline{1, H},$$

vectors $\boldsymbol{p}_i = (p(i, 1), \ldots, p(i, H))$, $i \geq 0$, $\boldsymbol{k}_i = (k(i, 1), \ldots, k(i, H))$, $i \geq 1$, and their generating functions

$$\boldsymbol{P}_r(z) = \sum_{i=j_{r-1}+1}^{j_r} \boldsymbol{p}_i z^i,\ r = \overline{1, n},\ \boldsymbol{K}(z) = \sum_{i=1}^{\infty} \boldsymbol{k}_i z^i,\ |z| \leq 1.$$

Theorem 1. Probability generating functions $\boldsymbol{P}_r(z)$, $r = \overline{1, n}$, and $\boldsymbol{K}(z)$ satisfy the functional equations

$$\sum_{r=1}^{n} \boldsymbol{P}_r(z)(zI - \Omega^{(r)}(z)) = (\boldsymbol{\pi}_0(\Psi(z) - I) + \tag{3}$$
$$+\boldsymbol{K}(1))\Omega^{(1)}(z),$$
$$\boldsymbol{K}(z) = (\boldsymbol{\pi}_0(\Psi(z) - I) + \boldsymbol{K}(1))S^{(1)}(z) + \tag{4}$$
$$+\sum_{r=1}^{n} \boldsymbol{P}_r(z)S^{(r)}(z),$$

where $\boldsymbol{\pi}_0 = \boldsymbol{p}_0 + \boldsymbol{K}(1)$.

Proof. Using the formula of total probability and Lemma 1, we derive the following system of equations:

$$\boldsymbol{p}_l = (\boldsymbol{p}_0 + \sum_{m=1}^{\infty} \boldsymbol{k}_m) \sum_{k=1}^{l+1} \Psi_k \Omega^{(1)}_{l-k+1} +$$
$$+ \sum_{i=j_{r-1}+1}^{l+1} \boldsymbol{p}_i \Omega^{(r)}_{l-i+1} + \sum_{m=1}^{r-1} \sum_{i=j_{m-1}+1}^{j_m} \boldsymbol{p}_i \Omega^{(m)}_{l-i+1},\ l \geq 0,$$
$$\boldsymbol{k}_l = (\boldsymbol{p}_0 + \sum_{m=1}^{\infty} \boldsymbol{k}_m) \sum_{k=1}^{l} \Psi_k S^{(1)}_{l-k} + \sum_{i=j_{r-1}+1}^{l} \boldsymbol{p}_i S^{(r)}_{l-i} +$$
$$+ \sum_{m=1}^{r-1} \sum_{i=j_{m-1}+1}^{j_m} \boldsymbol{p}_i S^{(m)}_{l-i},$$
$$l > 0,\ j_{r-1} \leq l < j_r,\ r = \overline{1, n}.$$

Multiplying the equations by a corresponding degree of z and summing them up we obtain the theorem statement. Theorem 1 is proved.

Equation (3) involves n unknown functions $\boldsymbol{P}_r(z)$, $r = \overline{1, n}$, and vectors $\boldsymbol{\pi}_0$, $\boldsymbol{K}(1)$. Generating functions $\boldsymbol{P}_r(z)$, $r = \overline{1, n-1}$, can be determined by expanding the both sides of (3) in series at the point $z = 0$:

$$\boldsymbol{P}_r(z) = \boldsymbol{\pi}_0 Y_r(z) + \boldsymbol{K}(1) Q_r(z), \quad r = \overline{1, n-1}, \tag{5}$$

where

$$Y_r(z) = \sum_{i=j_{r-1}+1}^{j_r} Y_i z^i,\ Q(z) = \sum_{i=j_{r-1}+1}^{j_r} Q_i z^i,$$

and matrices Y_i, Q_i, $i = \overline{0, j_{n-1}}$, are calculated recurrently $Y_0 = I$, $Q_0 = -I$,

$$Y_{i+1} = \Bigg(Y_i - \sum_{k=1}^{i+1} \Psi_k \Omega^{(1)}_{i-k+1} - \sum_{k=j_{r-1}+1}^{i} Y_k \Omega^{(r)}_{i-k+1} -$$
$$- \sum_{m=1}^{r-1} \sum_{k=j_{m-1}+1}^{j_m} Y_k \Omega^{(m)}_{i-k+1} \Bigg) \left(\Omega^{(r)}_0\right)^{-1},$$
$$Q_{i+1} = \Bigg(Q_i - \sum_{m=1}^{r-1} \sum_{k=j_{m-1}+1}^{j_m} Q_k \Omega^{(m)}_{i-k+1} -$$
$$- \sum_{k=j_{r-1}+1}^{i} Q_k \Omega^{(r)}_{i-k+1} \Bigg) \left(\Omega^{(r)}_0\right)^{-1},$$
$$i = \overline{j_{r-1}, j_r - 1},\ r = \overline{1, n-1}.$$

Then using (3) and (5) we can derive the generating function $\boldsymbol{P}_n(z)$. So we have only two vectors $\boldsymbol{\pi}_0$ and $\boldsymbol{K}(1)$ unknown. To get the value $\boldsymbol{K}(1)$ from (4), the value $\boldsymbol{P}_n(1)$ needs to be calculated. Substitution $z = 1$ into equations (3)–(4) and their summation give the relation

$$\sum_{r=1}^{n} \boldsymbol{P}_r(1)(I - A_r) + \boldsymbol{K}(1)(I - A_1) = \boldsymbol{\pi}_0(\Psi(1) - I)A_1, \tag{6}$$

where $A_r(1) = \Omega^{(r)}(1) + S^{(r)}(1)$, $r = \overline{1, n}$. Note that matrices $A_r(1)$, $r = \overline{1, n}$, are stochastic, so the matrices $I - A_n + \mathbf{1}\boldsymbol{\rho}$, $r = \overline{1, n}$, are nonsingular, where $\boldsymbol{\rho}$ is row eigenvector of the matrix A_n corresponding to the eigenvalue 1.

Adding the expression

$$\left(\sum_{r=1}^{n} \boldsymbol{P}_r(1) + \boldsymbol{K}(1) \right) \mathbf{1}\boldsymbol{\rho} = \boldsymbol{\rho}$$

to the both sides of (6), we obtain

$$\boldsymbol{P}_n(1) = \boldsymbol{\rho} + \boldsymbol{\pi}_0(\Psi(1) - I)A_1 Z + \sum_{r=1}^{n-1} \boldsymbol{P}_r(1)(A_r - \tag{7}$$
$$-I - \mathbf{1}\boldsymbol{\rho})Z + \boldsymbol{K}(1)(A_1 - I - \mathbf{1}\boldsymbol{\rho})Z,$$

where $Z = (I - A_n + \mathbf{1}\boldsymbol{\rho})^{-1}$. Then using (4), (5) and (7), the dependence of the vector $\boldsymbol{K}(1)$ on $\boldsymbol{\pi}_0$ is derived

$$\boldsymbol{K}(1) = \boldsymbol{\rho} S^{(n)}(1) V + \boldsymbol{\pi}_0 T V, \tag{8}$$

where

$$T = (\Psi(1) - I)A^*_1 + \sum_{r=1}^{n-1} Y_r(1) A^*_r,$$
$$V = (I - A^*_1 - \sum_{r=1}^{n-1} Q_r A^*_r)^{-1},$$
$$A^*_r = S^{(r)}(1) + (A_r - I - \mathbf{1}\boldsymbol{\rho}) Z S^{(n)}(1),\ r = \overline{1, n-1}.$$

Finally, to calculate unknown vector $\boldsymbol{\pi}_0$ we exploit the functional equation (3) and the property of the equation $det(zI - \beta^{(n)}(z)) = 0$ to have exactly $H = (W + 1)(N + 1)M$ roots inside a unit disc $|z| < 1$ of a complex plane (Theorem 3 in (Gail et al. 1997)). Denote these roots as z_k with corresponding multiplicities l_k, $k = \overline{1, J}$, $\sum_{k=1}^{J} l_k = H$, where J is the number of different root. Using the analyticity property of the function $\boldsymbol{P}_n(z)$ inside the disc $|z| < 1$ (see e.g. (Dudin 1998)) we obtain the following system of equations for vector $\boldsymbol{\pi}_0$ entries:

$$\boldsymbol{\pi}_0 \frac{d^l}{dz^l}\Big\{\Big[(\Psi(z)-I+TV)\Omega^{(1)}(z)+\sum_{r=1}^{n-1}(Y_r(z)+ +TVQ_r(z))(\Omega^{(r)}(z)-zI)\Big]\times \times adj(zI-\Omega^{(n)}(z))\Big\}\Big|_{z=z_k} =$$

$$= -\boldsymbol{\rho} S^{(n)} V\frac{d^l}{dz^l}\Big\{\Big[\Omega^{(1)}(z)+\sum_{r=1}^{n-1}Q_r(z)(\Omega^{(r)}(z)-zI)\Big]\times \times adj(zI-\Omega^{(n)}(z))\Big\}\Big|_{z=z_k},$$

$$l=\overline{0,l_k-1},\ k=\overline{1,J}, \sum_{k=1}^{J} l_k = H.$$

Having known the value of vector $\boldsymbol{\pi}_0$, we calculate the value of $\boldsymbol{K}(1)$. Then substituting the values of these vectors into (3), (4) and (5) we get the final expressions for the generating functions $\boldsymbol{P}_r(z)$, $r=\overline{1,n}$, and $\boldsymbol{K}(z)$.

PERFORMANCE CHARACTERISTICS AND THE COST CRITERION VALUE

As some customers can leave the system unserved, the important performance characteristic is the probability of an arbitrary customer successful service. Denote this probability as P_+. Using the ergodic theorems for functionals defined on the Markov chains (Skorohod 1980) the formula for P_+ can be obtained

$$P_+ = \frac{\sum_{r=1}^{n} \boldsymbol{P}_r(1)\mathbf{1}}{\sum_{r=1}^{n} \boldsymbol{P}_r(1)\mathbf{1} + \boldsymbol{K}'(1)\mathbf{1}}.$$

and the performance characteristics involved in cost criterion (1) can be determined

$$L = \sum_{r=1}^{n} \boldsymbol{P}'_r(1)\mathbf{1},$$

$$\Lambda = \frac{\lambda^{(1)}}{\sum_{r=1}^{n} \boldsymbol{P}_r(1)\mathbf{1} + \boldsymbol{K}'(1) + \sum_{r=2}^{n}(\lambda^{(1)}-\lambda^{(r)})\boldsymbol{P}_r(1)B_r\mathbf{1}},$$

$$\Phi_r = \Lambda \boldsymbol{P}_r(1)B_r\mathbf{1},\ r=\overline{2,n}, \quad \Phi_1 = 1-\sum_{r=2}^{n}\Phi_r,$$

$$R = \sum_{r=1}^{n}\lambda^{(r)}\Phi_r(1-P_+).$$

where $B_r = \int_0^\infty e^{D^{(r)}(1)t}\otimes e^{F_0^{(r)}t}t \otimes dB^{(r)}(t) + \int_0^\infty e^{D^{(r)}(1)t}\otimes\big(te^{F_0^{(r)}t}F_1^{(r)}\big)\otimes(P^{(r)}-B^{(r)}(t))\,dt$, $r=\overline{2,n}$.

Having calculated performance characteristics we get the value of the cost criterion under the fixed set of thresholds $(j_1,j_2,\dots,j_{n-1})$. Having the algorithm for calculation of the cost criterion value for any fixed set of thresholds we can find the optimal value of thresholds $(j_1^*,j_2^*,\dots,j_{n-1}^*)$ minimizing the cost criterion on the bounded region $A_J = \{(j_1,j_2,\dots,j_{n-1}) : 0 \le j_1 \le j_2 \le \dots \le j_{n-1} \le J\}$.

NUMERICAL EXAMPLE

To illustrate the obtained results we present bellow simple numerical example. Consider the queue with four modes of operation, $a=0.5$, $c_1=2$, $c_2=100$, $c_3=20$, $c_4=35$, $d=25$. BMAP-input of customers is given by the matrices

$$D_0^{(1)}=\begin{pmatrix}-5.52 & 0.52\\ 0.35 & -0.35\end{pmatrix}, D_1^{(1)}=\begin{pmatrix}4 & 0\\ 0 & 4.8\end{pmatrix},$$
$$D_2^{(1)}=\begin{pmatrix}1 & 0\\ 0 & 1.2\end{pmatrix}, D_0^{(2)}=\begin{pmatrix}-4.36 & 0.36\\ 0.21 & -3.2\end{pmatrix},$$
$$D_1^{(2)}=\begin{pmatrix}2.8 & 0\\ 0 & 2.1\end{pmatrix}, D_2^{(2)}=\begin{pmatrix}1.2 & 0\\ 0 & 0.09\end{pmatrix}.$$

$$D_0^{(3)}=\begin{pmatrix}-3.36 & 0.36\\ 0.21 & -2.21\end{pmatrix}, D_2^{(3)}=\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix},$$
$$D_0^{(4)}=\begin{pmatrix}-1.25 & 0.25\\ 0.04 & -2.04\end{pmatrix}, D_1^{(4)}=D_2^{(4)}=\begin{pmatrix}0.5 & 0\\ 0 & 1\end{pmatrix}.$$

The semi-Markovian kernel characterizing the service process in the rth mode has the form

$$B^{(r)}(t)=\begin{pmatrix}0.6B_1^{(1)}(t) & 0.4B_2^{(1)}(t)\\ 0.35B_1^{(2)}(t) & 0.65B_2^{(2)}(t)\end{pmatrix},$$

where $B_i^{(r)}(t)=\int_0^t \frac{b_i^{(r)}(b_i^{(r)}\tau)^{k_i^{(r)}-1}}{(k_i^{(r)}-1)!}e^{-b_i^{(r)}\tau}d\tau$, $b_1^{(1)}=18$, $b_1^{(2)}=b_2^{(2)}=20$, $b_2^{(1)}=6$, $b_1^{(3)}=24$, $b_2^{(3)}=30$, $b_1^{(4)}=10$, $b_2^{(4)}=15$, $k_1^{(1)}=k_2^{(2)}=k_1^{(4)}=2$, $k_2^{(1)}=k_2^{(4)}=3$, $k_1^{(2)}=1$, $k_1^{(3)}=4$, $k_2^{(3)}=5$.

MAP-input of disasters is given by the matrices

$$F_0^{(1)}=\begin{pmatrix}-0.61 & 0.21\\ 0.32 & -0.68\end{pmatrix}, F_1^{(1)}=\begin{pmatrix}0.4 & 0\\ 0 & 0.36\end{pmatrix},$$
$$F_0^{(2)}=\begin{pmatrix}-0.26 & 0.16\\ 0.27 & -0.35\end{pmatrix}, F_1^{(2)}=\begin{pmatrix}0.1 & 0\\ 0 & 0.08\end{pmatrix},$$

$$F_0^{(3)}=\begin{pmatrix}-0.26 & 0.21\\ 0.32 & -0.4\end{pmatrix}, F_1^{(3)}=\begin{pmatrix}0.05 & 0\\ 0 & 0.08\end{pmatrix},$$
$$F_0^{(4)}=\begin{pmatrix}-0.25 & 0.21\\ 0.32 & -0.33\end{pmatrix}, F_1^{(4)}=\begin{pmatrix}0.04 & 0\\ 0 & 0.01\end{pmatrix}.$$

Table contains the values of customer input intensity $\lambda^{(r)}$, disaster flow intensity $\varphi^{(r)}$ and mean service time $b^{(r)}$ when rth mode is used, $r=\overline{1,n}$.

Table: Characteristics of operation modes

number of mode	$\lambda^{(r)}$	$\varphi^{(r)}$	$b^{(r)}$
1	6.71	0.384	0.131
2	4.37	0.092	0.184
3	4.73	0.062	0.166
4	2.79	0.028	0.20

Denote by C_r the cost criterion value when the rth operation mode is used only, $r = \overline{1,4}$. For our system $C_1 = 39.09$, $C_2 = 116.14$, $C_3 = 37.39$, $C_4 = 38.67$. The values of C_r, $r = \overline{1,4}$, are calculated by the formulas from (Dudin and Nishimura 1999).

When all operation modes are in use the optimal criterion value is $C^* = 27.780$ and optimal thresholds are $j_1^* = j_2^* = 1$, $j_3^* = 3$. Note that the second mode is not used under optimal system operation. Comparing the value C^* and $min\{C_1, C_2, C_3, C_4\}$ we can conclude that optimal control allows to reduce the system operation cost more than 27%.

The algorithm was realized as the module of software "SIRIUS+" developed in Laboratory of Applied Probabilistic Analysis of Belarus State University, see (Dudin et al. 2000, Dudin et al. 2004).

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QUEUES WITH DISRUPTIVE AND NON-DISRUPTIVE VACATIONS

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ABSTRACT

We consider a queueing system with disruptive and non-disruptive vacations. Both disruptive and non-disruptive vacations may interrupt service of customers. The customer repeats its service after a disruptive vacation and continues its service after a non-disruptive vacation. Using a transform approach, we obtain various performance measures such as the moments of the queue content and waiting times. We illustrate our approach by means of some numerical examples.

INTRODUCTION

Consider an (abstract) production system that is subject to failures. Failures occur during production or when the system is idle and may either lead to the complete loss of the item that is in production, or to a temporary production halt. In the former case, the production of the "interrupted" item has to start all over. In the latter case, production may continue after the failure is fixed.

In queueing theory parlance, periods of temporary service unavailability are referred to as server vacations or server interruptions. These queueing models have proved to be a useful abstraction in situations where a service facility is shared by multiple queues or where the facility is subject to failure. Systems where several queues share a single facility include a.o. priority queueing systems (Fiems, 2004; Walraevens , 2002) and polling systems (Takagi, 1986). From the vantage point of a single queue, the facility leaves for a vacation whenever it attends the other queues. When the facility is subject to failures, the vacations correspond to the time one needs to fix the failure or the time one spends on preventive maintenance (Lee et al., 2003).

According to Ibe and Trivedi (1990), White and Christie (1958) were the first to study queues with service interruptions. They consider an $M/M/1$ queueing system with exponentially distributed vacations. Generally distributed service times and vacations are considered by Avi-Itzhak and Naor (1963) and also by Thiruvengadam (1963). The former authors assume that the server remains with the queue for an exponentially distributed amount of time. Federgruen and Green (1986) then relax this assumption to a phase-type distributed amount of time. Van Dijk (1988) provides an approximate analysis of a system with exponentially distributed service times but with generally distributed on- and off-periods whereas Takine and Sengupta (1997) and Masuyama and Takine (2003) study queueing systems with service interruptions in a Markov-modulated environment. These authors also allow correlation in the arrival process. A processor sharing queueing system with exponentially distributed on-periods and generally distributed off-periods is studied by Núñez Queija (2000). All these contributions assume that customers resume service after the interruption. Gaver (1962) also considers the cases where service is repeated or repeated and resampled after the interruption. The latter operation mode is also studied by Ibe and Trivedi (1990) for a two station polling system.

In the present contribution we consider in particular $M/G/1$ queueing systems with combined disruptive and non-disruptive renewal-type interruptions. We consider two variants of the system. The system with resampling assumes that the service time after a disruptive vacation is resampled. For the system without resampling, the service time after a disruptive vacation equals the original service time. In terms of the abstract production system introduced earlier, the system with resampling may capture the uncertainty in the production process whereas the system without resampling allows to consider a system that produces different types of products with unequal production time requirements.

The remainder of this contribution is organised as follows. In the following two sections, the queueing model is described in detail and analysed respectively. We then illustrate our approach by means of some numerical examples and draw conclusions.

QUEUEING MODEL

We consider an infinite capacity queueing system that adheres a first-in-first-out service discipline. Customers arrive in accordance with a Poisson process with arrival rate λ and their service times constitute a series of independent and identically distributed (i.i.d.) random variables with common density function $s(t)$, $t \geq 0$ and corresponding Laplace-Stieltjes transform (LST) $S(\zeta)$, $\zeta > -R_S$. Here $-R_S$ is the boundary of the region of convergence of the LST.

The customers are served by a single server that leaves for a vacation from time to time. Vacations arrive in accordance with a Poisson process with rate ν while the server is serving customers and with rate ν_i when the server is idle. No new vacations arrive when the server is on vacation. When a vacation interrupts a customer's service, the vacation is either disruptive (with probability p_d) or non-disruptive (with probability $p_n = 1 - p_d$). The server will continue the interrupted service time if the vacation is non-disruptive and will repeat the entire service time if the vacation is disruptive. For further use, we also introduce the arrival rates $\nu_d = \nu p_d$ and $\nu_n = \nu p_n$ of the disruptive and the non-disruptive vacations respectively. The lengths of the consecutive disruptive (non-disruptive, idle time) vacations constitute a series of i.i.d. positive random variables with density function $v_d(t)$ ($v_n(t)$, $v_i(t)$), $t \geq 0$ and corresponding LST $V_d(\zeta)$ ($V_n(\zeta)$, $V_i(\zeta)$) .

Two different queueing models are considered. The model with resampling assumes that the service time after a disruptive vacation is resampled. For the model without resampling, the service time after a disruptive interruption equals the original service time.

ANALYSIS

To simplify the analysis, we first consider the effective service times for the models with and without resampling. This then allows for a unified queueing analysis afterwards.

Effective service time

Let a customer's effective service time be defined as the amount of time between the epoch where the server starts serving this customer for the first time and the epoch where this customer leaves the system. As such, the effective service time includes all service vacations during the customer's service time as well as time that is lost due to service repetitions.

With resampling

We first focus on the model with resampling. Let $T(x)$ denote the effective service time of a customer with service time x and let S denote the service time of a random customer. Further, let Λ denote an independent exponentially distributed random variable with mean $1/\nu$ and let Q denote an independent random variable that takes the values d and n with probability p_d and p_n respectively. The variable Λ corresponds to the time between the start of the customer's service and the following vacation and the variable Q corresponds to the type of this vacation. We then find the following expression for the effective service time,

$$T(S) = \begin{cases} S & \text{for } \Lambda \geq S, \\ \Lambda + V + T^*(S^*) & \text{for } \Lambda < S \text{ and } Q = d, \\ \Lambda + V + T(S - \Lambda) & \text{for } \Lambda < S \text{ and } Q = n. \end{cases}$$

Here $T^*(\cdot)$ and S^* are independent random variables that are distributed as $T(\cdot)$ and S respectively.

Let $T(\zeta|x)$ denote the LST of the effective service time, given that the service time equals x (before being resampled, if this is the case) and let $T(\zeta)$ denote the LST of the effective service time of a random customer. In view of the former expressions and by conditioning on Λ, we find that $T(\zeta|x)$ obeys following integral equation,

$$\begin{aligned} T(\zeta|x) &= e^{-(\zeta+\nu)x} + \nu_d V_d(\zeta) T(\zeta) \frac{1 - e^{-(\zeta+\nu)x}}{\zeta + \nu} \\ &\quad + \nu_n V_n(\zeta) \int_0^x e^{-(\zeta+\nu)y} T(\zeta|x - y) dy \\ &= e^{-(\zeta+\nu)x} + \nu_d V_d(\zeta) T(\zeta) \frac{1 - e^{-(\zeta+\nu)x}}{\zeta + \nu} \\ &\quad + \nu_n V_n(\zeta) e^{-(\zeta+\nu)x} \int_0^x e^{(\zeta+\nu)y} T(\zeta|y) dy \, , \end{aligned}$$

for $x \geq 0$. One then easily shows that a solution of the former equation has the form,

$$T(\zeta|x) = \frac{\nu_d V_d(\zeta) T(\zeta)}{\zeta + \nu - \nu_n V_n(\zeta)} + C(\zeta) e^{-(\zeta+\nu-\nu_n V_n(\zeta))x} \, .$$

where $C(\zeta)$ is an unknown function. Since the effective service time of a zero length service time equals zero, we have $T(\zeta|0) = 1$. Therefore, plugging $x = 0$ in the former equation and solving for $C(\zeta)$ yields,

$$C(\zeta) = 1 - \frac{\nu_d V_d(\zeta) T(\zeta)}{\zeta + \nu - \nu_n V_n(\zeta)} \, .$$

Combining the former equations and integrating over all possible service times with respect to the service time distribution then yields the following expression for the LST of the effective service times,

$$T(\zeta) = \frac{\nu_d V_d(\zeta) T(\zeta)}{\chi(\zeta)} + \left(1 - \frac{\nu_d V_d(\zeta) T(\zeta)}{\chi(\zeta)}\right) S(\chi(\zeta)) \, .$$

Here we introduced $\chi(\zeta) = \zeta + \nu - \nu_n V_n(\zeta)$ to simplify notation. Solving for $T(\zeta)$ finally leads to,

$$T(\zeta) = \frac{S(\chi(\zeta))\chi(\zeta)}{\chi(\zeta) - \nu_d V_d(\zeta)(1 - S(\chi(\zeta))} \, . \qquad (1)$$

By means of the moment generating property of LSTs, we may obtain expressions for the various moments of the effective service times. In particular, the first moment is given by,

$$\mathrm{E}[T] = \frac{1 - S(\nu_d)}{\nu_d S(\nu_d)} (1 + \nu_d \, \mathrm{E}[V_d] + \nu_n \, \mathrm{E}[V_n]) \, , \qquad (2)$$

where $\mathrm{E}[V_d]$ and $\mathrm{E}[V_n]$ denote the mean length of a disruptive and non-disruptive vacation respectively. Similar expressions can be obtained for higher moments. In particular, the kth moment can be expressed in terms of the LST $S(\zeta)$ and its derivatives, evaluated in $j\nu_d$ ($j = 1 \dots k$) and the moments up to order k of the vacations. Notice that for $\nu_d > 0$ existence of the moments of the effective service time does not require the existence of the moments of the service time.

Without resampling

For the model without resampling, we may proceed analogously. It is however more convenient to rely on the results of the model with resampling.

As before, let $T(\zeta|x)$ denote the effective service time of a customer given that the customer's service time equals x. It is then easy to see that $T(\zeta|x)$ also equals the LST for the model with resampling under the assumption that the service times are fixed and equal to x. This notion immediately leads to the following expression for the conditional LST of the effective service times,

$$T(\zeta|x) = \frac{e^{-\chi(\zeta)x}\chi(\zeta)}{\chi(\zeta) - \nu_d V_d(\zeta)(1 - e^{-\chi(\zeta)x})} .$$

Integrating over all possible service times with respect to the service time distribution, then yields an expression for the LST of the effective service times for the model without resampling,

$$T(\zeta) = \int_0^\infty \frac{s(x)e^{-\chi(\zeta)x}\chi(\zeta)}{\chi(\zeta) - \nu_d V_d(\zeta)(1 - e^{-\chi(\zeta)x})} dx . \quad (3)$$

The moment generating property of LSTs, then allows us to obtain expressions for the various moments of the effective service times. In particular, the first moment is given by,

$$\mathrm{E}[T] = \frac{S(-\nu_d) - 1}{\nu_d}(1 + \nu_d \mathrm{E}[V_d] + \nu_n \mathrm{E}[V_n]) , \quad (4)$$

for $\nu_d < R_S$. Higher moments can also be expressed in terms of the LST of the service times. In particular, the kth moment can be expressed in terms of the LST $S(\zeta)$ and its derivatives, evaluated in $-j\nu_d$ ($j = 1 \dots k$) and the moments up to order k of the vacations. Therefore, the kth moment exist whenever the kth moments of the vacations exist and whenever,

$$k\nu_d < R_S . \quad (5)$$

The existence of the kth moments for $k\nu_d = R_S$ depends on the behaviour of the LST $S(\zeta)$ and its derivatives for $\zeta = -R_S$.

Queueing analysis

The former expressions for the LSTs of the effective service times now enables us to present a unified queueing analysis of both models.

Queue content

Let U_k denote the number of packets in the queue upon departure of the k the customer. We then find,

$$U_{k+1} = \begin{cases} U_k + \Omega_{k+1} - 1 & \text{for } U_k > 0, \\ \Gamma_{k+1} + \Omega_{k+1} & \text{for } U_k = 0. \end{cases}$$

Here Ω_k denotes the number of customer arrivals during the kth customer's effective service time and Γ_k denotes the number of arrivals during the "remaining vacation time" upon arrival of the kth customer. The remaining vacation time of the kth customer is defined as the time between the arrival instant and the end of the vacation if the server is on vacation when the customer arrives or equals 0 if the server is available. Let $U_k(z)$ denote the probability generating function of U_k. In view of the former expression, we find,

$$\begin{aligned} U_{k+1}(z) = & \frac{1}{z}(U_k(z) - U_k(0))T(\lambda(1 - z)) \\ & + U_k(0)\Theta(\lambda(1 - z))T(\lambda(1 - z)) , \end{aligned}$$

where $\Theta(s)$ is the LST of the remaining vacation time upon arrival of a random customer that arrives in an empty system and where $T(\zeta)$ is given by (1) or (3) depending on the model under consideration.

One can show that the queueing system under consideration reaches steady state whenever $\lambda \mathrm{E}[T] < 1$ (where $\mathrm{E}[T]$ is given by (1) or (3) depending on the model under consideration). Let $U(z) = \lim_{k \to \infty} U_k(z)$ denote the probability generating function of the queue content at departure epochs in steady state. From the former expression we obtain,

$$U(z) = U(0)\frac{\Theta(\lambda(1 - z))z - 1}{z - T(\lambda(1 - z))}T(\lambda(1 - z)) . \quad (6)$$

The normalisation condition $U(1) = 1$ then yields the following expression for the remaining unknown $U(0)$,

$$U(0) = \frac{1 - \lambda \mathrm{E}[T]}{1 + \lambda \mathrm{E}[\Theta]} , \quad (7)$$

with $\mathrm{E}[\Theta]$ the average remaining vacation time.

We now determine the LST $\Theta(\zeta)$. Let Γ_A and Γ_V denote exponentially distributed random variables with mean $1/\nu_i$ and $1/\lambda$ respectively. The former variable corresponds to the time that the server leaves for a vacation after becoming idle. The latter random variable corresponds to the arrival time of the first arrival after the epoch where the server becomes idle. We may now decompose the remaining vacation time as follows,

$$\Theta = \begin{cases} 0 & \text{for } \Gamma_A < \Gamma_V , \\ \Gamma_V + V - \Gamma_A & \text{for } \Gamma_V \le \Gamma_A < \Gamma_V + V , \\ \Theta^* & \text{for } \Gamma_A > \Gamma_V + V . \end{cases}$$

Here Θ^* is an independent copy of Θ. In view of the former expressions, the LST of Θ satisfies,

$$\Theta(\zeta) = \frac{\lambda}{\lambda + \nu_i} + \nu_i \lambda \frac{V_i(\zeta) - V_i(\lambda)}{(\lambda + \nu_i)(\lambda - \zeta)} + \Theta(\zeta) \frac{\nu_i V_i(\lambda)}{\lambda + \nu_i} .$$

Solving for $\Theta(\zeta)$ leads to,

$$\Theta(\zeta) = \frac{\lambda(\lambda - \zeta) + \nu_i \lambda (V_i(\zeta) - V_i(\lambda))}{(\lambda + \nu_i - \nu_i V_i(\lambda))(\lambda - \zeta)} . \quad (8)$$

The moment-generating property of LSTs further yields,

$$\mathrm{E}[\Theta] = \frac{\nu_i}{\lambda} \frac{\lambda \, \mathrm{E}[V_i] - (1 - V_i(\lambda))}{\lambda + \nu_i (1 - V_i(\lambda))} , \quad (9)$$

where $\mathrm{E}[V_i]$ denotes the mean length of a vacation that starts when the server is idle.

Finally, combining expressions (6), (7), (8) and (9) then yields,

$$U(z) = \frac{1 - \lambda \, \mathrm{E}[T]}{\lambda(1 + \nu_i \, \mathrm{E}[V_i])} T(\lambda(1-z)) \times \frac{\lambda(1-z) + \nu_i(1 - V_i(\lambda(1-z)))}{T(\lambda(1-z)) - z} .$$

By the Burke-Takács theorem (see e.g. (Cooper, 1981, pp. 187)), $U(z)$ is also the probability generating function of the number of customers in the queue upon arrival of a customer. By PASTA – Poisson arrivals see time averages – $U(z)$ is also the probability generating function of the queue content at a random point in time.

The moment generating property of probability generating functions then yields expressions for the various moments of the queue content at random points in time. In particular, the mean queue content is given by,

$$\mathrm{E}[U] = \lambda \, \mathrm{E}[T] + \frac{1/2 \, \lambda^2 \, \mathrm{E}[T^2]}{1 - \lambda \, \mathrm{E}[T]} + \frac{1/2 \, \nu_i \lambda \, \mathrm{E}[V_i^2]}{1 + \nu_i \, \mathrm{E}[V_i]} . \quad (10)$$

Delay

Let customer delay be defined as the time between a customer's arrival and departure instant and let $D(\zeta)$ denote the LST of a random customer's delay. Since the queue content upon departure of a customer equals the number of arrivals that arrived during this customer's delay, we find,

$$U(z) = D(\lambda(1-z)) ,$$

or equivalently,

$$\begin{aligned} D(\zeta) &= U\left(1 - \frac{\zeta}{\lambda}\right) \\ &= \frac{1 - \lambda \, \mathrm{E}[T]}{1 + \nu_i \, \mathrm{E}[V_i]} \frac{\zeta + \nu_i(1 - V_i(\zeta))}{\zeta - \lambda(1 - T(\zeta))} T(\zeta). \end{aligned}$$

The moment generating property of LSTs then yields the various moments of the customer delay. In particular, the mean customer delay is given by,

$$\mathrm{E}[D] = \mathrm{E}[T] + \frac{\lambda \, \mathrm{E}[T^2]}{2(1 - \lambda \, \mathrm{E}[T])} + \frac{\nu_i \, \mathrm{E}[V_i^2]}{2(1 + \nu_i \, \mathrm{E}[V_i])} .$$

Notice that the latter expression also follows from equation (10) and Little's result.

NUMERICAL EXAMPLES

We now illustrate our approach by means of some numerical examples. Consider in particular the case where customer service times are Gamma distributed and where the different vacations are exponentially distributed. Recall that the gamma distribution is completely specified by the mean μ and standard deviation σ and has the following LST,

$$F(\zeta) = \left(1 + \zeta \frac{\sigma^2}{\mu}\right)^{-\frac{\mu^2}{\sigma^2}} .$$

for $\zeta > -\mu/\sigma^2$. Also notice that the Gamma distribution reduces to an exponential distribution for $\mu = \sigma$.

For further use we introduce the fraction θ that the server is available while there are customers in service and the mean length of the sum of an available and vacation period κ,

$$\begin{aligned} \theta &= \frac{1/\nu}{1/\nu + p_n \, \mathrm{E}[V_n] + p_d \, \mathrm{E}[V_d]} , \\ \kappa &= \frac{1}{\nu} + p_n \, \mathrm{E}[V_n] + p_d \, \mathrm{E}[V_d] . \end{aligned}$$

The latter is a measure for the absolute lengths of the available periods and vacations and is therefore referred to as the time scale parameter of the interruption process (while customers receive service).

In Figures 1 and 2 the mean and standard deviation of the effective service times are depicted versus the time scale parameter κ of the interruption process. Non-disruptive and disruptive vacations share the same (exponential) distribution and the server is available during $\theta = 80\%$ of the time. The mean and standard deviation of the customer service times are equal to $E[S] = 20$ and $\sigma[S] = 40$ respectively. Different values of the probability p_n that a vacation is non-disruptive are assumed as depicted and we consider both the systems with (lower curves) and without resampling (upper curves).

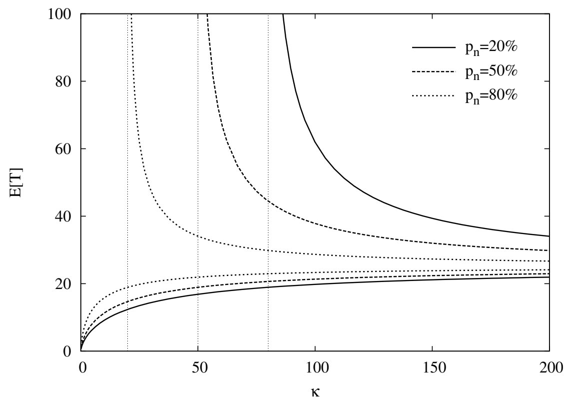

Figure 1: Mean Effective Service Time $\mathrm{E}[T]$ vs. the Time Scale Parameter κ

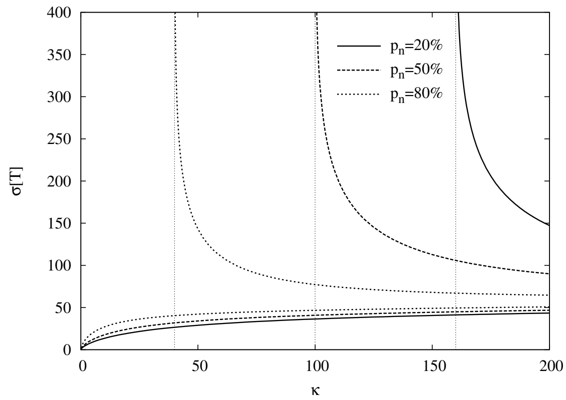

Figure 2: Standard Deviation $\sigma[T]$ of the Effective Service Time vs. the Time Scale Parameter κ

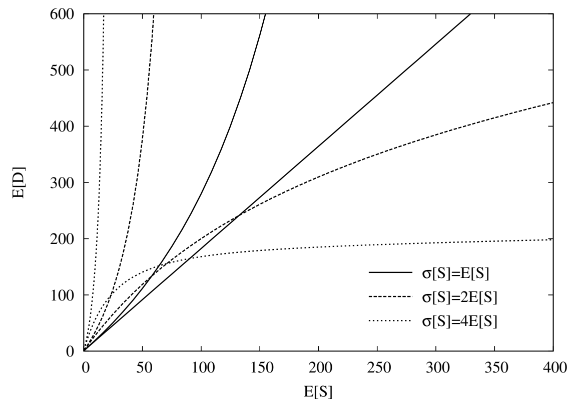

Figure 3: Mean Customer Delay $\mathrm{E}[D]$ vs. the Mean Customer Service Time $\mathrm{E}[S]$ for Fixed Load

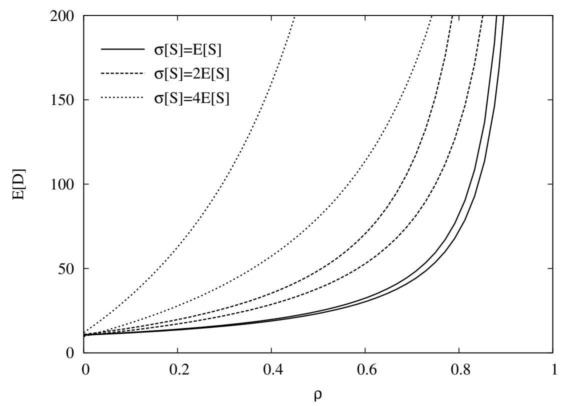

Figure 4: Mean Customer Delay $\mathrm{E}[D]$ vs. Load ρ for Fixed Mean Customer Service Time

For increasing values of κ, one observes that all curves converge. Large κ implies few but long vacations. Whenever there is such an interruption, neither resampling nor the disruptiveness of the interruption comes in to play compared to the length of the vacation. For decreasing values of κ, performance of the systems with and without resampling is completely different. For the system with resampling, all curves converge to 0. This comes from the fact that low κ implies many short interruptions such that service times get resampled until they are very small. For the system without resampling, the service times remain the same after an interruption. Since decreasing κ implies that sufficiently long available periods become less probable, the mean and standard deviation of the effective service times converge to ∞. Finally notice that the vertical asymptotes for the system without resampling come from the condition (5) for the existence of the moments.

In figures 3 and 4 the mean customer delay is depicted versus the mean customer service time for fixed load and versus the load for fixed mean customer service time respectively. Disruptive and non-disruptive vacations share the same (exponential) distribution, the server is available during $\theta = 95\%$ of the time while serving customers and the time scale parameter equals $\kappa = 400$. Half of the vacations during service are disruptive ($p_d = 50\%$) and the server does not leave for vacations during its idle periods ($\nu_i = 0$). In figure 3 the arrival load equals $\rho = \lambda\,\mathrm{E}[S] = 40\%$ whereas the mean service time equals $\mathrm{E}[S] = 10$ in figure 4. Finally, different values for the standard deviation of the customer service times are considered as depicted. For each value of the standard deviation the upper and lower curves correspond to the system without and with resampling respectively.

In figure 3 one observes that performance deteriorates whenever one increases the mean service time. This comes from the fact that longer service times imply higher probabilities that service is interrupted by a disruptive vacation. For the system with resampling, the loss of service time due to the disruptive vacations is somewhat mitigated by the fact that the new service time might be shorter than the original one. This is not the case for the system without resampling where one notes a severe performance deterioration whenever one increases the average service time. Finally, one observes that higher val-

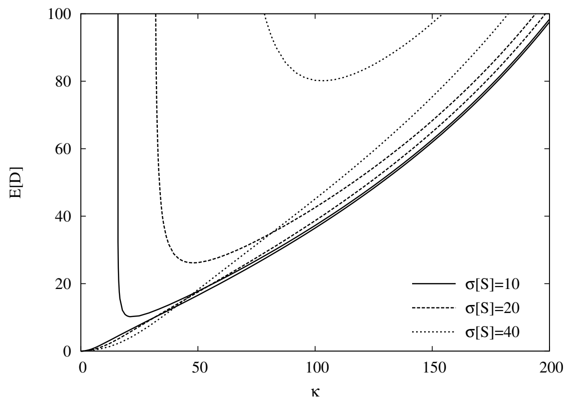

Figure 5: Mean customer delay $\mathrm{E}[D]$ vs. the Time Scale Parameter κ

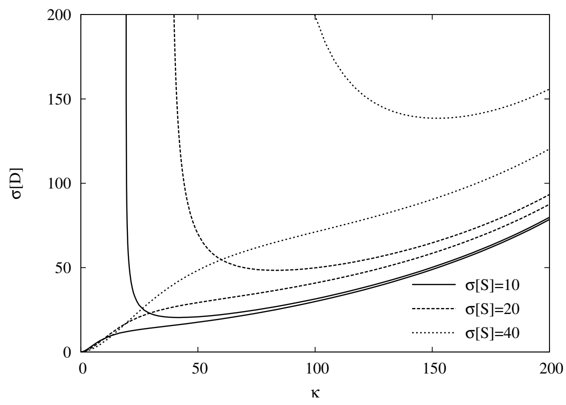

Figure 6: Standard Deviation $\sigma[D]$ of the Customer Delay vs. the Time Scale Parameter κ

ues of the standard deviation of the service times lead to worse performance for the system without resampling and for better performance for the system with resampling. This can be explained by noting that customers with long service times have a severe impact on the performance of the system without resampling since they need to wait for a sufficiently long period during which the server does not leave for a disruptive vacation. Customers with longer service times do not have this impact for the system with resampling since their service times get resampled. In figure 4, the mean and standard deviation of the service times are fixed and therefore the moments of the effective service times are fixed as well. Increasing the load leads to an exponential increase of the mean customer delay which is typical queueing behaviour.

Figures 5 and 6 depict the mean value and the standard deviation of the customer delay versus the time scale parameter κ of the vacation process. Disruptive and non-disruptive vacations and vacations that start during idle times share a common (exponential) distribution and the arrival rate of the vacations does not depend on whether a customer receives service or not ($\nu_i = \nu$). The server is available during 80% of the time and 20% of the vacations that interrupt the service of a customer are disruptive. Further, the arrival load equals $\rho = 20\%$ and the mean service time equals $\mathrm{E}[S] = 20$. Different values of the standard deviation $\sigma[S]$ of the service times are assumed as depicted. Again the upper and lower curves correspond to the system without and with resampling respectively for each value of the standard deviation $\sigma[S]$.

For the system with and without resampling one sees that mean and standard deviation converge to 0 and ∞ respectively for decreasing values of the time scale parameter κ. This comes from the fact that the effective service times tend to 0 and ∞ (see figures 1 and 2) when the time scale parameter decreases. Also, higher values of the standard deviation of the service times lead to a performance improvement for the system with resampling and for a performance degradation for the system without resampling as we already observed in figures 3 and 4. Finally, for increasing values of the time scale parameter, performance degrades and the influence of resampling disappears. Customer waiting times now mainly follow from the longer vacations during which the queue builds up.

CONCLUSIONS

We considered a queueing system with disruptive and non-disruptive vacations. After a disruptive vacation, service of the interrupted customer is either repeated or repeated and resampled. By means of an LST approach, we obtained performance measures such as the moments of queue content and customer delay. We finally illustrated our approach by means of some numerical examples.

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SESSION 2

PERFORMANCE MODELS

COMPATIBILITY OF MULTICAST AND SPATIAL TRAFFIC DISTRIBUTION FOR MODELING MULTICORE NETWORKS

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KEYWORDS
stochastic modeling, network-on-chip, multicore processor, multicast distribution, spatial traffic distribution

ABSTRACT
Processors with multiple cores came to the center of interest in recent years. These cores are connected on-chip by an appropriate network. To investigate how a multicore processor behaves dependent on the chosen network-on-chip topology, a corresponding model must be established for performance evaluation. Instead of modeling the entire system including the cores, it is more reasonable to reduce the model complexity and to simply stochastically model the detached network. Then, traffic generators at the network inputs must provide reasonable multicore processor traffic which usually consists of multicasts and a particular spatial distribution. Because the traffic is not exactly known, both, multicasts and spatial traffic, are described as stochastic distributions for model input. But not all multicast distributions can be achieved with a particular desired spatial distribution and vice versa. Thus, it is important to check for the compatibility of the spatial distribution and the multicasts that the modeler is willing to investigate. Such a compatibility check is provided by the algorithm presented in this paper. It prevents from inconsistent traffic parameters while modeling.

1 INTRODUCTION

The ongoing improvement in VLSI technology leads to a further increasing number of devices per chip. Since this increased density cannot longer be used to improve the performance of uniprocessor chips as in previous years, multicore processors come to the center of interest [1].

To allow cooperating cores on such a multicore processor, an appropriate communication structure between them must be provided. In case of a low number of cores (e.g. a dual core processor), a shared bus may be sufficient. But in the future, hundreds or even thousands of cores will collaborate on a single chip. Then, more advanced network topologies will be needed. Many topologies are proposed for these so called networks-on-chips (NoCs) [2, 3, 4, 5, 6]. For instance, this paper will use meshes and multistage interconnection networks (MINs) as examples. But most other topologies are also covered by the paper.

To map the communication demands of the cores onto predefined topologies like meshes, MINs, and other topologies, Bertozzi et al. [6] invented a tool called NetChip (consisting of SUNMAP [7] and xpipes [8]). This tool provides complete synthesis flows for NoC architectures.

Another example where MINs deal as NoC is given by Guerrier and Greiner [4] who established a fat tree structure using Field Programmable Gate Arrays (FPGAs). They called this on-chip network with particular router design and communication protocol Scalable, Programmable, Integrated Network (SPIN). Its performance for different network buffer sizes was compared.

Alderighi et al. [5] used MINs with the Clos structure. Multiple parallel Clos networks connect the inputs and outputs to achieve fault tolerance abilities. Again, FPGAs serve as basis for realization.

But previous papers only considered unicast traffic in the NoC. It is obvious that multicore processors also have to deal with multicast traffic. For instance, if a core changes a shared variable that is also stored in the cache of other cores, multicasting the new value to the other cores keeps them up to date. Thus, multicast traffic builds a non-negligible part of the traffic.

Further on, it is very likely that traffic in multicore processors will reveal some locality in its spatial distribution. Usually, an application will be distributed to some of the cores. But due to many available cores, more than a single application can be processed in parallel. Then, there will be much more communication between cores that process the same application than between cores of different applications. Thus, cores for the same application are chosen such that they are close together to achieve low communication latency. In consequence, local traffic dominates.

As a result, networks for multicore systems should outstandingly support multicast traffic and local traffic. Investigating whether networks are suitable for multicore processors is usually performed by stochastically modeling them. Therefore, offering multicast traffic and localities specifies an important feature of the modeling technique.

Because the traffic is not exactly known, both, multicasts and spatial traffic, are described as stochastic distributions for model input. But not all multicast distributions can be achieved with a particular desired spatial distribution and vice versa. Thus, it is important to check for the compatibility of the spatial distribution and the multicasts that the modeler is willing to investigate. Such a compatibility check is provided by the algorithm presented in this paper. It prevents from inconsistent traffic parameters while modeling.

The paper is organized as follows. Section 2 introduces the architectures of networks-on-chips, particularly multistage interconnection networks and meshes. Multicast traffic, localities, and their modeling is described in Section 3. Based on this, Section 4 presents a new algorithm to check for the compatibility of the desired spatial distribution and the multicasts. Section 5 summarizes and gives conclusions.

2 NETWORK-ON-CHIP ARCHITECTURES

This section gives two examples for network-on-chip architectures. First, bidirectional multistage interconnection networks are discussed and then, mesh networks as a second approach are described.

2.1 Bidirectional Multistage Interconnection Networks

Multistage Interconnection Networks (MIN) are dynamic networks which are based on switching elements (SE). SEs are arranged in stages and connected by interstage links. The link structure and amount of SEs characterizes the MIN.

MINs [9] of size $N \times N$ (N inputs and N outputs) consist of $c \times c$ switching elements. The number of stages is given by $n = \log_c N$ (with n, c, $N \in \mathbb{N}$) in case of MINs with the banyan property which provide a unique path for an input-output pair. Bidirectional MINs (BMIN) [10] consist of at least $n = \log_c N$ stages to allow connections between each input and each output.

Their interstage links and their SEs are bidirectional. That means packets can be transferred in both directions. In consequence, each input also represents the corresponding output. Figure 1 depicts the structure of a bidirectional MIN. If packet switching is applied buffers can

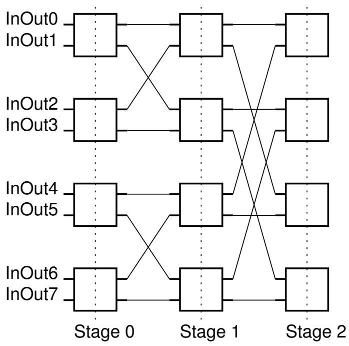


Figure 1: Bidirectional MIN

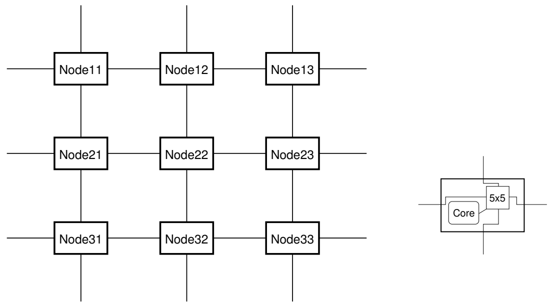
be introduced. A packet is first routed from the network input to the right, denoted as forward direction. As soon as it reaches a stage from which a path exists in backward direction (that means from right to left) to its destination output, it turns around. This stage is called turnaround stage. Finally, the packet proceeds its way in backward direction to the desired output.

During its movement in forward direction, the packet may choose any arbitrary SE output because each SE output offers a path to the network destination output via a turnaround stage. Moreover, all paths that a particular packet may choose reveal the same stage as turnaround stage due to the MIN structure. That means all redundant paths are of equal length.

In backward direction, only a single path through the network exists to reach a particular output.

2.2 Mesh Networks

A static network architecture for NoCs is a mesh [11]. In such an architecture, the cores are located at the crosspoints of the mesh. Three kinds of meshes are distinguished: one-dimensional meshes (also called chains), two-dimensional meshes (2-D meshes, grids), and three-dimensional meshes (3-D meshes). Figure 2 shows a 2-D mesh. The nodes of the mesh incorporate a core and a



(a) Mesh (b) Mesh node

Figure 2: 2-D mesh architecture

5×5 SE (Figure 2(b)), optional with buffers. The SE connects all inputs and outputs of the node to allow packets to pass the node. Further on, the core is linked via the SE to the rest of the mesh.

Each node is connected to its two nearest neighbors in each dimension. For instance, four bidirectional links handle all communication of a node in a 2-D mesh (Figure 2(a)). The number of links per node does not change if additional cores (i.e. nodes) are added to the mesh. Therefore, a mesh offers very good scalability. Its blocking behavior reveals one of the most important disadvantages of meshes. Usually, messages pass several nodes and links until they reach their destination. As a result, the same link is demanded by many connections: blocking occurs. Thus, messages are mostly transferred by packet switching to deal with the blocking by introducing buffers.

Meshes as well as BMINs reveal some locality. The next section discusses this locality and shows how to profit from it.

3 MULTICAST AND LOCALITY

As motivated in the introduction, networks for multicore processors must be able to deal with multicast traffic. Further on, traffic locality will be observed and thus must also be represented in a network model. Both issues will be discussed in the following.

3.1 Multicast

Multicasting can be efficiently performed by copying the packets within the switches of a multicore processor network instead of copying them before they enter the network. This scheme is called message replication while routing (MRWR). For instance, Figure 3 shows such a scenario for an 8×8 BMIN consisting of 2×2 SEs. A packet is sent by Core 0 (Input 0) and is destined to Core 2 (Output 2) and Core 4 (Output 4). The packet enters the network and is not copied until it reaches the middle stage. Then, two copies of the packet proceed their way through the remaining stages.

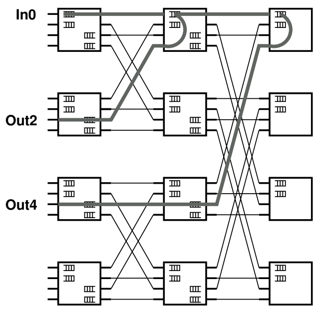


Figure 3: Multicast in bidirectional MINs

Message replication before routing (MRBR) applied to the example above would copy the packet and send it twice into the network. Compared to this, message replication while routing reduces the amount of packets in the first stages of the path through the network.

Applying MRWR, the packets are copied as late as possible to optimally reduce the number of packets in the network.

3.2 Locality

Two aspects of locality have to be considered. First, the locality of network traffic due to applications that are distributed to different set of cores. Traffic within a set of cores can be assumed to be more intensive than traffic between different sets representing different applications.

Second, the network topology reveals some locality in its structure. Figure 4 points out the locality of bidirectional MINs [12]. The structural locality for Core 0 (connected to Input/Output 0) is demonstrated. There is a very high locality for Core 0 with Core 1 (dark grey area). The communication path is very short (just a turnaround at Stage 0).

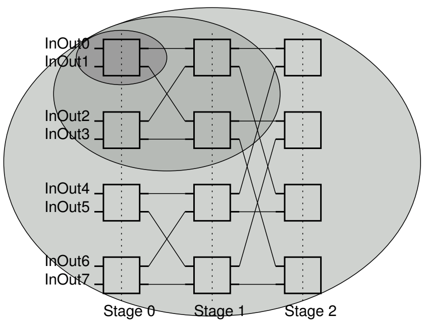


Figure 4: Locality in bidirectional MINs

A slighter locality can be found between Core 0 and Core 2 or Core 3 (medium grey area). Here, packets must pass three stages to reach the destination: Stage 0, a turnaround in Stage 1, and finally backwards via Stage 0. No locality can be seen for Core 0 when communicating with one of the cores numbered from 4 to 7 (light grey area) is initiated. All network stages are involved.

In meshes, it is obvious that the communication path to neighbor cores is much shorter than for instance the path between two cores in opposite corners.

In consequence, both aspects of locality should be mapped when applications are distributed to different cores: The cores should be chosen such that they reveal structural locality resulting in fast communication. However, sometimes it may not be possible to chose the cores in this way because either cores of structural locality are already occupied by other applications or the application is distributed to more cores than locally connected ones.

3.3 Modeling local multicast traffic

To investigate how a multicore processor behaves dependent on the chosen network topology, a corresponding stochastic model must be established for performance evaluation. Instead of modeling the entire system including the cores, it is more reasonable to reduce the model complexity and to simply model the detached network. But then, it has to be ensured that the network traffic represents the traffic in a multicore processor. Thus, traffic generators at the network inputs are needed that produce the assumed spatial and multicast traffic distribution.

The traffic generators must be provided with the desired kind of spatial distribution and multicasts. Because the future multicore traffic is not exactly known, both, multicasts and spatial traffic, are described as stochastic distributions for model input. The easiest way is to specify the spatial distribution of the traffic and the kind of multicasts independent of each other [13]. But not all multicast distributions can be achieved with a particular desired spatial distribution and vice versa: there are multicast distributions and spatial distributions that are incompatible. For instance, if the multicast distribution is such that only broadcasts to all network outputs occur: Then, all outputs receive the same amount of packets and any other than a uniform spatial distribution cannot be reached.

Thus, it is important to check for the compatibility of the spatial distribution and the multicasts that the modeler is willing to investigate. Such a compatibility check avoids inconsistent traffic parameters while modeling.

4 COMPATIBILITY OF MULTICAST AND SPATIAL DISTRIBUTION

In this section, an algorithm is invented that checks whether a network input is able to generate traffic with a given stochastic multicast distribution by fulfilling the desired stochastic spatial distribution. It is assumed that the input can reach N outputs of the network. The multicast distribution is described by the multicast probabilities $a(i)$ with $i \in \mathbb{N}, 1 \leq i \leq N$. The multicast probability $a(i)$ gives the probability with which a generated packet is destined to i outputs. The spatial distribution is described by the local probabilities $\ell(h)$ with $h \in \mathbb{N}, 0 \leq h < N$. The local probability $\ell(h)$ gives the probability with which a generated packet is destined to Output h.

4.1 Compatibility

To demonstrate the idea of the algorithm, the particular multicast distribution that there are only multicasts to g outputs and unicasts (to a single output) is considered first as an example:

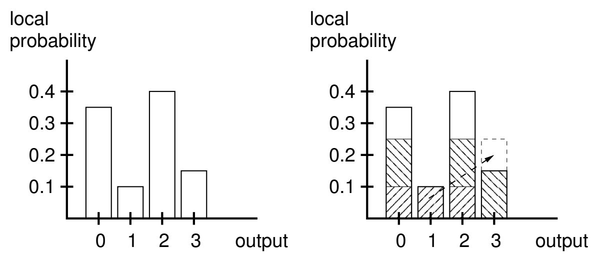
$$a(i = g) \quad := \quad a(g) \tag{1}$$

$$a(1) \quad := \quad 1 - a(g) \tag{2}$$

$$a(1 \neq i \neq g) \quad := \quad 0 \tag{3}$$

Then, the only critical issue is the multicast to g outputs. It has to be checked whether the multicasts can be performed in the required ratio without violating the given spatial distribution. The unicasts can always be distributed among the outputs for any given spatial distribution.

Figure 5(a) gives another example. The figure depicts



(a) Desired distribution (b) Feasible with $a(3)$

Figure 5: Spatial distribution for a network with 4 outputs

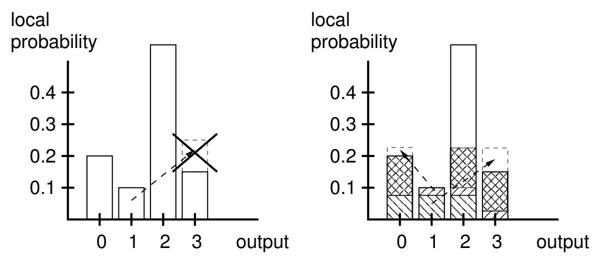
the desired spatial distribution for a network with 4 outputs: Output 0 shall be one of the packet destinations with a probability of 35% ($\ell(0) = 0.35$), Output 1 with 10% ($\ell(1) = 0.1$), Output 2 with 40% ($\ell(2) = 0.4$), and Output 3 with 15% ($\ell(3) = 0.15$). For instance, if there were *only* multicasts to $g = 4$ outputs (that means $a(4) = 1$ and $a(1) = 0$) then, the distribution of Figure 5(a) could not be reached because all packets were destined to *all* outputs and that means the spatial distribution was a uniform one (all bars of Figure 5(a) were of the same height of 0.25).

A similar problem would arise if there were only multicasts to $g = 3$ outputs ($a(3) = 1$ and $a(1) = 0$). Then, the uniform spatial distribution could be avoided, for instance if half of the packets is sent to Outputs 0, 1, and 2 and half of them is sent to Outputs 0, 2, and 3. In this case, Outputs 0 and 2 would be twice as often the destination of packets than Outputs 1 and 3: the spatial distribution $\ell(0) = 0.\overline{3}$, $\ell(1) = 0.1\overline{6}$, $\ell(2) = 0.\overline{3}$, and $\ell(3) = 0.1\overline{6}$ would result. But again, the distribution of Figure 5(a) could not be reached by this multicast distribution. The filled part of the bars (Figure 5(b)) would be feasible but not the unfilled part by multicasts to 3 outputs.

In consequence, the filled part of the bars of Figure 5(b) give the maximum amount of multicast traffic to 3 outputs (received by the network outputs) that is allowed if the given spatial distribution shall be fulfilled (the unfilled parts can be contributed by multicasts to 2 outputs and to a single output). The term "amount of multicast traffic to i outputs" denotes the fraction of received traffic (by all network outputs) divided by i that originates from sending multicast traffic to i outputs.

The figure also gives the answer to the question how this maximum amount of multicast traffic to $g = 3$ outputs can be determined: Out of the four bars, $g = 3$ "new bars" must be build such that the smallest bar has maximum height (dashed arrow). The height of the smallest bar (0.25) gives the maximum amount of multicast traffic to 3 outputs that is feasible. For instance, if the unfilled remaining parts in Figure 5(b) are covered by unicasts (amount: $0.1 + 0.15$), the generated traffic must consist of multicast traffic to 3 outputs with probability $a(3) = 0.25/(0.25+0.1+0.15) = 0.5$ and of unicast traffic with probability $a(1) = (0.1+0.15)/(0.25+0.1+0.15) = 0.5$.

But Figure 5(b) only represents a special case. If the sum of the probabilities of Output 1 and Output 3 would be greater than the probability of any other output (Output 0 in the example of Figure 6(a)), then a distribution of the probabilities of Output 1 to the bar of this other output and to Output 3 must be performed such that their bars are of equal height. Figure 6(b) shows the optimal



(a) Desired distribution (b) Feasible with $a(3)$

Figure 6: Spatial distribution by "distributing" Output 1

distribution of the multicasts to 3 outputs among the four available outputs. Each filled-in pattern represents one of the output combinations. The dashed boxes depict previously mentioned distribution of the probabilities of Output 1 resulting in an equal height of the "new bars" of Output 0 and 3 of 0.225 which is the maximum amount of multicast to $g = 3$ outputs.

Previous examples are summarized and generalized in the following conclusion describing how to find the maximum allowed multicast traffic to g outputs that does not violate a desired spatial traffic distribution defined by $\mathcal{D}_W = \{\ell(h) | h \in \mathbb{N}, 0 \leq h < N\}$: A set distribution $\mathcal{S} = \{\mathcal{S}_l\}$ of the outputs and their probabilities to g sets $\mathcal{S}_l$ with $1 \leq l \leq g$ must be found. Each set element $s_l \in \mathcal{S}_l$ is identified by its corresponding output h and represents the local probability $\ell(h)$ of this output. That means $s_l(h) = \ell(h)$ if Output h belongs to set $\mathcal{S}_l$. The probability $\ell(h)$ of a particular output can be split and instead of the entire probability, the resulting probabilities $\ell_r(h)$ (with $\ell(h) = \sum_r \ell_r(h)$) can be part of the sets and distributed among them which yields $s_l(h) = \ell_r(h)$. The set distribution must be found such that the smallest of the set probability sums

$$x_{\text{small}}(\mathcal{S}) = \min_l \left\{ \sum_{s_l(h) \in \mathcal{S}_l} s_l(h) \right\} \tag{4}$$

(the corresponding minimum sum set $\mathcal{S}_l$ is denoted as $\mathcal{S}_{\text{small}}$) is maximal

$$x_{\max}(\mathcal{D}_W) = \max_{\mathcal{S}} \{x_{\text{small}}(\mathcal{S})\} \tag{5}$$

(the corresponding set distribution is called $\mathcal{S}_{\max}$) and that all outputs h_{small} of the set $\mathcal{S}_{\text{small}}$ can be combined with other outputs h (where $h \neq h_{\text{small}}$) of the other sets $\mathcal{S}_l \in \mathcal{S}_{\max}$ (where $\mathcal{S}_l \neq \mathcal{S}_{\text{small}}$) according to their probability. That means combinations not consisting of g different outputs are not allowed.

Theorem 1: A splitting of the g highest probabilities when building the g different sets is not allowed to avoid combinations that include a particular output more than once. Thus, $\mathcal{S}_{\max}$ is found by ordering the local probabilities $\ell(h)$:

$$\ell(h_1) \leq \ell(h_2) \leq \cdots \leq \ell(h_j) \leq \cdots \leq \ell(h_N) \tag{6}$$

Each set $\mathcal{S}_l$ (with $1 \leq l \leq g$) is initialized with a different one of the g highest probabilities: $\ell(h_{N+1-l}) \in \mathcal{S}_l$. Then, the $N - g$ lowest probabilities $\ell(h_1)$ to $\ell(h_{N-g})$ are distributed (and splitted) among the f sets $\mathcal{S}_l$ ($f \in \mathbb{N}, 1 \leq f \leq g$) with the f lowest probabilities ($g - f + 1 \leq l \leq g$). f is chosen such that all sums $\sum_{s_l(h) \in \mathcal{S}_l} s_l(h)$ are equal for $g - f + 1 \leq l \leq g$ and are less than the sum of all other sets that only consist of $\ell(h_j)$ with $1 \leq j \leq g - f$.

Remark: Figure 6(b) gives an example for $N = 4$ and $g = 3$. The dashed boxes depict how $\mathcal{S}_{\max}$ was found: $\mathcal{S}_1 = \{\ell(2) = 0.55\}$, $\mathcal{S}_2 = \{\ell(0) = 0.2, \ell_1(1) = 0.025\}$, and $\mathcal{S}_3 = \{\ell(3) = 0.15, \ell_2(1) = 0.075\}$. The $N - g = 1$ smallest probability is distributed and splitted to the $f = 2$ smallest probabilities of the $g = 3$ highest ones.

Proof: Part 1: This part proves that Theorem 1 produces a set distribution where all outputs h_{small} of the set $\mathcal{S}_{\text{small}}$ with the smallest probability sum can be combined with other outputs h (where $h \neq h_{\text{small}}$) of the other sets $\mathcal{S}_l \in \mathcal{S}_{\max}$ (where $\mathcal{S}_l \neq \mathcal{S}_{\text{small}}$) according to their probability without violating the allowed output combinations. Theorem 1 causes that the highest probability of the $N - g$ smallest probabilities is always smaller or equal to the smallest of the g highest probabilities: $\ell(h_{N-g}) \leq \ell(h_{N-g+1})$. In the worst case, $\ell(h_{N-g})$ will be split to all g sets, with fraction $\frac{\ell(h_{N-g})}{g}$ to each of them. To avoid invalid output combinations, these fractions are not allowed to be somehow combined in any output combination. This can always be realized because the probability $\ell(h_{N-g+1})$ (and thus all others of the g highest ones) can be divided into g parts, all of them with value $\frac{\ell(h_{N-g+1})}{g}$. Further on, $\ell(h_{N-g}) \leq \ell(h_{N-g+1})$ is equivalent to $\frac{\ell(h_{N-g})}{g} \leq \frac{\ell(h_{N-g+1})}{g}$. In consequence, each of the splitted parts of $\ell(h_{N-g})$ can be combined with a different part of $\ell(h_{N-g+1})$ (and thus with different parts of all others of the g highest probabilities).
Because this is true for the highest probability of the $N - g$ smallest ones, it is also true for all others of the $N - g$ smallest ones and a distribution of them is allowed.
Part 2: If, instead of one of the $N - g$ smallest probabilities, one of the g highest probabilities (say: $\ell(h_y)$) would be distributed (and split), then
a) either the same set probability sums would result if $\ell(h_y)$ was smaller or equal to $x_{\max}$
b) or if $\ell(h_y) > x_{\max}$, this probability is not allowed to be split because Output h_y must be included in *each* output combination to reach the highest amount (consider for instance that $\ell(2)$ of Figure 6(b) would be split). Splitting it to other sets would cause output combinations in which it is represented multiple times: it would be the only element of one $\mathcal{S}_l$ and at least part of another $\mathcal{S}_l$.
Because Theorem 1 gives an allowed set distribution (Part 1) and no better set distribution can be found (Part 2), Theorem 1 is proven to determine $\mathcal{S}_{\max}$. □

Theorem 1 can now be used to determine $x_{\max}(\mathcal{D}_W)$ which gives the amount of feasible multicasts to g outputs: The smallest $N - g$ probabilities sum up to

$$\ell_{\text{todist}} = \sum_{j=1}^{N-g} \ell(h_j) \tag{7}$$

This amount can be distributed to the f sets with the smallest f of the g highest probabilities. That means the probability sum of the smaller $f - 1$ sets reaches the probability $\ell(h_{N-g+f})$ of the f-th set and then, all of them are risen with the remaining probability ℓ_{remain} to meet the probability amount of Equation (7). That means

$$\begin{aligned}
\ell_{\text{todist}} &= \sum_{j=N-g+1}^{N-g+f-1} \left(\ell(h_{N-g+f}) - \ell(h_j)\right) + f \cdot \ell_{\text{remain}} \\
&= (f-1) \cdot \ell(h_{N-g+f}) \\
&\quad - \sum_{j=N-g+1}^{N-g+f-1} \ell(h_j) + f \cdot \ell_{\text{remain}}
\end{aligned} \tag{8}$$

Replacing ℓ_{todist} by Equation (7) and combining the sums leads to

$$\begin{aligned}
\sum_{j=1}^{N-g+f-1} \ell(h_j) &= (f-1) \cdot \ell(h_{N-g+f}) + f \cdot \ell_{\text{remain}} \\
&\geq (f-1) \cdot \ell(h_{N-g+f})
\end{aligned} \tag{9}$$

The maximum $f \in \mathbb{N}$ that fulfills Equation (9) gives the number of sets to which the probability ℓ_{todist} is distributed to:

$$f_{\max} = \max_{f \leq g} \left\{ f \,\middle|\, \sum_{j=1}^{N-g+f-1} \ell(h_j) \geq (f-1) \cdot \ell(h_{N-g+f}) \right\} \tag{10}$$

Using $f_{\max}$, the remaining probability ℓ_{remain} can be calculated by using Equation (9). This remaining probability additionally rises the probability sum of all $f_{\max}$ sets above the value $\ell(h_{N-g+f_{\max}})$ resulting in a value

$$x_{\max}(\mathcal{D}_W) = \frac{\sum_{j=1}^{N-g+f_{\max}-1} \ell(h_j) - (f_{\max}-1)\cdot \ell(h_{N-g+f_{\max}})}{f_{\max}} + \ell(h_{N-g+f_{\max}}) \tag{11}$$

describing the maximum amount of multicast to g outputs for the given spatial traffic distribution $\mathcal{D}_W$.

In the example of Figure 6(b) with $\mathcal{D}_W = \{0.2, 0.1, 0.55, 0.15\}$ and a desired multicast traffic to $g = 3$ outputs, $x_{\max}(\mathcal{D}_W)$ results with $f_{\max} = 2$ in

$$\begin{aligned} x_{\max}(\mathcal{D}_W) &= \frac{0.1 + 0.15 - 1\cdot 0.2}{2} + 0.2 \\ &= 0.225 \end{aligned} \tag{12}$$

It should be noted that the sum of the local probabilities $x_{\max}(\mathcal{D}_W) = 0.225$ does *not* give the maximum value of the multicast probability $a(3)$. It has to be normalized because a multicast to $g = 3$ outputs produces 3 packets leaving an output compared to a unicast producing only a single packet. The general form of the normalization leads to

$$\frac{a(g)}{\sum_{i=1}^{N} i\cdot a(i)} \leq x_{\max}(\mathcal{D}_W) \tag{13}$$

Considering the example of Equations (1) to (3),

$$\frac{a(3)}{3\cdot a(3) + 1\cdot a(1)} \leq x_{\max}(\mathcal{D}_W) = 0.225 \tag{14}$$

is obtained. That means $a(3) \leq 0.4\overline{09}$ is required to fulfill the given spatial distribution.

4.2 Algorithm

Equations (10), (11), and (13) can now be used to determine by induction whether a given stochastic multicast distribution can fulfill a desired stochastic spatial distribution. In step 1, it is started with checking the amount of multicasts to the largest number of outputs, which means multicasts to N outputs. This multicast is possible if

$$\frac{a(N)}{\sum_{i=1}^{N} i\cdot a(i)} \leq x_{\max}(\mathcal{D}_W^{[N]}) \tag{15}$$

with $\mathcal{D}_W^{[N]} = \{\ell^{[N]}(h)\} = \{\ell(h)\} = \mathcal{D}_W$. To check whether a given amount of multicasts to any other number g of outputs is additionally possible, the already checked multicasts to N outputs must be considered: they produce already $\frac{a(N)}{\sum_{i=1}^{N} i\cdot a(i)}$ of the desired traffic probabilities $\ell(h)$ to each output h. Only the remaining part can be used by the multicasts that are checked now (for instance the unfilled parts of the bars in the example of Figure 8(a)).

It is now assumed that all multicasts from N outputs to $g+1$ outputs have already been proved to be possible. A multicast to g outputs is then additionally possible if

$$\frac{a(g)}{\sum_{i=1}^{N} i\cdot a(i)} \leq x_{\max}(\mathcal{D}_W^{[g]}) \tag{16}$$

where $\mathcal{D}_W^{[g]}$ results by subtracting the amount of multicasts to $g+1$ outputs from the previously checked spatial distribution $\mathcal{D}_W^{[g+1]}$. It has to be taken care that a multicast to $g+1$ outputs can be uniformly distributed among the N outputs (see the filled parts of the bars in Figure 5(b) or Figure 6(b) as examples for distributing a multicast to 3 outputs among all 4 outputs). Thus, $\mathcal{D}_W^{[g]}$ is yield as

$$\mathcal{D}_W^{[g]} = \left\{\ell^{[g]}(h)\right\} = \left\{\ell^{[g]^*}(h)\right\} = \mathcal{D}_W^{[g]^*} \text{ if all } \ell^{[g]^*}(h) \geq 0 \tag{17}$$

where

$$\mathcal{D}_W^{[g]^*} = \left\{\ell^{[g]^*}(h)\right\} = \left\{\ell^{[g+1]}(h) - \frac{a(g+1)}{\sum_{i=1}^{N} i\cdot a(i)}\frac{g+1}{N}\right\} \tag{18}$$

Some $\ell^{[g]^*}(h)$ may become negative because in case of low values of $\ell^{[g+1]}(h)$ (for instance if $\ell^{[g+1]}(h) = 0$ which means there is no more traffic to this output allowed), the subtraction results in negative values. Then, a correction is needed setting the negative values to zero. The traffic that was previously assumed to pass these outputs (denoted as $\ell_{\text{neg}}^{[g]}$) must additionally be distributed to the other outputs:

$$\ell^{[g]^*}(h) \longleftarrow \begin{cases} 0 & \text{if } \ell^{[g]^*}(h) \leq 0 \\ \ell^{[g]^*}(h) - \frac{\ell_{\text{neg}}^{[g]}}{||\ell^{[g]^*}(h)>0||} & \text{otherwise} \end{cases} \tag{19}$$

with

$$\ell_{\text{neg}}^{[g]} = -\sum_{\ell^{[g]^*}(h)<0} \ell^{[g]^*}(h) \tag{20}$$

and $||\ell^{[g]^*}(h) > 0||$ denoting the number of $\ell^{[g]^*}(h)$ that are greater than zero.

If all $\ell^{[g]^*}(h) \geq 0$ then the spatial distribution $\mathcal{D}_W^{[g]}$ to be considered for checking the amount of multicasts to g outputs is found: $\mathcal{D}_W^{[g]} = \left\{\ell^{[g]}(h)\right\} = \left\{\ell^{[g]^*}(h)\right\} = \mathcal{D}_W^{[g]^*}$. Otherwise, the negative values must be corrected again using Equation (19). This must be performed till all $\ell^{[g]^*}(h)$ become greater or equal zero.

Figure 7 summarizes the entire algorithm. If the algorithm processes the entire for-loop without an error message and break, the given multicast distribution can fulfill the desired spatial distribution. The case $g = 1$ need not to be checked because unicasts can fulfill any remaining distribution.

4.3 Examples

Two examples shall demonstrate the algorithm. The first example network consists of $N = 4$ outputs, for instance a 4×4 BMIN. The desired spatial distribution shall be this one of Figure 5(a): $\ell(0) = 0.35$, $\ell(1) = 0.1$, $\ell(2) = 0.4$,

for $g = N$ **to** 2 **step** -1
  if $g = N$ **then**
    $\mathcal{D}_W^{[N]} = \mathcal{D}_W$
  else
    $\mathcal{D}_W^{[g]^*} = \left\{ \ell^{[g]^*}(h) \right\} = \left\{ \ell^{[g+1]}(h) - \frac{a(g+1)}{\sum_{i=1}^{N} i \cdot a(i)} \frac{g+1}{N} \right\}$
    repeat
      $\ell_{\text{neg}}^{[g]} = -\sum_{\ell^{[g]^*}(h)<0} \ell^{[g]^*}(h)$
      if $\ell_{\text{neg}}^{[g]} \neq 0$ **then**
        $\ell^{[g]^*}(h) \longleftarrow \begin{cases} 0 & \text{if } \ell^{[g]^*}(h) \leq 0 \\ \ell^{[g]^*}(h) - \frac{\ell_{\text{neg}}^{[g]}}{||\ell^{[g]^*}(h)>0||} & \text{else} \end{cases}$
      else
        $\mathcal{D}_W^{[g]} = \mathcal{D}_W^{[g]^*}$
      endif
    until $\ell_{\text{neg}}^{[g]} = 0$
  endif
  if $\frac{a(g)}{\sum_{i=1}^{N} i \cdot a(i)} > x_{\text{max}}(\mathcal{D}_W^{[g]})$ **then**
    print "multicast incompatible to spatial distribution"
    break
  endif
endfor

Figure 7: Algorithm to check multicast compatibility

and $\ell(3) = 0.15$. The multicast distribution is given by $a(1) = 0.2$, $a(2) = 0.5$, $a(3) = 0.1$, and $a(4) = 0.2$. It is investigated now whether an input that can reach all outputs is able to generate network traffic that fulfills both, the desired spatial distribution and the given multicast distribution. The algorithm starts with the multicast to all $g = N = 4$ outputs. Then,

$$\mathcal{D}_W^{[4]} = \mathcal{D}_W = \{0.35, 0.1, 0.4, 0.15\} \tag{21}$$

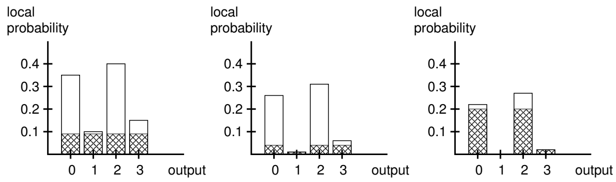
directly results. Ordering the local probabilities leads to $\ell(h_1 = 1) = 0.1 \leq \ell(h_2 = 4) = 0.15 \leq \ell(h_3 = 0) = 0.35 \leq \ell(h_4 = 2) = 0.4$. Because for $g = 4$, there are $N - g = 0$ probabilities to be distributed to the remaining, it is clear that $x_{\text{max}}(\mathcal{D}_W^{[4]})$ is determined by the smallest $\ell(h)$, which yields $x_{\text{max}}(\mathcal{D}_W^{[4]}) = 0.1$. More formally, Equation (10) can also be used to calculate $x_{\text{max}}(\mathcal{D}_W^{[4]})$: It is started with $f = 1$. Then, Equation (9) is fulfilled: $0 \geq 0 \cdot 0.1$. For $f = 2$, this equation resulting in $0.1 \geq 1 \cdot 0.15$ is not fulfilled. Thus, $f_{\text{max}} = 1$ holds and Equation (11) yields

$$x_{\text{max}}(\mathcal{D}_W^{[4]}) = 0.1 \tag{22}$$

as previously concluded. The term

$$\frac{a(4)}{\sum_{i=1}^{4} i \cdot a(i)} = \frac{0.2}{2.3} \approx 0.087 \tag{23}$$

is less than $x_{\text{max}}(\mathcal{D}_W^{[4]}) = 0.1$ and thus, the given amount of multicasts to 4 outputs (depicted by the filled parts of the bars of Figure 8(a)) can be realized without violating the desired spatial distribution.



(a) $g = 4$ (b) $g = 3$ (c) $g = 2$

Figure 8: Remaining spatial distribution and used amount for multicasts to g outputs

In the next step, $g = 3$ is checked. Considering the multicast to 4 outputs, each $\ell^{[4]}(h)$ must be decreased by $\frac{a(4)}{\sum_{i=1}^{4} i \cdot a(i)} \cdot \frac{4}{4} \approx 0.087$ which yields

$$\mathcal{D}_W^{[3]^*} = \{0.263, 0.013, 0.313, 0.063\} \tag{24}$$

All elements of the sum are not negative, thus

$$\mathcal{D}_W^{[3]} = \mathcal{D}_W^{[3]^*} \tag{25}$$

With $N - g = 1$, Equation (9) is again checked with the ordered probabilities of $\mathcal{D}_W^{[3]}$ to determine f_{max}: For $f = 1$, the equation is always true because the right side results always in zero. For $f = 2$, $0.013 + 0.063 \geq 1 \cdot 0.263$ is wrong and thus, $f_{\text{max}} = 1$ and

$$x_{\text{max}}(\mathcal{D}_W^{[3]}) = \frac{0.013}{1} + 0.063 = 0.076 \tag{26}$$

results. Due to

$$\frac{a(3)}{\sum_{i=1}^{4} i \cdot a(i)} = \frac{0.1}{2.3} \approx 0.04348 \tag{27}$$

which is less than $x_{\text{max}}(\mathcal{D}_W^{[3]})$, the given amount of multicasts to 3 outputs can also be realized (depicted by the filled parts of the bars of Figure 8(b); the low amount of remaining traffic to Output 1 and thus the increased traffic to the other ones is already incorporated but calculated in the sequel of the example).

For $g = 2$, each $\ell^{[3]}(h)$ must be decreased by $\frac{a(3)}{\sum_{i=1}^{4} i \cdot a(i)} \cdot \frac{3}{4} \approx 0.033$ which yields

$$\mathcal{D}_W^{[2]^*} = \{0.23, -0.02, 0.28, 0.03\} \tag{28}$$

Here, one of the elements became negative (the previously mentioned Output 1) and thus, it has to be set to zero while correcting the remaining 3 outputs by $\frac{\ell_{\text{neg}}^{[2]}}{||\ell^{[2]^*}(h)>0||} = \frac{0.02}{3}$. No further correction is necessary and

$$\mathcal{D}_W^{[2]} = \{0.2233, 0.0, 0.2733, 0.0233\} \tag{29}$$

results (see Figure 8(c)). These probabilities are ordered and f_{max} is determined for $N - g = 2$: Equation (9) is always true for $f = 1$. In case of $f = 2$, the equation

$0.0 + 0.0233 + 0.2233 \geq 1 \cdot 0.2733$ becomes false and thus, $f_{\max} = 1$ and

$$x_{\max}(\mathcal{D}_W^{[2]}) = \frac{0.0233}{1} + 0.2233 = 0.2466 \tag{30}$$

is yield. Due to

$$\frac{a(2)}{\sum_{i=1}^{4} i \cdot a(i)} = \frac{0.5}{2.3} \approx 0.21739 \tag{31}$$

which is less than $x_{\max}(\mathcal{D}_W^{[2]})$, the given amount of multicasts to 2 outputs can also be realized (depicted by the filled parts of the bars of Figure 8(c)).

Considering the previous results and because the not yet checked unicasts can be distributed in any distribution to the outputs, the given multicast distribution can be realized without violating the desired spatial distribution.

The second example is a short one. Again, the network consists of $N = 4$ outputs. The desired spatial distribution is defined by $\ell(0) = 0.25$, $\ell(1) = 0.25$, $\ell(2) = 0.3$, and $\ell(3) = 0.2$. The multicast distribution is given by $a(1) = 0.1$, $a(2) = 0.9$, $a(3) = 0$, and $a(4) = 0$ what means that there is only multicast to 2 outputs and unicast. It is investigated whether an input that can reach all outputs is able to generate network traffic that fulfills both, the desired spatial distribution and the given multicast distribution.

Because there is no multicast to $g = 4$ or $g = 3$ outputs, the algorithm starts with $g = 2$. Then,

$$\mathcal{D}_W^{[2]} = \mathcal{D}_W = \{0.25, 0.25, 0.3, 0.2\} \tag{32}$$

directly results. Ordering the local probabilities leads to $\ell(h_1 = 3) = 0.2 \leq \ell(h_2 = 0) = 0.25 \leq \ell(h_3 = 1) = 0.25 \leq \ell(h_4 = 2) = 0.3$. Equation (10) with $N - g = 2$ is used to calculate $x_{\max}(\mathcal{D}_W^{[2]})$: It is started with $f = 1$. Then, Equation (9) is always fulfilled. For $f = 2$, this equation is also true with $0.2 + 0.25 + 0.25 \geq 0.3$. Due to $f \leq g$, the maximum f results in $f_{\max} = 2$ and Equation (11) yields

$$x_{\max}(\mathcal{D}_W^{[2]}) = \frac{0.2 + 0.25 + 0.25 - 0.3}{2} + 0.3 = 0.5 \tag{33}$$

The term

$$\frac{a(2)}{\sum_{i=1}^{4} i \cdot a(i)} = \frac{0.9}{1.9} \approx 0.4737 \tag{34}$$

is less than $x_{\max}(\mathcal{D}_W^{[2]}) = 0.5$ and thus, the given amount of multicasts to 2 outputs can be realized without violating the desired spatial distribution.

Because unicasts can be distributed in any distribution to the outputs, the given multicast distribution can be realized without violating the desired spatial distribution.

4.4 Benefit

With the given algorithm, the modeler becomes able to check whether a particular desired spatial traffic distribution and multicast traffic distribution are compatible. Any inconsistent traffic parameters are prevented before the simulation or analysis of the model starts.

5 CONCLUSIONS AND FUTURE WORK

In this paper, a new algorithm for checking the compatibility of a given stochastic spatial traffic distribution and stochastic multicast traffic distribution is presented. Spatial traffic distributions and multicast traffic distributions are provided to traffic generators in stochastic network models if, for instance, network-on-chip architectures for multicore processors are investigated. The compatibility check avoids inconsistent traffic parameters while modeling.

The new algorithm can be performed before the simulation or analysis of the NoC in question is started. If the algorithm detects any incompatibility of the traffic distributions, simulation or analysis is aborted and the modeler is able to correct the traffic parameters. To the best of the authors' knowledge, this is the first algorithm that checks for consistent multicast and spatial distribution in network traffic models.

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ANALYTICAL INTERCONNECTION NETWORKS MODEL FOR MULTI-CLUSTER COMPUTING SYSTEMS

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Analytical Modeling, Multi-Cluster, Communication Network, Latency, Performance Analysis.

ABSTRACT

This paper addresses the problem of interconnection networks performance modeling of large-scale distributed systems with emphases on multi-cluster computing systems. The study of interconnection networks is important because the overall performance of a distributed system is often critically hinged on the effectiveness of its interconnection network. We present an analytical model that considers stochastic quantities as well as processor heterogeneity of the target system. The model is validated through comprehensive simulation, which demonstrates that the proposed model exhibits a good degree of accuracy for various system sizes and under different operating conditions.

1. INTRODUCTION

An increasing trend in the high performance computing (HPC) development is towards the networked distributed systems such as commodity-based cluster computing and grid computing systems. These network-based systems have proven to be cost-effective parallel processing tools for solving many complex scientific, engineering and commercial applications as compared to the conventional supercomputing systems (Abawajy and Dandamudi 2003).

In this paper, we focus on the interconnection networks for multi-cluster computing systems. The study of interconnection networks is important because the overall performance of a distributed system is often critically hinged on the effectiveness of its interconnection network. Also, the interconnection network design plays a central role in the design and development of multi-cluster computing systems. Simulation has been used to investigate the performance of various components of multi-cluster computing systems (Abawajy and Dandamudi 2003). Instead, we focus on analytical model. An accurate analytical model can provide quick performance estimates and will be a valuable design tool. The significant advantage of analytical models over simulation is that they can be used to obtain performance results for large systems which may not feasible to study using simulation due to the excessive computation demands.

Several analytical performance models of multi-computer systems have been proposed in the literature for different interconnection networks and routing algorithms (e.g., Sarbazi-Azad et al. 2002; Boura and Das 1997; Drapper and Ghosh 1994). Unfortunately, little attention has been given to cluster computing systems. Most of the existing researches are based on homogenous cluster systems and the evaluations are confined to a single cluster (Du et al. 2000; Hu and Kleinrock 1995). In contrast, we focus on heterogeneous multi-cluster computing environment.

To this end, we present an analytical performance model of interconnection networks for multi-cluster computing systems. The model is based on probabilistic analysis and queuing network to analytically evaluate the performance of interconnection networks for multi-cluster systems. The model takes into account stochastic quantities as well as processor heterogeneity among clusters. The model is validated through comprehensive simulation, which demonstrated that the proposed model exhibits a good degree of accuracy for various system sizes and under different operating conditions.

The rest of the paper is organized as follows. In Section 2, we give a brief overview of the multi-cluster systems. In Section 3, we give detailed description of the proposed analytical model. We present the model validation experiments in Section 4. We summarize our findings and conclude the paper in Section 5.

2. SYSTEM OVERVIEW

The system under study in this paper is a multi-cluster computing systems which is made up of C clusters, each cluster i is composed of N_i processors of type τ_i, $i \in \{0,1,...,C-1\}$. Also, each cluster has two communication networks, an Intra-Communication Network (*ICN1*), which is used for the purpose of message passing between processors, and an intEr-

Communication Network (*ECN1*), which is used to transmit messages between clusters, management of the system, and also for the scalability of the system.

It should be noted that, ECN1 can be accessed directly by the processors of each cluster without going through the ICN1 (see Fig. 1). To interconnect ECN1 and ICN2, a set of Concentrators/Dispatchers (Dally and Towles 2004) are used, which combine message traffic from/to one cluster to/from other cluster.

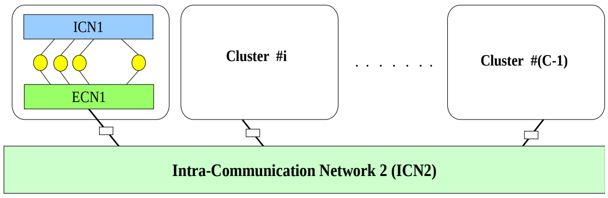

Fig. 1. The Multi-Cluster Computing System

The main factors which have impact on the performance of a multi-cluster system are *Topology, Flow control mechanism, and Routing algorithm*. The connection pattern of nodes defines the network's *topology*. Most current cluster systems (Boas 2003) employ fat-tree for scalability and high bandwidth inter-processor communication. In this paper we adopted *m*-port *n*-tree (Lin 2003) as a fixed arity switches to construct the topology for each cluster system. An *m*-port *n*-tree topology consists of N processing nodes and N_{sw} communication switches which can be calculated with Eqs.(1) and (2), respectively.

$$N = 2\times\left(\frac{m}{2}\right)^n \qquad (1)$$

$$N_{sw} = (2n-1)\times\left(\frac{m}{2}\right)^{n-1} \qquad (2)$$

In addition, each communication switch itself has m communication ports $\{0,1,2,...,m-1\}$ that are attached to other switches or processing nodes.

In regards of flow control mechanism, since the dedicated cluster network technologies, e.g., Myrinet, Infiniband and QsNet are using wormhole flow control, so we adopt this mechanism to outline the analytical model. Also, these commercial networks adopt deterministic routing algorithms. Of this, we used a deterministic routing based on Up*/Down* routing (Schroeder 1990) which is proposed in (Javadi et al. 2006). In this algorithm, each message experiences two phases, an *ascending phase* to get to a Nearest Common Ancestor (NCA), followed by a *descending phase*.

Unlike most works on heterogeneous parallel systems, we express the speeds of various nodes in each cluster relatively to a fixed reference machine (Clematis and Corana 1999), and not relatively to the fastest node. Although the latter choice may appear more natural since it makes it possible to obtain the speed-up by comparing performance of the parallel system with that of the fastest single node available, we think that choosing a fixed reference allows clearer performance analysis, especially if we vary the number and/or the power of nodes. Since we consider the processor heterogeneity between each cluster, the total relative speed and the average relative speed of the C clusters in the system is as follows, respectively:

$$S = \sum_{i=0}^{C-1} s^{(i)} \qquad (3)$$

$$\bar{s} = \frac{S}{C} \qquad (4)$$

Where $s^{(i)}$ is the relative speed of a processor in the cluster *i*.

3. THE ANALYTICAL MODEL

In this section, we develop an analytic model for the above mentioned multi-cluster system. The proposed model is built on the basis of the following assumptions which are widely used in the similar studies (Javadi et al. 2005, Sarbazi-Azad et al. 2002; Boura and Das 1997; Hu and Kleinrock 1995):

1. Each processor in cluster *i* generates packets independently, which follows a Poisson process with a mean rate of $\lambda_g^{(i)}$.
2. The arrival process at a given communication network is approximated by an independent Poisson process.
3. The destination of each request would be any node in the system with uniform distribution.
4. The number of processors in all clusters are equal ($N_0 = N_1 ... = N_{C-1}$) and the clusters' nodes are heterogeneous in their speed ($\tau_i = s^{(i)}$).
5. The communication switches are input buffered and each channel is associated with a single flit buffer.
6. Message length is fixed (*M* flits).

3.1. Traffic Analysis

The traffic pattern affects mainly the average message distance, d_{avg}, which is expected number of links that a message makes to reach its destination. The average message distance is generally given by

$$d_{avg} = \sum_{j=1}^{n} 2j \times P_j \qquad (5)$$

Where P_j is the probability of a message crossing 2*j*-link (*j*-link in ascending and *j*-links in descending phase) to reach its destination in a *m*-port *n*-tree topology. As it is mentioned in assumption 3, we take into account the uniform traffic pattern so, based on the *m*-port *n*-tree topology, we can define this probability as follows:

$$P_j = \begin{cases} \dfrac{\left(\dfrac{m}{2}-1\right)\left(\dfrac{m}{2}\right)^{j-1}}{N_0 - 1} & j = 1, 2, \ldots, n-1 \\[2ex] \dfrac{(m-1)\left(\dfrac{m}{2}\right)^{j-1}}{N_0 - 1} & j = n \end{cases} \quad (6)$$

With substituting of Eq.(6) in Eq.(5), the average message distance is obtained as,

$$d_{avg} = \frac{(nm - 2n - 1)\left(\dfrac{m}{2}\right)^n + 1}{\left[\left(\dfrac{m}{2}\right)^n - \dfrac{1}{2}\right]\left(\dfrac{m}{2} - 1\right)} \quad (7)$$

3.2. Arrival Message Rate

The message flow model of the system is shown in Fig. 2, where the path of a flit through various communication networks is illustrated. The processor requests will be directed to the ICN1 and the ECN1 by probabilities $1 - P_o$ and P_o respectively.

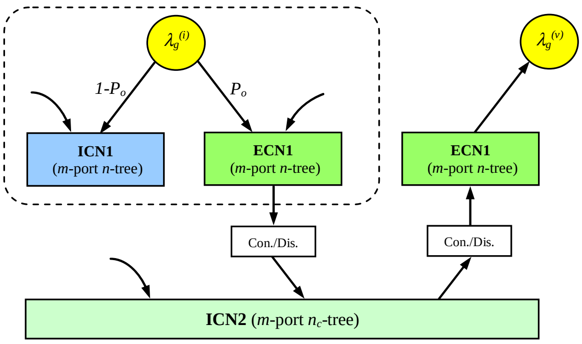

Fig. 2. Message Flow Model in each Communication Network

The external request (out of cluster) of cluster *i* goes through the ECN1 with probability P_o and then ICN2. In the return path, it again accesses the ECN1 in cluster *v* to get to the destination node. So, the message rate received by ICN1 and ECN1 in cluster *i* (to cluster *v*) can be calculated as follows:

$$\lambda_{I1}^{(i)} = (1 - P_o)\lambda_g^{(i)} \quad (8)$$

$$\lambda_{E1}^{(i,v)} = P_o \lambda_g^{(i)} + P_o \lambda_g^{(v)} \qquad v \neq i \quad (9)$$

In the second stage, the input message rate of ICN2 can be computed by following equation:

$$\lambda_{I2}^{(i)} = N_0 P_o \lambda_g^{(i)} \quad (10)$$

We now derive the rate of received messages in each channel, which can be written as:

$$\eta_{I1}^{(i)} = \frac{\left(1 - P_o^{(i)}\right)\lambda_g^{(i)} \times d_{avg(I1)}}{4n} \quad (11)$$

$$\eta_{E1}^{(i,v)} = \frac{P_o\left(\lambda_g^{(i)} + \lambda_g^{(v)}\right) \times d_{avg(E1)}}{4n} \qquad v \neq i \quad (12)$$

$$\eta_{I2}^{(i)} = \frac{N_0 P_o \lambda_g^{(i)} \times d_{avg(I2)}}{4n_c} \quad (13)$$

Where n_c is the number of tree in ICN2 and can be computed by Eq(1). In the Eq.(8) to Eq.(13), the probability P_o has been used as the probability of outgoing request within a cluster. According to assumption 3, this parameter is obtained by the following equation:

$$P_o = \frac{\sum_{i=1}^{C-1} N_i}{N - 1} = \frac{(C-1) \times N_0}{C \times N_0 - 1} \quad (14)$$

3.3. Average Network Latency

In this section, we find the average latency of each communication network from cluster *i* point of view, $\overline{T}^{(i)}$. Since each message may cross different number of hops to reach its destination, we consider the network latency of an 2*j*-hop message as $T_j^{(i)}$, and averaging over all the possible nodes destined made by a message yields the average message latency as:

$$\overline{T}^{(i)} = \sum_{j=1}^{n}\left(P_j \times T_j^{(i)}\right) \quad (15)$$

Where P_j can be calculated from Eq.(6). Our analysis begins at the last stage and continues backward to the first stage. The network stage numbering is based on location of switches between the source and the destination nodes. It is obvious that in *m*-port *n*-tree topology, the number of stages for 2*j*-hop journey is $K = 2j - 1$. It should be noted that, in this topology we have two types of connections, node to switch (or switch to node) and switch to switch. Each type of connection has a service time which is approximated as follows:

$$t_{cn} = \frac{1}{2}\alpha_{net} + L_m \beta_{net} \quad (16)$$

$$t_{cs} = \alpha_{sw} + L_m \beta_{net} \quad (17)$$

Where t_{cn} and t_{cs} represent times to transmit from node to switch (or switch to node) and switch to switch connection, respectively. α_{net} and α_{sw} are the network and switch latency, β_{net} is the transmission time of one byte (inverse of bandwidth) and L_m is the length of each flit in bytes.

The destination, stage $K-1$, is always able to receive a message, so the service time given to a message at the final stage is t_{cn}. At stage *k*, the average amount of time that a message waits to acquire a channel for cluster *i*, $W_{k,j}^{(i)}$, is given by the product of the channel blocking probability, $P_{B_{k,j}}{}^{(i)}$, and the average service time, $S_{k,j}{}^{(i)}/2$ (Dally and Towles 2004):

$$W_{k,j}{}^{(i)} = \frac{1}{2} S_{k,j}{}^{(i)} P_{B_{k,j}}{}^{(i)} \quad (18)$$

The value of $P_{B_{k,j}}{}^{(i)}$ is determined using a birth-death Markov chain (Kleinrock 1975). As it can be seen in Fig. 3, the rate of transition out and into the first state is $\eta_k{}^{(i)}$ and $1/S_{k,j}{}^{(i)} - \eta_k{}^{(i)}$ respectively.

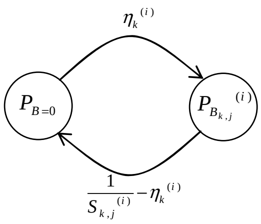

Fig. 3. Markov Chain to Calculate Blocking Probabilities

Solving this chain for the steady state probabilities gives:

$$P_{B_{k,j}}{}^{(i)} = \eta_k{}^{(i)} S_{k,j}{}^{(i)} \quad (19)$$

The average service time of a message at stage k is equal to the message transfer time and waiting time at subsequent stages to acquire a channel, so:

$$S_{k,j}{}^{(i)} = \begin{cases} Mt_{cn} & k = K-1 \\ \sum_{l=k+1}^{K-1} \left(W_{l,j}{}^{(i)} \right) + Mt_{cs} & \text{otherwise} \end{cases} \quad (20)$$

According to this equation, the average service time of a message, not being in the first stage, is equal to average network latency. Hence, $T_j{}^{(i)} = S_{0,j}{}^{(i)}$.

A message originating from a given source node in cluster i sees a network latency of $\overline{T}^{(i)}$. Due to blocking situation that takes place in the network, the distribution function of message latency becomes general. Therefore, a channel at source node is modeled as an M/G/1 queue. The average waiting time for an M/G/1 queue is given by (Kleinrock 1975):

$$\overline{W}_s{}^{(i)} = \frac{\rho^{(i)} \overline{x}^{(i)} \left(1 + \frac{\sigma_{\overline{x}}^{2(i)}}{\overline{x}^{2(i)}} \right)}{2\left(1 - \rho^{(i)}\right)} \quad (21)$$

$$\rho^{(i)} = \lambda^{(i)} \overline{x}^{(i)} \quad (22)$$

Where $\lambda^{(i)}$ is the average arrival rate on the network, $\overline{x}^{(i)}$ is the average service time, and $\sigma_{\overline{x}}^{2(i)}$ is the variance of the service time distribution. Since the minimum service time of a message at the first stage is equal to Mt_{cn}, the variance of the service time distribution is approximated based on a method proposed in (Draper and Ghosh 1994),

$$\sigma_{\overline{x}}^{2(i)} = \left(\overline{T}^{(i)} - Mt_{cn} \right)^2 \quad (23)$$

As a result, the average waiting time in the source queue becomes,

$$\overline{W}_s{}^{(i)} = \frac{\lambda^{(i)} \left(\overline{T}^{(i)} \right)^2 \left(1 + \frac{\left(\overline{T}^{(i)} - Mt_{cn} \right)^2}{\left(\overline{T}^{(i)} \right)^2} \right)}{2\left(1 - \lambda^{(i)} \overline{T}^{(i)}\right)} \quad (24)$$

Finally, the average message latency, $\overline{L}^{(i)}$, seen by the message crossing from source node from cluster i to its destination, consists of three parts; the average waiting time at the source queue ($\overline{W}_s{}^{(i)}$), the average network latency ($\overline{T}^{(i)}$), and the average time for the tail to reach the destination. Therefore,

$$\overline{L}^{(i)} = \overline{W}_s{}^{(i)} + \overline{T}^{(i)} + \sum_{k=1}^{d_{avg}-2} t_{cs} + t_{cn} \quad (25)$$

The average message latency in the ICN1 from cluster i point of view would be found by Eq.(25) with substitution of $\eta_k{}^{(i)} = \eta_{I1}{}^{(i)}$, $\lambda^{(i)} = \lambda_{I1}{}^{(i)}$, and $d_{avg} = d_{avg(I1)}$.

3.3.1. Average Latency in the Inter-Cluster Networks

As mentioned before, external messages cross through both networks, ECN1 and ICN2, to get to the destination in other cluster. Since the flow control mechanism is wormhole, the latency of these networks should be calculated as a merge one. Of this and based on the Eq.(15) we can write,

$$\overline{T}_{E1\&I2}{}^{(i)} = \frac{1}{C-1} \sum_{v=0, v \neq i}^{C-1} \left(\sum_{j=1}^{n} \sum_{h=1}^{n_c} \left(P_{j+h} \times T_{j+h}{}^{(i)} \right) \right) \quad (26)$$

Where the probability P_{j+h} is,

$$P_{j+h} = P_j \times P_h \quad (27)$$

Where P_j and P_h can be calculated from Eq.(6). The average network latency of inter-cluster networks can be founded with the equations which are presented in the previous section by following substitutions:

$$K = 2(j+h) - 1 \quad (28)$$

$$\eta_k{}^{(i)} = \begin{cases} \eta_{I2}{}^{(i)} & j \le k < j + 2h - 1 \\ \eta_{E1}{}^{(i,v)} & \text{otherwise} \end{cases} \quad (29)$$

$$\lambda^{(i)} = \lambda_{E1}{}^{(i,v)} \quad (30)$$

$$d_{avg} = d_{avg(E1)} + d_{avg(I2)} \quad (31)$$

3.3.2. Average Waiting Time at the Concentrator/Dispatcher

The average waiting time at the concentrator/dispatcher is calculated in a similar manner to that for the source queue (Eq.(21)). By modeling the injection channel in the concentrator/dispatcher as an M/G/1 queue, the average arrival rate and average waiting time are given by following equations:

$$\overline{W}_{con}^{(i)} = \frac{\lambda_{I2}^{(i)}\left(\overline{T}_{I2}^{(i)}\right)^2\left(1+\frac{\left(\overline{T}_{I2}^{(i)}-Mt_{cs}\right)^2}{\left(\overline{T}_{I2}^{(i)}\right)^2}\right)}{2\left(1-\lambda_{I2}^{(i)}\overline{T}_{I2}^{(i)}\right)} \quad (32)$$

Where $\overline{T}_{I2}^{(i)}$ is the average network latency of the ICN2 from cluster *i* point of view. Also, we model the ejection channel in the concentrator/dispatcher as an M/G/1 queue, with the same rate of injection channel. So, the service time of the queue would be Mt_{cs} and there is no variance in the service time, since the messages length is fixed. Hence,

$$\overline{W}_{dis}^{(i)} = \frac{\lambda_{I2}^{(i)}\left(Mt_{cs}\right)^2}{2\left(1-\lambda_{I2}^{(i)}Mt_{cs}\right)} \quad (33)$$

The sum of the two above mentioned waiting times gives average waiting time at the concentrators/dispatchers as follows:

$$\overline{W}_{c}^{(i)} = \overline{W}_{con}^{(i)} + \overline{W}_{dis}^{(i)} \quad (34)$$

Putting all together, we could find the average message latency of cluster *i* based on Fig. 2 with the following equation:

$$\overline{\ell}^{(i)} = \left(1-P_o\right)\left(\overline{L}_{I1}^{(i)}\right) + P_o\left(\overline{L}_{E1\&I2}^{(i)} + \overline{W}_{c}^{(i)}\right) \quad (35)$$

To calculate the total average of message latency, we use a weighted arithmetic average as follows:

$$\overline{\ell} = \sum_{i=0}^{C-1}\left(\frac{s^{(i)}}{S}\times\overline{\ell}^{(i)}\right) \quad (36)$$

At last, to perform our analysis we chose to express the degree of heterogeneity of the system through a single parameter, i.e., the standard deviation of relative speeds as follows:

$$H = \sqrt{\frac{1}{C}\sum_{i=0}^{C-1}\left(s^{(i)}-\overline{s}\right)^2} \quad (37)$$

4. VALIDATION OF THE MODEL

In order to validate the proposed model and justify the applied approximations, the model was simulated. Requests are generated randomly by each processor with an exponential distribution of inter-arrival time with a rate of $\lambda_g^{(i)}$. The destination node is determined by using a uniform random number generator. For each simulation experiment, statistics were gathered for a total number of 100,000 messages. Statistic gathering was inhibited for the first 10,000 messages to avoid distortions due to the *warm-up* phase.

Extensive validation experiments have been performed for several combinations of clusters sizes, network sizes, message length, and degree of heterogeneity. The general conclusions have been found to be consistent across all the cases considered. After all, to illustrate the result of some specific cases to show the validity of our model, the items which were examined carefully are as follows:

- *System size: N=2^9 and N=2^{10}*
- *Cluster size: C=2^4 and C=2^5*
- *Total relative speed: S=C*
- *Switch size: m=4 and m=8 ports*
- *Message length: M=64 flits*
- *Flit length: Lm=256 and 512 bytes*
- *Network technology bandwidth: 500/time unit*
- *Network latency: 0.02 time unit*
- *Switch latency: 0.01 time unit*

The results of simulation and analysis for the systems with above mentioned parameters are depicted in Fig. 4 to Fig. 7 in which the average message latencies are plotted against the offered traffic with different values for degree of heterogeneity.

The figures reveal that the analytical model predicts the average message latency with a good degree of accuracy when the system is in the steady state region, that is, when it has not reached the saturation point. However, there are discrepancies in the results provided by the model and the simulation when the system is under heavy traffic and approaches the saturation point. This is due to the approximations that have been made in the analysis to ease the model development. Since, the most evaluation studies focus on network performance in the steady state regions, so we can conclude that the proposed model can be a practical evaluation tool that can help system designer to explore the design space and examine various design parameters.

5. CONCLUSIONS

Analytical models play a crucial role in evaluation of a system under various design issues. In this paper, an analytical model of interconnection networks for multi-cluster computing systems is discussed. The proposed model has been validated with versatile configurations and design parameters. Simulation experiments have proved that the model predicts message latency with a reasonable accuracy. For future work, we intent to take the non-uniform traffic pattern into account, which is closer to the real traffic in such systems.

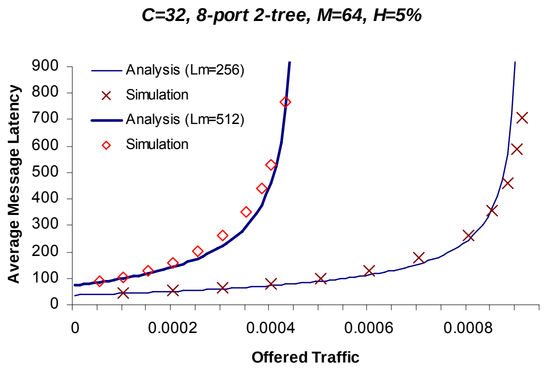

Fig. 4. Avg. Latency in a System with $N=2^{10}$, M=64, H=5%

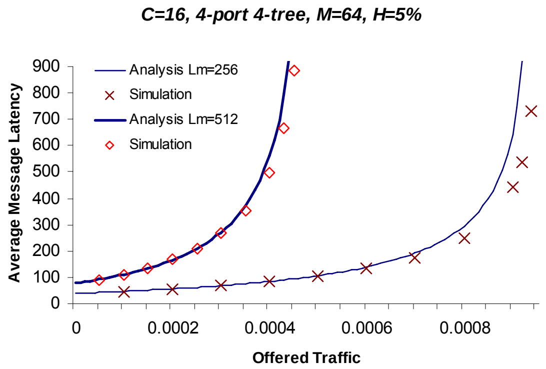

Fig. 5. Avg. Latency in a System with $N=2^9$, M=64, H=5%

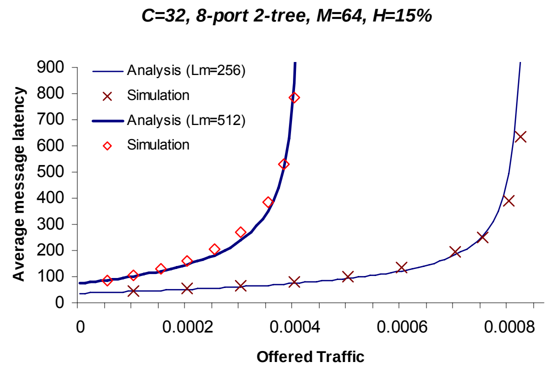

Fig. 6. Avg. Latency in a System with $N=2^{10}$, M=64, H=15%

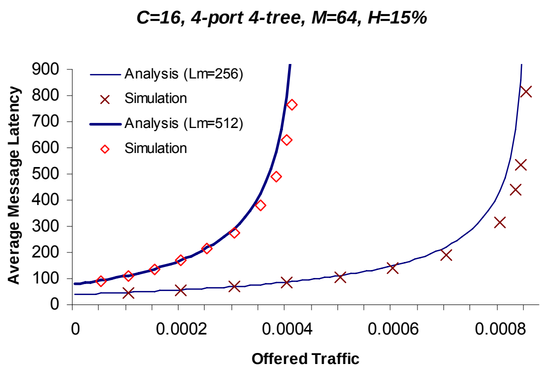

Fig. 7. Avg. Latency in a System with $N=2^9$, M=64, H=15%

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QUEUEING MODEL OF THE AAL2 MULTIPLEXER IN UTRAN

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KEYWORDS

UMTS, ATM, AAL2 multiplexer, Timer_CU, BMAP/D/1 queue, matrix analytic solution

ABSTRACT

This paper presents a queueing model for the performance evaluation of the AAL2 multiplexer in UTRAN (UMTS Terrestrial Radio Access Network). Based on the model analytical expressions are provided for the distribution of the waiting time and for the multiplexing efficiency. The performance of the AAL2 multiplexer is studied, especially the impact of the Timer_CU on the QoS parameters. The analytical results are verified with simulations.

1 INTRODUCTION

As UMTS (Universal Mobile Telecommunication System) is currently under deployment, the system's performance improvement and optimization is of special interest. In addition to introducing improvements that aim to increase the system capacity and the users' peak throughput for data services, the optimal usage of transport resources must be also achieved. The WCDMA radio control functions are imposing strict delay requirements [3] both for RT (real-time) and NRT (non real-time) services over the transport network between the RNC (Radio Network Controller) and Node B (Base Station). These requirements should be guaranteed with the maximal utilization of the transport capacity that is a limited resource especially on the last mile links (typically 1xE1 or 2xE1).

ATM/AAL2 (Asynchronous Transfer Mode/ATM Adaptation Layer Type 2 [1]) has been selected as transport network layer for the UMTS Iub interface that connects the RNC and the Node B, because it is able to multiplex several voice and data connections into one VCC (Virtual Circuit Connection); improving in this way the utilization of the transport network. The delay on the transport network consists of the AAL2 multiplexing delay (the delay on the AAL2 layer) and the delay on the ATM layer. Assuming CBR (Constant Bit-Rate) VCCs we can consider that the former is the dominating delay component on the transport network. If no AAL2 switching is implemented in the transport network, the delay on the AAL2 multiplexer should be analyzed at the RNC for the downlink traffic. Since the traffic on the AAL2/ATM transport network is delay sensitive, the buffering delay caused by the AAL2 multiplexer is allowed to exceed a given value (maximum allowed delay) only with low probability. The maximum allowed delay and the probability of exceeding it are the basis of the ATM VCC and link capacity dimensioning, and also the reference value when the quality of the transport network is evaluated. Additionally, the multiplexing efficiency is evaluated with a measure called packing density. The efficiency is the highest when the payload of the ATM cells is fully utilized (does not contain padding). In order to increase the efficiency, a timer (called Timer_CU) is introduced at the AAL2 multiplexer.

The performance of AAL2 multiplexing with different traffic types and multiplexer settings was examined analytically in [12], [6], [13] and [10], by simulation [8], [9], [11], [15] or by both analytical models and simulation [7].

While some papers are focusing on the buffer requirement and delay issues [15] others are investigating the differences between Assembly Before Transmission (ABT) and Combined Assembly and Transmission (CAT) multiplexing method and studying the effect of changing the Timer_CU value [9], [7] or the various scheduling algorithms that can be used [8]. Analytical model of the ABT architecture with Timer_CU assuming Poisson arrivals is described in [12]. Another analytical model is provided by [13] with Poisson arrivals and no Timer_CU. Batch Bernoulli traffic model, no Timer_CU and frame sizes multiple of the ATM cell payload is assumed in [10] and [6]. Some papers are comparing the performance of different adaptation layers (AAL1, AAL2, AAL5) in case of transporting voice over ATM network [11], [15]. According to our knowledge, there are no research results published on the performance analysis of CAT AAL2 multiplexers with less restrictive traffic models.

In this paper we introduce and analyze a CAT AAL2 multiplexer with Timer_CU. The traffic model is a batch markovian arrival process (BMAP). We show that the embedded process at departures is a Markov chain of M/G/1 type, which can be efficiently analyzed by matrix geometric methods. We provide methods to compute the two most crucial performance measures of the AAL2 multiplexer, namely the distribution of the waiting time and the multiplexing efficiency.

The rest of the paper is organized as follows. Section II discusses the AAL2 multiplexer. Section III provides a detailed overview of the analysis of the BMAP/D/1-Timer queuing system. Numerical results are summarized in Section IV. Section V concludes the paper.

2 THE AAL2 MULTIPLEXER

The user plan protocol stack of RNC - Node B interface (Iub) consists of Radio Network and Transport Network Layers [1]. FP (Frame Protocol) creates frames out of the user traffic mapped into Dedicated Channels (DCH) and sends them through the transport bearers i.e. AAL2 connections (Figure 2) at each Transmission Time Interval (TTI).

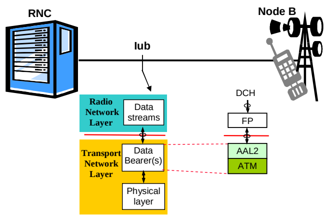

Figure 1: Protocol stack of the Iub interface

The AAL2 layer is multiplexing up to 248 connections into one ATM VCC. The incoming FP frames (AAL2-SDUs) are segmented by the Service Specific Convergence Sublayer (SSCS) into maximum 45 byte segments. The Common Part Sublayer (CPS) encapsulates these segments by adding a 3 bytes header. The encapsulated segments are called CPS-Packets and their size is at maximum 48 bytes. The AAL2 multiplexer puts the CPS-Packets into CPS-PDUs (with 1 byte header referred to as STF), which are in fact the payload of the ATM cells and their maximum size is 48 bytes. This means that a full sized CPS-Packet can only be transported in two ATM cells. In order to increase the multiplexing gain a timer (Timer_ CU) is initialized whenever the CPS-PDU is smaller than 48 bytes i.e. the CPS-Packets under assembly are not filling an ATM cell.

The ATM cell is padded and sent when Timer_CU expires and there is no new CPS-Packet arriving to the multiplexer. Setting larger value for the Timer_CU results in larger delay and higher multiplexing gain, i.e., the load of ATM links is reduced.

This paper focuses on the analysis of the AAL2 multiplexing delay and multiplexing efficiency as the function of the available bandwidth and on the impact of the Timer CU in case of CBR VCC.

3 ANALYSIS OF THE AAL2 MULTIPLEXER MODEL

Based on the description presented in the previous section, we have created a queueing model for the AAL2 multiplexer.

In this paper we assume that the inter-arrival time and size variation of the CPS-Packets is given by a BMAP traffic descriptor. The CPS-Packets are stored in the multiplexing buffer. In our model, the buffer size is measured in bytes, thus if we say that the queue length is k, it means that k bytes are waiting in the buffer.

The server transmits the CPS-Packets multiplexed into CPS-PDUs. The size of the CPS-PDU payload L is constant ($L = 47$ bytes). As soon as a CPS-PDU is assembled, it is encapsulated into an ATM cell, and transmitted (see Figure 2). Since the transmission happens on a constant bit rate channel (CBR VCC), the service time is deterministic (Δ).

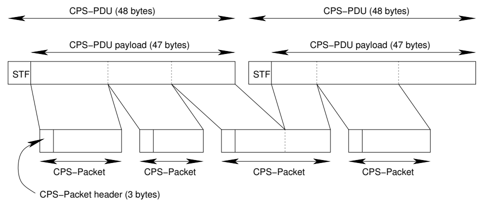

Figure 2: Multiplexing CPS-Packets into ATM cells

To model the buffer of the multiplexer, we apply the BMAP/D/1-Timer queueing model. This system is introduced in [5]. In that paper the derivation and a numerical model for the waiting time distribution is provided. In the following sections we briefly summarize the results of that paper, and extend it to compute the multiplexing efficiency.

3.1 Traffic Model

As mentioned above, the traffic arriving to the multiplexer is described by a continuous-time batch markovian arrival process with m phases. Its $m \times m$ generator is denoted by D. The arrival process itself is characterized by a set of matrices D_i ($\sum_{i=0}^{K} D_i = D$), where $[D_k]_{i,j}$ corresponds to the arrival of a k bytes long CPS-Packet followed by a state transition from i to j.

The probability that there are n arrivals in time t with the arrival process being in state i at the beginning and in state j at time t is denoted by $[P(k,t)]_{i,j}$.

The BMAP is a strong modeling tool for traffic description in markovian analysis. It can be created based on measured or approximated real traffic behavior (see [4]). Although deterministic traffic can not be captured accurately by BMAPs, former research ([14]) pointed out that the superposition of deterministic sources behaves as a Poisson process (the more on-off modulated deterministic traffics are superposed, the closer is the aggregate to the Poisson process).

The BMAP characterization of the traffic in UTRAN is out of scope of this paper, it is the subject of further research.

3.2 Behavior of the Timer

The timer ensures that when data reaches the head of the queue (positions 1 to L, thus it will be included in the next CPS-PDU), it will be served in time T, even if there is not enough data to fill a complete CPS-PDU when the timer expires. In that case, the server pads up the a partially filled CPS-PDU, and transmits it with the same service time (Δ).

To give a better description of the service mechanism, we summarize the behavior in the following three points. These three points will be referred many times in the sequel, and will be used as three cases requiring different treatments during the analysis:

P1. While there are $\geq L$ bytes in the queue, the timer is not started. The server takes the first L bytes out of the queue, compiles the CPS-PDU, and starts the transmission.

P2. If the queue size is $< L$ but > 0 when the transmission begins (after taking out the ones whose transmission has started), the timer is started. When the service is ready and the queue is still $< L$, no new service begins until the timer elapses, or the sufficient amount of data arrives.

P3. If an arrival happens when the queue is empty, and the batch size of the arrival is below L, the timer is started immediately. The first service can start when the the queue size exceeds L, or when the timer elapses. Of course the service can not start until the CPS-PDU under transmission does not leave the multiplexer.

3.3 Queue Length Distribution at Packet Departures

The state of the multiplexer buffer can be characterized by a discrete time Markov chain at CPS-PDU assembly instants. These CPS-PDU assembly instants can be translated as data departure events from the multiplexer buffer. Embedding at departures usually leads to a so called M/G/1 type structure, as it does in our case, too. The transition probability matrix builds up as follows:

$$\boldsymbol{\mathcal{X}} = \begin{bmatrix} \boxed{\quad\quad\quad\boldsymbol{\mathcal{B}}\quad\quad\quad} & \dots \\ \boxed{\quad\quad\quad\boldsymbol{\mathcal{A}}\quad\quad\quad} & \dots \\ \quad\boxed{\quad\quad\boldsymbol{\mathcal{A}}\quad\quad} & \dots \\ \quad\quad\boxed{\quad\boldsymbol{\mathcal{A}}\quad} & \dots \\ \quad\quad\quad\boxed{\boldsymbol{\mathcal{A}}} & \dots \\ \ddots & \vdots \end{bmatrix}. \qquad (1)$$

Usually M/G/1 type matrices are defined by their quadratic matrix blocks. Now we define the matrix by its block rows, because it simplifies the definition. The states inside the $m \times m$ blocks are reflecting the state of the arrival process; a transition from block i to block j means that the queue size changed from i to j since the last departure instant.

Matrix row $\boldsymbol{\mathcal{A}}$ corresponds to case P1 of Section 3.2. In this case the inter departure time is exactly Δ, since the timer does not play a role. At the next embedded point the server will decrease the queue by L, so the queue size change equals to the arrivals during Δ minus L. Thus, $\boldsymbol{\mathcal{A}}$ is defined by:

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} P(0,\Delta) & P(1,\Delta) & P(2,\Delta) & \dots \\ & P(0,\Delta) & P(1,\Delta) & \dots \\ & \ddots & & \vdots \\ & & P(0,\Delta) & \dots \end{bmatrix}$$

(This matrix has $L \times m$ rows).

The definition of $\boldsymbol{\mathcal{B}}$ is more complex due to the effect of the timer. We further divide $\boldsymbol{\mathcal{B}}$, to distinguish between the completely idle (P3) and not completely idle (P2) cases:

$$\boldsymbol{\mathcal{B}} = \left[\begin{array}{c} \hat{\boldsymbol{\mathcal{B}}} \\ \hline \tilde{\boldsymbol{\mathcal{B}}} \end{array}\right]. \qquad (2)$$

The first block row (with height m) describes the transitions from the idle buffer (these matrices are denoted by $\hat{\boldsymbol{\mathcal{B}}}$, and correspond to P3 in Section 3.2), the other rows describe transitions from buffer levels $1 - L$-1 (P2 in Section 3.2).

In both cases the timer is started, and the next transition may begin no sooner than Δ. The evolution of the queue size between levels 1 and L is important to capture the effect of the timer. Therefore we define the continuous time Markov chain generator $\boldsymbol{Q}$, that follows the queue size increase process between 1 and L:

$$\boldsymbol{Q} = \begin{bmatrix} D_0 & D_1 & \dots & D_{L-2} \\ & D_0 & \dots & D_{L-3} \\ & & \ddots & \vdots \\ & & & D_0 \end{bmatrix}, \qquad (3)$$

and $\boldsymbol{Z}(t)$ which is the transition probability matrix of the buffer size increase process during time t, thus, $\boldsymbol{Z}(t) = e^{\boldsymbol{Q}t}$.

The behavior of the timer is described by $\boldsymbol{\Pi}(t)$. $[\boldsymbol{\Pi}(t)]_{i,j}$ is the probability that starting with i as initial state (with buffer length between 1 and $L-1$) the state of system will be j just after the start of the next service. The next service can start when the necessary number of bytes have arrived until time t (first term in eq. (4)), or at time t the buffer content is served even if a full CPS-PDU can not be created (second term of eq. (4)). This probability matrix is computed by:

$$\mathbf{\Pi}(t) =$$

$$\int_0^t e^{\boldsymbol{Q}\tau} d\tau \cdot \begin{bmatrix} D_{L-1} & \dots & D_K & & & \\ D_{L-2} & \dots & D_{K-1} & D_K & & \\ \vdots & \ddots & \dots & \dots & D_K & \\ D_1 & \dots & \dots & \dots & \dots & D_K \end{bmatrix} \qquad (4)$$

$$+ e^{\boldsymbol{Q}t} \cdot \begin{bmatrix} I_{m\times m} & 0 & 0 & \dots \\ I_{m\times m} & 0 & 0 & \dots \\ \vdots & \ddots & & \\ I_{m\times m} & 0 & 0 & \dots \end{bmatrix}.$$

If the buffer size is between 1 and $L-1$ ($\tilde{\boldsymbol{\mathcal{B}}}$), the number of arrivals during the Δ interval (during which the server is busy) has to be investigated. If the arriving CPS-Packets increase the buffer above L, the assembly and transmission of a new CPS-PDU starts just after finishing the previous one (this events are expressed by the first matrix term of eq. (5)). If the buffer level is still below L (second term of eq. (5), an additional delay follows, with a maximal length of $(T-\Delta)^+$, since the timer started at the moment when the buffer decreased below L. Thus:

$$\tilde{\boldsymbol{\mathcal{B}}} = \begin{bmatrix} P(L-1,\Delta) & P(L,\Delta) & \dots \\ P(L-2,\Delta) & P(L-1,\Delta) & \dots \\ \vdots & \vdots & \dots \\ P(1,\Delta) & P(2,\Delta) & \dots \end{bmatrix} + \qquad (5)$$
$$+ \boldsymbol{Z}(\Delta) \cdot \mathbf{\Pi}((T-\Delta)^+).$$

If the buffer empties at the beginning of the service of a packet ($\hat{\boldsymbol{\mathcal{B}}}$), we have two cases. First, it is possible that there are no arrivals during the server occupancy time (Δ). In this case the next arrival can bring the queue above L, immediately causing a start of the transmission of a CPS-PDU (first term of eq. (6)); or the queue remains below L, and a waiting period for more CPS-Packets follows with a maximum length given by T, since the arrival into the empty queue initiated the timer (second term of eq. (6)). The second case is when there were arrivals during the server occupancy time. In this case the arrivals can bring the queue above L, and the service of a new CPS-PDU starts just after the end of service of the previous one (third term of eq. (6)). The arrival can leave the system below L, and a waiting period is started with a maximal length of $(T-(\Delta-\tau))^+$, since the timer started at the first arrival time (τ) and from the timer $\Delta-\tau$ time already expired when the server becomes empty (fourth term of eq. (6)). Thus we have:

$$\begin{aligned} \hat{\boldsymbol{\mathcal{B}}} = {} & P(0,\Delta)(-D_0)^{-1} \; D_L \quad D_{L+1} \quad \dots \quad D_K \; + \\ & + P(0,\Delta)(-D_0)^{-1} \; D_1 \quad D_2 \quad \dots \quad D_{L-1} \; \cdot \mathbf{\Pi}(T) + \\ & + \; P(L,\Delta) \quad P(L+1,\Delta) \quad \dots \; + \\ & + \int_0^\Delta e^{D_0\tau} \cdot \; D_1 \quad D_2 \quad \dots \quad D_{L-1} \; \cdot \boldsymbol{Z}(\Delta-\tau) \cdot \\ & \cdot \mathbf{\Pi}((T-\Delta+\tau)^+) d\tau. \end{aligned} \qquad (6)$$

The steady state distribution of the embedded Markov chain (1) is partitioned the following way:

$$\boldsymbol{x} = \Big[\underbrace{p_0 \quad p_1 \quad \dots \quad p_{L-1}}_{\boldsymbol{x}_0} \; \underbrace{p_L \quad p_{L+1} \quad \dots \quad p_{2L-1}}_{\boldsymbol{x}_1} \; \dots \Big],$$

where p_i is a vector of size m. The steady state probability vector can be efficiently obtained by a matrix analytic method summarized in [5].

3.4 Waiting Time Distribution

The waiting time distribution $P(W > w)$ is the probability that the waiting time of an arriving CPS-Packet (measured from its arrival to its departure) exceeds a given threshold w. It can be easily computed if some parameters are kept fixed.

These parameters are: the length of the buffer at the arrival (k), the remaining server occupation time (t_1), and the maximal departure time measured from the point when the the buffer descends below level L (t_2). If we know the particular values of these parameters, the waiting time distribution $P_W(k,t_1,t_2)$ can be computed by the following way:

$$P_W(k,t_1,t_2) =$$
$$\begin{cases} h & \text{if } w < t_1 + \frac{k}{L}\ \Delta, \\ 0 & \text{if } k < L, w \geq t_1 + t_2 + \frac{k}{L}\ \Delta, \\ & \text{if } k \geq L, w \geq t_1 + (T-\Delta)^+ + \frac{k}{L}\ \Delta, \\ \left[e^{\boldsymbol{Q}(W^*-\Delta)} h \right]_{\{k/L\}} & \text{otherwise,} \end{cases}$$

where h is a vector of ones with size m.

The first item corresponds to the case when the server occupancy time plus the service time of the CPS-Packets in the queue exceeds the waiting time requirement. In this case $P(W > w)$ equals to one.

The second item covers the case when the waiting time requirement is surely satisfied. This happens if the waiting time requirement is larger than the server occupancy time, plus the service time of the packets in the queue, plus the maximal possible delay caused by the timer. The latter quantity is t_2 if the buffer size is less than L (this is the definition of t_2). It is $(T-\Delta)^+$ if $k \geq L$, because the server will be occupied when the last CPS packet gets below L in the buffer.

The third item means that the waiting time exceeds w if the L long block in the queue, in which k belongs to, is still not filled up until $w-\Delta$. $[e^{\boldsymbol{Q}t}]_i$ is the probability that the arrival process did not generate enough arrivals to leave this block until time t, if there were i bytes in the block at the beginning. $\{k/L\}$ (where $\{\}$ denotes the remainder of the division) is the buffer position inside the L long block after the arrival.

To obtain the waiting time distribution, we have to multiply $P_W(k,t_1,t_2)$ by the probability of the given k, t_1 and t_2 parameters. For all the details and an efficient numerical method see [5].

3.5 Multiplexing efficiency

The multiplexing efficiency is characterized by $\eta \in (1/L, 1]$. It equals to 1, if the payloads of all departing ATM cells are fully utilized, and η is small if the departing ATM cells are containing only few bytes as useful payload. The multiplexing efficiency is calculated from of the departure rate μ and the arrival rate λ as $\eta = \lambda/(L\mu)$.

In the rest of this section we compute the mean departure time $E(D)$ that is the inverse of μ. Again three cases are distinguished, according to P1, P2 and P3. To model the effect of the timer to the mean departure time, we introduce the $m \times L$–1 sized vector $E(W_T(t))$. The kth m sized block in $E(W_T(t))$ is the mean waiting time till the buffer size increases above L, or it is t if it is still less than L at t. $E(W_T(t))$ is computed by:

$$E(W_T(t)) = \int_0^t \tau e^{\boldsymbol{Q}\tau} d\tau \begin{bmatrix} \sum_{k=L-1}^{\infty} D_k h \\ \sum_{k=L-2}^{\infty} D_k h \\ \vdots \\ \sum_{k=1}^{\infty} D_k h \end{bmatrix} + t e^{\boldsymbol{Q}t} h_{m \times L-2}$$

P1. $E(D_1)$ is the mean departure time under the condition that at the last departure the queue size was not less than L. The departure time is Δ, since there are enough bytes to assemble a new CPS-PDU just after finishing the last one:

$$E(D_1) = \Delta h_m,$$

where h_m means a vector of ones with size m.

P2. $E(D_2)$ is the mean departure time under the condition that at the last departure the queue size was between 1 to $L-1$. The server is occupied for time Δ (first term of eq. (7)). If the queue size remains $< L$ when the server becomes idle, the departure time increases due to the timer (second term of eq. (7)):

$$E(D_2) = \Delta h_{m \times L-2} + Z(\Delta) E(W_T((T-\Delta)^+)). \quad (7)$$

P3. $E(D_3)$ is the mean departure time under the condition that the last departure has left the queue empty. The server is busy for time Δ (first term of eq. (8)). If the first arrival arrives in $(0, \Delta)$, and the queue is still below L at Δ, the departure time is increased by and additional delay caused by the timer. This is reflected by the second term of eq. (8). If the first arrival arrives after Δ, and its size is less than L, the timer is started. In this case the departure time is increased by the time of the first arrival (its mean value is $(-D_0)^{-1}$, and by the additional delay of the timer (third term of eq. (8)):

$$\begin{aligned} E(D_3) &= \Delta h_m + \\ &+ \int_0^{\Delta} e^{D_0 t} [D_1 \dots D_{L-1}] Z(\Delta - t) E(W_T((T-\Delta+t)^+) dt + \\ &+ e^{D_0 \Delta} \left((-D_0)^{-1} h_m + (-D_0)^{-1} [D_1 \dots D_{L-1}] E(W_T(T)) \right). \end{aligned} \quad (8)$$

Using these conditional mean departure times, the departure intensity is expressed by:

$$\mu = \frac{1}{p_0 E(D_3) + [p_1 \dots p_{L-1}] E(D_2) + \sum_{k=L}^{\infty} p_k E(D_1)}$$

4 NUMERICAL RESULTS

We have implemented the computation method in MATLAB, and also wrote a simulation tool in Omnet++ ([2]) to check the correctness of both the expressions and the MATLAB implementation. Most of the figures in this section are showing both the MATLAB (with lines) and the simulation results (indicated with points).

In the numerical examples, the BMAP of the packet arrivals is a "naive" model of N AMR12.2 voice channels ($N = 5$). These voice channels have an "on-off" behavior, with exponentially distributed "on" and "off" durations. The mean "on" period is 1.5 sec and the mean "off" period is 1.0 sec. During the "on" period the inter arrival times are exponentially distributed with a mean of 20 milliseconds. A (deterministic) 37 byte long data frame (including 5 byte FP header) is generated at each arrival. The 37 bytes long data frame together with the 3 bytes long CPS-Packet header gives 40 bytes long CPS-Packets.

To decrease the computation time, we compress all size-related quantities by 6. Thus, the size of the CPS-PDU is $L = 47/6 \approx 8$, and the size of the CPS-Packets is $40/6 \approx 7$ in the following examples.

In our first example the service time is varying between 0 and 5 ms, and we examine the probability of exceeding the $w = 5$ ms waiting time. The results are depicted in Figure 3.

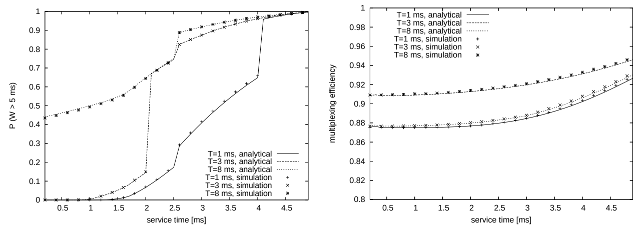

Figure 3: $P(W > 5$ ms) and the multiplexing efficiency vs. the service time

There are probability masses at $\Delta = w/2 = 2.5$ and at $\Delta = w - T$. For the exhaustive explanation of this phenomenon see [5]. From an engineering aspect, Figure 3 reflects also that if $T > w$ holds, the waiting time requirement can not be satisfied even by infinite link capacity (see the curve of $T = 8$ ms), it does not decrease to 0 as the service time tends to 0 (that is, the link capacity tends to infinity). Figure 3 shows that the multiplexing efficiency is better with higher timer values.

In the last example we investigate the effect of increasing the input traffic of the queue by increasing parameter N. Figure 4 shows that if $T + \Delta < w$ holds, the waiting time increases, because the queue size increases. But, if $T + \Delta > w$ (as at $T = 8$), the effect is the opposite, since with increasing traffic the probability that the packet is sent due to the timer – thus, the probability of exceeding the 5 ms requirement – decreases.

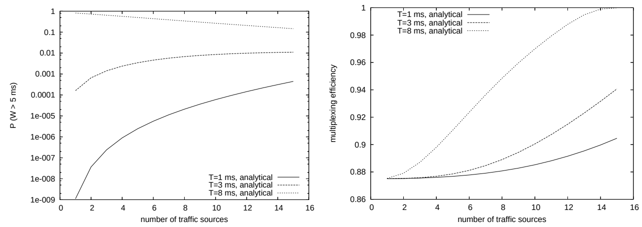

Figure 4: $P(W > 5$ ms) and the multiplexing efficiency vs. number of traffic sources

In all the examples the computation of one point of the waiting time distribution took few seconds on a Pentium4 2.4 GHz machine with our MATLAB implementation, while the simulation gave acceptable results only in few minutes. Of course, with a C implementation the speed of the analytical algorithm can be increased substantially.

5 CONCLUSION

This paper introduces a queuing model of the AAL2 multiplexer in UTRAN and its performance analysis.

The introduced analytical model reflects practically important properties of real systems. This way the numerical analysis of the model allows to investigate the effect of such crucial parameters like the Timer_CU. The analysis quantified the intuitive expectations, namely that the higher is the Timer_CU the better is the packing efficiency, but the higher is the CPS-Packet delay at the same time. Indeed, the effect of higher CPS-Packet delay is so significant that higher link capacity cannot compensate it as the Timer_CU tends to the maximal allowed delay.

The presented analytical results are verified by discrete event simulation, and the results show a good accuracy of the introduced analysis procedure.

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Approximation of the Variance of Waiting Time in a Two-Queue Time Dependent Priority System

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KEYWORDS

Time Dependent Priorities, Variance

ABSTRACT

In this paper we derive an approximation of the variance of waiting time in a two-queue time dependent priority system. The derivation method is based on a transformation of the time dependent priorities system onto a static priorities system, with partial class switching. The derivation technique, proof, numerical and simulation results are presented and discussed.

1 INTRODUCTION

In a variety of application areas, different customer classes are defined, for which different grades of service are to be provided (e.g. different waiting times). To achieve this objective, scheduling strategies beyond the simple FIFO strategy are required. Known scheduling strategies include weighted fair queueing, priority queueing and weighted round robin.

One key performance measure considered in the analysis of queueing systems is the waiting time measure. In addition to the mean waiting time, it is in many application cases advantageous to have more insight in the variance or the standard deviation of the waiting time. In a call center for example, it is often not sufficient to only consider the mean waiting times (in the context referred to as ASA - average speed to answer). More insight is required on how far the waiting times actually deviate from an average.

In this paper we present a new method, how to derive the variance of waiting time in a two-queue system using time dependent priorities.

This paper is structured as follows: we first give a brief overview of time dependent priorities. We then present the derivation technique, show numerical examples, simulation results, and conclude.

2 TIME DEPENDENT PRIORITIES

The priority queueing disciplines can be generally classified into static and time dependent priorities. In a static priority system, the priority of a customer is constant during its whole sojourn time in the system. In many cases it is advantageous for a customer priority to increase with time. Such systems are more flexible but need more expense for the administration.

We assume in the following text a queuing model with R classes of customers, where arriving customers belong to a priority class r ($r = 1, 2, \ldots, R$). The interarrival and services times in all classes are assumed as to be exponential.

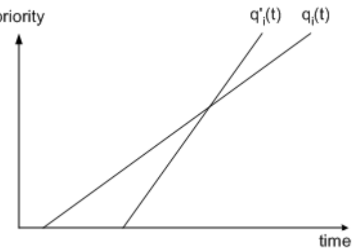


Figure 1: Priority functions with slopes b and b'

Each priority class is assigned a parameter b_r, which can be interpreted according to the priority function

$$q_r(t) = (t - t_0)b_r \tag{1}$$

as the increasing rate (slope) of the priority in the class r. A customer enters the system at time t_0 and then increases its priority at the rate b_r (see Figure 1). The priority of a higher class customer increases faster that the priority of a lower class customer, $0 \le b_1 \le b_2 \le \ldots \le b_R$.

Variants of this priority function, where exponents are assigned to the time component and/or slope components have also been introduced in the literature, e.g. in [5, 1].

3 DERIVATION OF THE VARIANCE OF WAITING TIME

3.1 Review of Related Research

The knowledge of the waiting time distribution or higher (central) moments of the waiting time function enables the system designer to perform more appropriate analysis of the queueing system.

The waiting time distribution has been extensively studied for single-class systems, however few results are available for multi-class systems [4]. In [4] an approximation formula is given for the waiting time distribution under several queueing policies, including static priorities and weighted fair queueing.

Several works study the behavior of waiting time in static priorities systems with multiple classes. Laplace transforms of the waiting time are provided e.g. in [5, Eqn. 3.32]. This equation can be differentiated and evaluated numerically to obtain higher moments of the mean waiting time. [8] addresses the waiting time distribution functions for a more general class of static priorities using preemption and preemption distances.

The analytical evaluation of the delay distributions in transform domains (e.g. Laplace transforms) usually requires complex mathematics and numerical approximations, especially for inversion. Furthermore, the models used for system analysis are often approximate models and exact distribution functions of the arrival or service processes are not known. In such cases, a possible approach is the use a two parameter description (mean and squared coefficient of variation) of the arrival and service processes.

The two-moment approximation was applied in the context of static priorities in [3] to derive the two first moments of waiting time. In [6], an approximation of the waiting time variance in an 2-class $M/M/1$ static priority system is given.

Other analysis approaches include modelling of the static priority system as a polling system, e.g. in [7] or using simulation.

The aim of our paper is to address the variance of waiting time in two-class time dependent priorities systems, which to the best of our knowledge has not been addressed in previous works.

3.2 Problem Statement

We consider an M/M/1 queueing system with two classes, where the strategy applied in time dependent priorities, as depicted by Figure 2.

The inter-arrival times in both classes are exponential, as described by the rates λ_1 and λ_2 respectively. The service times in both classes are also exponential, and described by the rates μ_1 and μ_2 respectively. ρ_i is used to denote the utilization in class i, and ρ for the total system utilization. Furthermore, we assume that the system is in stable condition, i.e. the total system utilization denoted by ρ is less than 1 ($\rho = \lambda 1/\mu_1 + \lambda 2/\mu_2$).

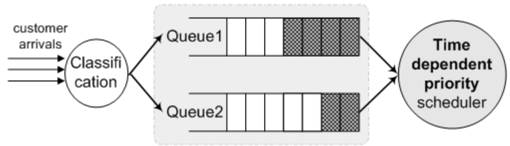


Figure 2: Initial Problem

We use the notation $W_i^{TDP}(\lambda_1, \lambda_2, \mu_1, \mu_2, b_1/b_2)$ or shortly W_i to refer to the waiting time of a customer of class i ($i = 1$ or $i = 2$), in a two-class time dependent priority system, characterized by the arrival rates λ_1 and λ_2, the service rates μ_1 and μ_2, and ratio of priority slopes b_1/b_2. $E[W_i]$ and $VAR[W_i]$ are used to refer the mean and variance of waiting time in class i.

In a time dependent priority system, the mean waiting times in class 1 and 2 are given by (see for example [5, 2, 1]):

$$E[W_1^{TDP}] = \frac{E[W_0]}{\Big(1-\rho\Big)\Big(1-\rho_2^{TDP}(1-\frac{b_1}{b_2})\Big)} \quad (2)$$

and

$$E[W_2^{TDP}] = \frac{E[W_0]}{(1-\rho)}\Big(1 - \frac{\rho_1^{TDP}(1-\frac{b_1}{b_2})}{1-\rho_2^{TDP}(1-\frac{b_1}{b_2})}\Big), \quad (3)$$

where $E[W_0]$ represents the mean remaining service time [2].

Our objective is to derive an approximation of the the variance of waiting time in class 1 and 2, denoted by $VAR[W_1]$ and $VAR[W_2]$.

3.3 Problem Transformation

The idea behind our approximative approach is to transform the initial problem using time dependent priorities, depicted by Figure 2, into a problem using static priorities, depicted by Figure 3, where results for the variance of waiting time have already been derived (see section 3.1).

The transformation is based on the idea that for customers waiting for service relatively long in class 1 (lower priority), a partial class switching to class 2 is performed, in order to enforce that they get served (also prior to other arriving class 2 customers).

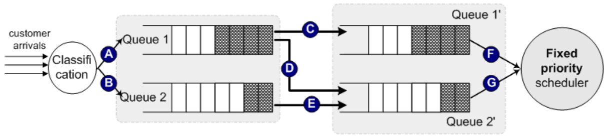


Figure 3: Transformed Problem

If we consider the long term (stationary) behavior of the time dependent priority system, we can generalize by saying, that a portion of class-1 customers, is "promoted" to class-2. This portion of the traffic, denoted by α, where $0\% \leq \alpha \leq 100\%$, is the class-1 traffic, which is served, even when class-2 customers are in the queue and waiting for service.

It can be expected that for the static priorities case (can be modelled by time dependent priorities by setting $b_1 = 0$ and $b_2 = 1$) that α must be equal to 0%, in order to keep both classes separate and keep the strict differentiation. In the first-come-first-served case and no differentiation (can be modelled by time dependent priorities by setting $b_1 = b_2 = 1$), that α must be equal to 100%, i.e. arrivals are merged in one stream and processed in FIFO order.

Considering the observation above and referring to Figure 2 and Figure 3, we can rephrase the idea behind the transformation as follows:

- Class-2 customers are generally served before class-1 customers, in the static priority scheduling sense. This is depicted in Figure 3 as B-traffic becoming E-traffic.
- A part of class-1 customers have to be served before other class-2 customers, due to their relatively long waiting time. This is depicted by D-traffic in Figure 3 and is the portion α of the original A-traffic, which undergoes the class switching.
- The rest of the A-traffic (i.e. excluding the D-part), is served last, in the static priority scheduling sense. This is denoted as C-traffic in Figure 3.

The transformed queueing system using static priorities is characterized by:

- customer arrival rate $(1 - \alpha) \cdot \lambda_1$ in Queue 1' being served at service rate μ_1,
- customer arrival rate $(\alpha \cdot \lambda_1) + \lambda_2$ in Queue 2' being served at service rate μ_1 for D-traffic and μ_2 for E-traffic.

In this transformed queueing system, the mean and variance in Queue 1' and Queue 2' can be calculated and used to determine the mean and variance in the original queueing system, using time dependent priorities.

3.4 Determination of α

One central element is to determine the part of the A-traffic, which is "promoted" and served as a high-priority traffic (i.e. class-2 traffic). In our notation this is referred to as D-traffic and represented as a portion α of the original A-traffic.

To determine α, we use two constraints based on the mean waiting times, which are known for both the original system (using time dependent priorities) and the transformed system (using static priorities).

The constraints can be formulated as:

- The mean waiting time of class-1 customers in the time dependent priority system is the weighted average of the mean waiting times of class 1 and 2 in the static priority system. α "percent" of the traffic has mean waiting time of Queue 2' and $(1-\alpha)$ "percent" has mean waiting time of Queue 1'.
- The mean waiting time of class-1 customers in the time dependent priority system is the same as the mean waiting time of customers in Queue 2' in the static priority system.

The two constraints can be expressed as:

$$E[W_1^{TDP}(\lambda_1, \lambda_2, \mu_1, \mu_2, b_1/b_2)] = \\ (1-\alpha)\, E[W_1^{SP}((1-\alpha)\lambda_1, \alpha\lambda_1 + \lambda_2, \mu_1, \mu_2)] \\ + \alpha\, E[W_2^{SP}((1-\alpha)\lambda_1, \alpha\lambda_1 + \lambda_2, \mu_1, \mu_2)] \tag{4}$$

and

$$E[W_2^{TDP}(\lambda_1, \lambda_2, \mu_1, \mu_2, b_1/b_2)] = \\ E[W_2^{SP}((1-\alpha)\lambda_1, \alpha\lambda_1 + \lambda_2, \mu_1, \mu_2)] \tag{5}$$

whereby the following notation is used:

$\mathbf{W_i^{TDP}(\lambda_1, \lambda_2, \mu_1, \mu_2, b_1/b_2)}$ denotes the waiting time of a customer of class i, in a two-class time dependent priority system, with class 1 characterized by arrival rate λ_1 and service rate μ_1, class 2 characterized by arrival rate λ_2 and service rate μ_2 and ratio of priority slopes b_1/b_2.

$\mathbf{W_i^{SP}(\lambda_1, \lambda_2, \mu_1, \mu_2)}$ denotes the waiting time of a customer of class i, in a two-class static priority system, with class 1 characterized by arrival rate λ_1 and service rate μ_1, class 2 characterized by arrival rate λ_2 and service rate μ_2.

It can be shown using several mathematical transformations that the portion of the traffic α is given by (refer also to Appendix 1 for the outline of the derivation):

$$\alpha = \frac{\frac{b_1}{b_2}}{1 - \rho(1 - \frac{b_1}{b_2})} \tag{6}$$

Figure 3.4 shows α as function of b_2 and the overall system utilization, where b_1 is equal to 1.

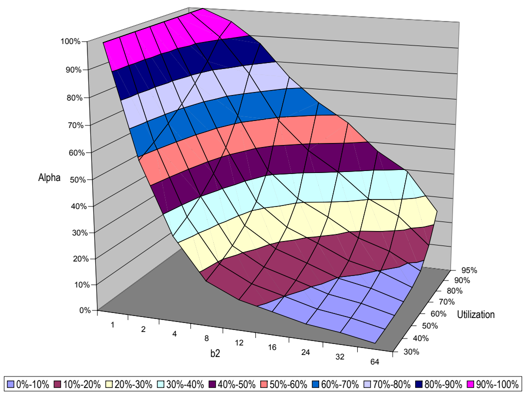


Figure 4: α as function of utilization and class 2 priority

We consider two limit cases of the formula above.

FIFO Case: To model a FIFO system using time dependent priorities, $\frac{b_1}{b_2}$ has to be set to 1. In this case α is equal to 1. This means that all class 1 traffic is pushed to the same queue like class 2. In other words, there is no difference between class 1 and class 2 anymore in the equivalent static priority system, i.e. a first-come-first-served mode is applied.

Strict Priority Case: To model a static priority system using time dependent priorities, $\frac{b_1}{b_2}$ has to be set to 0, in order for the priority of class 1 customers not to increase with time. In this case, α is equal to 0. Which means that no class 1 traffic is "promoted" to class 2 and a static priority order is maintained.

3.5 Waiting Time Variance in Time Dependent Priorities Systems

The variance of waiting time of customers in queue 1 of the time dependent priority system is composed of two parts: first the customers served in queue 1' of the static priority system (C-traffic) and second the customers served in queue 2' (D-traffic). The portions of both parts are $(1-\alpha)$ and α respectively. For the customers in queue 2 of the time dependent priority system, the variance is equal to the variance of queue 2' in the static priority system.

In the following we use the following two properties of the variance of two random variables X and Y:

$$VAR[aX] = a^2 VAR[X], \tag{7}$$

for a constant a, and if X and Y are stochastically independent,

$$VAR[X+Y] = VAR[X] + VAR[Y]. \tag{8}$$

For the variance in static priority systems, we discussed in Section 3.1 several methods how it can be derived. In [6], an approximation of the waiting time variance in an $M/M/1$ 2-class static priority system, with arrival rates λ_1 and λ_2 respectively and service rate μ in both classes, is derived as:

$$VAR[W_1] \approx \frac{2\lambda\mu^2 - \lambda_2\lambda^2 - \lambda^2\mu}{(\mu-\lambda)^2(\mu-\lambda_2)^3} \tag{9}$$

and

$$VAR[W_2] \approx \frac{2\lambda\mu - \lambda^2}{\mu^2(\mu-\lambda_2)^2} \tag{10}$$

Using the approximations and the variance properties above, he variance of waiting time in class 1 and class 2 in the time dependent priority system can be approximated by:

$$\begin{aligned} \mathrm{VAR}[W_1] &\approx (1-\alpha)^2 \ \frac{2\lambda\mu^2 - \alpha\lambda_1\lambda^2 - \lambda_2\lambda^2 - \lambda^2\mu}{(\mu-\lambda)^2(\mu-\alpha\lambda_1-\lambda_2)^3} \\ &+ \alpha^2 \ \frac{2\lambda\mu-\lambda^2}{\mu^2(\mu-\alpha\lambda_1-\lambda_2)^2} \end{aligned} \tag{11}$$

and

$$\mathrm{VAR}[W_2] \approx \frac{2\lambda\mu - \lambda^2}{\mu^2(\mu - \alpha\lambda_1 - \lambda_2)^2} \tag{12}$$

In Table 1, we present three examples, where the mean waiting times in class 1 and class 2 and the corresponding variances in a time dependent priority system are calculated:

3.6 Validation of the Derived Results

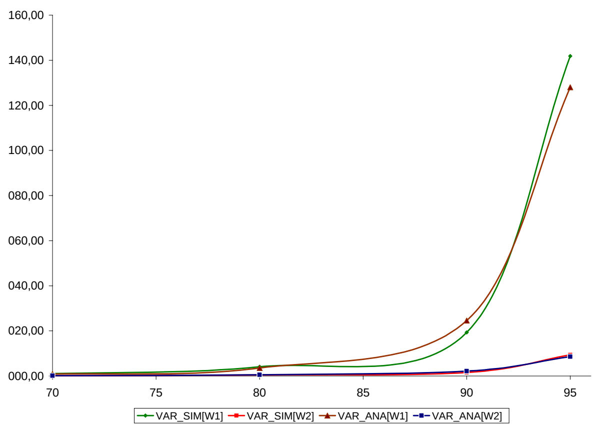
We conducted an extensive simulation study with 944 runs to study and validate the accuracy of the derived

Table 1: Waiting time variance using time dependent priorities

ID	Measure	Ex. 1	Ex. 2	Ex. 3
1	λ_1	1	1	1
2	λ_2	3	3	3
3	μ	6.67	5.71	4.44
4	b_2/b_1	8	16	2
5	α	20%	18%	91%
6	$E[W_1]$	0.371	0.804	3.057
7	$E[W_2]$	0.176	0.276	1.681
8	$VAR[W_1]$	0.386	1.672	9.500
9	$VAR[W_2]$	0.073	0.142	3.454

approximation. Our analysis revealed a good accuracy of the approximation, mostly within 20% deviation, considering the absolute and relative errors. For illustration purposes, we show sample results for the case where $b_1 = 1$ and $b_2 = 4$ at two different load distributions (LD) between class 1 and class 2 in Figures 5 and 6.

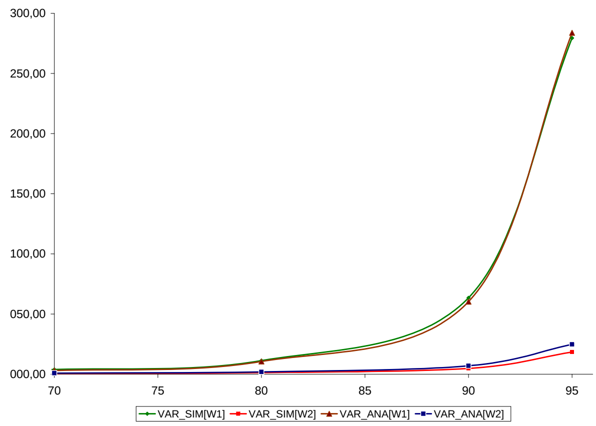
Figure 5: class1 25%, class2 75%



4 CONCLUSION

In this paper we presented a new technique for the derivation of the variance of waiting time in two-queue system using time dependent priorities. This derivation technique is based on a transformation of the original queueing problem using time dependent priorities onto a queueing problem using static priorities with partial class switching, for which results exist. We further showed numerical examples and selected results the simulation study we conducted in order to validate the accuracy of the derived approximations. The results presented can be applied to a variety of applications, e.g. for the call center design and optimization, or for jitter analysis in packet switched networks.

Figure 6: class1 75%, class2 25%



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Appendix 1 - Outline of the Proof of Equation 6

For the proof of the result in Equation (6), the following basic results are used:

For a static priority system which two priority classes, the closed formula for the mean waiting times are given by:

$$E[W_1^{SP}] = \frac{E[W_0]}{(1-\rho)(1-\rho_2^{SP})} \tag{13}$$

and

$$E[W_2^{SP}] = \frac{E[W_0]}{(1-\rho_2^{SP})}; \tag{14}$$

and for a time dependent priority system by equations (2) and (3).

The utilizations quantities in the original queueing system (i.e. using time dependent priorities) are given by

$$\rho_1^{TDP} = \frac{\lambda_1}{\mu_1} \tag{15}$$

and

$$\rho_2^{TDP} = \frac{\lambda_2}{\mu_2}. \tag{16}$$

For the transformed queueing system the following utilizations are applicable:

$$\rho_1^{SP} = \frac{(1-\alpha)\lambda_1}{\mu_1} = (1-\alpha)\frac{\lambda_1}{\mu_1} = (1-\alpha)\rho_1^{TDP} \tag{17}$$

and

$$\rho_2^{SP} = \frac{\alpha\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} = \alpha\rho_1^{TDP} + \rho_2^{TDP}. \tag{18}$$

Reference is made here also to Equations (3.1) to (3.4) in [5].

We use the constraint equation given in 4 and substitute with Equations (13), (14) and (2) to get:

$$\begin{aligned} \frac{E[W_0]}{1-\rho}\Big(1-\rho_2(1-\frac{b_1}{b_2})\Big)^{-1} &= \\ (1-\alpha)\ \frac{E[W_0]}{1-\rho}(1-\alpha\rho_1-\rho_2)^{-1} & \\ +\ \alpha\ E[W_0](1-\alpha\rho_1-\rho_2)^{-1} & \end{aligned}$$

This expression can further be simplified and resolved get α as stated in equation 6:

$$\alpha = \frac{\frac{b_1}{b_2}}{1-\rho(1-\frac{b_1}{b_2})}$$

It can be shown that for the value of α derived above, that the two terms given in the second constraint function in Equation (5) are equal.

SESSION 3

COMMUNICATION SYSTEM MODELS

ERROR CONTROL IN VOICE OVER IP OVER BLUETOOTH

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KEYWORDS

Bluetooth, Voice over IP, header compression, Personal Area Network, error control.

ABSTRACT

Header compression is a key aspect to voice over IP over Bluetooth in Personal Area Networks.

The adequacy of Bluetooth in carrying audio information in the presence of radio interface was investigated.

Speech quality was estimated subjectively for different retransmission parameters for network error control. Error control depended on a trade off between an acceptable latency and error rate.

The results were explained analytically

INTRODUCTION

Bluetooth is a wireless technology with the purpose to replace wires. However, its usage has evolved to a multipath environment in a Personal Area Network (PAN) (Bluetooth SIG. 2002). Here the communication flows on router links: Slave-Master-Slave-Master ....

The chief principle of Voice over Internet Protocol (VoIP) is to consider audio information as data packets that are sent in a long-range wired environment as 20 Bytes of data encapsulated in a 60-Byte header. This overhead is used to minimise the number of corrupted packets, because there is no packet retransmission (Postel 1980).

In contrast, short-range wireless VoIP over Bluetooth (VoIPoB) needs only the necessary information for adequate system performance (Kapoor et al. 2002). Header compression and retransmission may be used and these techniques and parameters are discussed here.

VOICE OVER IP OVER BLUETOOTH

Bluetooth data are sent in different types of Asynchronous ConnectionLess (ACL) packets. The types are categorised by type of Forward Error Correction (FEC) and packet length. The Medium rate packet and the minimum FEC type is a single frequency-slot Data packet (called *DM1*) (Miller and Bisdikian 2001). This DM1 packet is the smallest data packet type. It comprises 30 Bytes of payload, of which 10 Bytes is FEC. The remaining 20 Bytes have a single Byte payload header, 3-Bytes Bluetooth Network Encapsulation Protocol (BNEP) and a 2-Byte Cyclical Redundancy Check (CRC).

It is proposed that there should be a single Byte VoIPoB header, with source and destination addresses being defined in a higher-level protocol (named "L2CAP"). This means that each VoIPoB data packet of type DM1 can contain up to 13 Bytes of audio data. This is approximately 80 kbps of audio data in each direction (assuming a duplex packet requires 1.25 ms). This data rate is adequate for 64 kbps audio data with no compression.

HEADER COMPRESSION FOR VOIPOB

The Network Access Point (NAP) is the interface between wired and wireless networks. There is a translation of VoIP addressing into Bluetooth addressing at the NAP. Source and destination IP addresses can be obtained from a Session Initiation Protocol (SIP) message. There is no need to send this information further than the NAP.

The only extra information that are needed in DM1 packets are a *flow label* (3-bits), a *hop-limit* (2-bits) and a *sequence-number* (3-bits). These characteristics allow the recompression of the header from its original 60-Byte size down to just one Byte as shown in Figure 1.

3-Bit Flow Label	2-Bit Hop Limit	3-Bit Sequence Number	AUDIO DATA

Figure 1: VoIPoB Header Format.

The flow-label of 3-bits allows eight simultaneous conversations in a conference connection. The 2-bit hop-limit means that up to 4 different links can be used by the PAN routers. A 3-bit sequence-number allows for packet re-ordering after a set of 8 packets have been received correctly. Packet reordering within a group may be necessary because of retransmission or differing routing delays.

SIMULATION MODEL

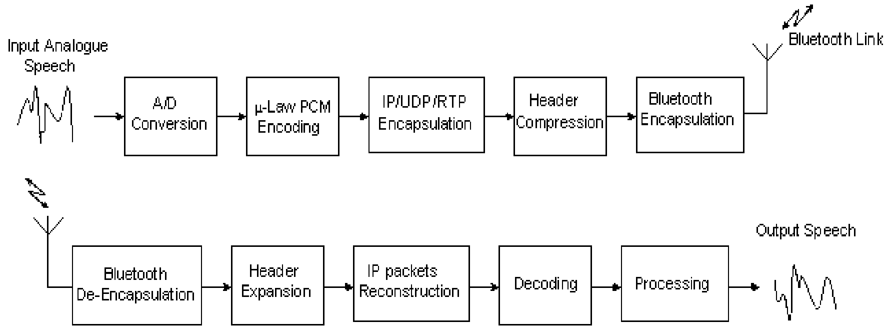

Figure 2: Block Diagram of Simulator

A block diagram of the functions carried out by a VoIPoB simulation model is shown in Figure 2.

Analogue input speech is converted to digital samples that are encoded as Pulse-Coded Modulation (PCM) signals. This is followed by framing into VoIP packets with encapsulation (Internet Protocol/User datagram Protocol/Real-Time Protocol header). This is replaced by a VoIPoB header and other Bluetooth encapsulation.

The packets are sent over a communication channel, where errors are inserted to represent non-ideal channel conditions.

At the receiver, the Bluetooth encapsulation is stripped and the VoIPoB header is replaced by the VoIP header. These VoIP frames are processed and finally the output is the audio information available to the user. The output audio signal may contain gaps or disturbances.

RETRANSMISSION

One of the main advantages of voice that is sent as data over Bluetooth is the ability to retransmit packets that get lost, delayed or corrupted during transmission. The number of allowed retransmissions, however, depends on the ACL buffer size before processing and an acceptable latency.

An acceptable latency to the human ear is about 150ms (ITU-T Recommendation G.114) (De Vleeschauer et al. 2000). A more serious problem is audio delay jitter, which can be as high as 50 ms (Roychoudhuri et al. 2003). However, this is a problem over the wired network rather than over short-range wireless.

The conclusion is that retransmission latencies of 15 ms per point-to-point link over Bluetooth will cause no perceptible degradation of audio reception.

SIMULATION RESULTS

Simulation was performed using the model described in Figure 2 using a 35s stereo audio *.WAV* file, which was PCM coded. As shown in Figure 2, VoIP packets were first constructed and then were converted to VoIPoB DM1 packets containing 13 Bytes of audio data.

The audio DM1 packets were passed through a simulated Bluetooth channel and packets were injected with errors according to a Poisson probability distribution.

Two different re-transmission schemes were simulated one with 2 retransmissions and the other with 4 retransmissions in a buffer size of 8 packets. The most convenient solution is to have a buffer size of 8 packets and 4 retransmissions. For this buffering scheme, 10 ms was required for transmission of 8 packets and an extra 5ms latency was required for retransmission.

At the simulator output, sets of 8 packets were examined together. Those packets in error in each set were retransmitted through the simulator and errors were injected in the channel with a Poisson distribution once more. For each group of size 8, up to 4 retransmissions were allowed.

This meant that if there was a single error in the set then up to 4 attempts were made to pass the packet through the simulator. This set with a single error resulted in an almost negligible residual error probability.

In contrast, a set of 8 packets containing 5 packets in error would be certain to have at least one packet in error after four packet retransmission attempts.

Average Packet Error Rates (PER) of 20%, 10%, 5%, 1% and 0.1% were used before correction. The results have shown a considerable reduction in PER after retransmission as shown in Figure 3

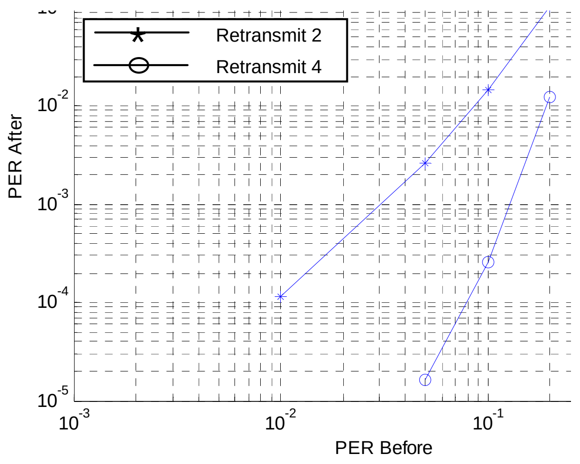

Figure 3: Error Control Simulation Results.

It can be seen from Figure 3 that the Packet Error Rate, PER was reduced from 20% to about 1% by taking packets in groups of 8 and retransmitting any error packets four times/group. For an initial PER of 10%, the result after retransmission was a PER of about $3x10^{-4}$.

These results were confirmed subjectively: Music with a 20% PER is barely recognisable above the audio noise but after the retransmission regime the rate of audio "clicks" is small. The remaining errors of music with an original 10% PER or less was a "perfect" reproduction after 4 retransmissions of error packets. These audio files are available and can be heard over the World-Wide Web (Pollard 2006).

MODELLING RESULTS

The model uses theoretical calculations of Poisson distributed errors present in a set of size S, where p is the initial average Packet Error Rate (PER):

The Probability of e errors after r retransmissions is given by:

$$P_S[e][r] = eE_S[e]p^{r-e+1} \text{ for r} \geq \text{e , or}$$

$$P_S[e][r] = E_S[e](rp + e - r) \text{ for r < e}$$

Where Probability of e errors in set of size S is: $E_S[e] = \binom{S}{e} p^e (1-p)^{S-e}$

The initial Packet Error Rate, p is compared with the resulting PER after retransmission for different combinations of buffer sizes and number of retransmissions.

The results for group size of 8 packets and 1, 2 and 4 retransmissions are given in Figure 4.

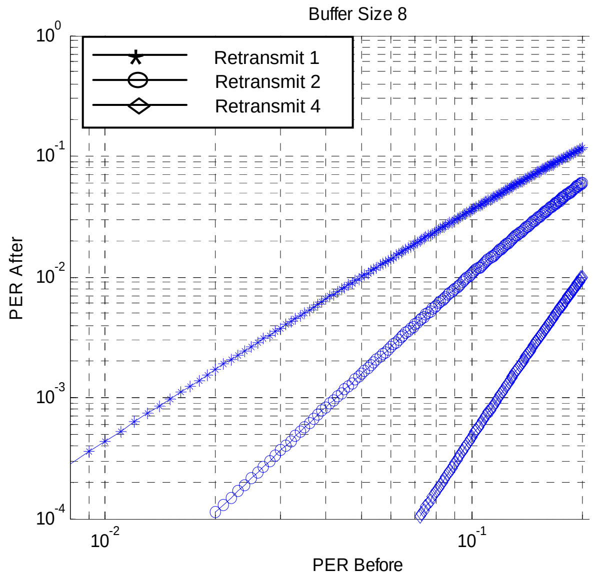

Figure 4: Original PER vs Corrected PER (Group Size 8 and 1, 2 and 4 Retransmissions)

A single retransmitted packet resulted in a reduction of PER from 20% to 10% and an original PER of 1% to $5x10^{-4}$. Two retransmitted packets reduced the PER from 20% to 6% and 2% to 10^{-4}. Four retransmissions reduced the PER from 20% to 1% and 10% to $3x10^{-4}$. These modelling results were in agreement with the simulation results shown in Figure 3.

A comparison was made for a group size of 1, which is commonly used for voice transmissions over Bluetooth. Results are shown for no retransmission, a single retransmission, 2 retransmissions and 4 retransmissions in Figure 5

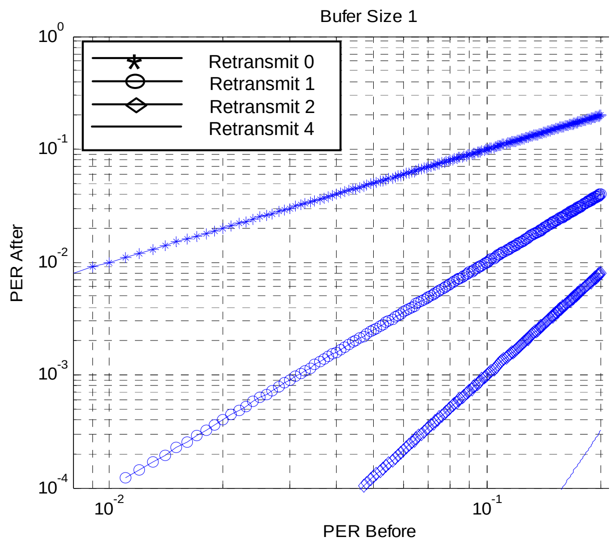

Figure 5: Original PER vs Corrected PER (Group Size 1 and 0, 1, 2 and 4 retransmissions)

The modelling methodology was repeated for a larger group size 16 in order to gain an insight into the affect of group size. Results are shown in Figure 6.

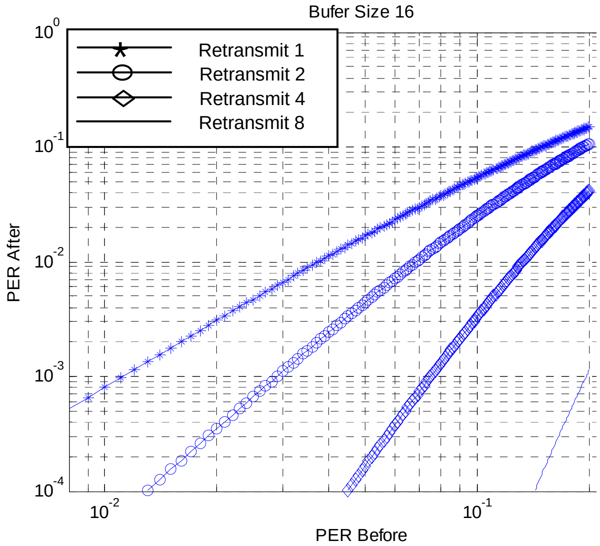

Figure 6: Original PER vs Corrected PER (Group Size 16 and 1, 2, 4 and 8 retransmissions)

Comparison of Figure 4, 5 and 6 shows that the effect of increasing group size for constant number of retransmissions is not as dramatic as to increase the number of retransmissions at constant group size. However, increasing the number of retransmissions (as well as increasing the group size) increases the added latency of a link.

CONCLUSION

The results given in this paper have shown that Bluetooth presents an adequate environment of implementing VoIP applications.

It was shown by means of simulation and modelling that the chosen retransmission technique (4 retransmissions in a buffer of 8 packets) can reduce packet error rates and is within an acceptable latency for the human ear.

In the future, a converged environment of voice and data can be integrated into one Bluetooth service.

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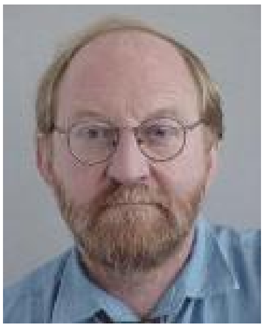

John Pollard obtained a PhD in Electronics from Southampton University in 1968 and a MSc in Computing from Macquarie University, Australia in 1986.
His background is in the design of communication systems and software systems. In recent years, he has been interested in the use of the World-Wide Web as an enabling technology for teaching and for distributed modeling software. An integrated combination of hardware and software is necessary to connect a distributed system of computers and mobile input/output electronics (such as mobile telephones and Bluetooth-enabled devices) with databases and real, physical apparatus.

DIMENSIONING OF A DEJITTERING BUFFER FOR VARIABLE BIT RATE VIDEO STREAMS

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KEYWORDS

Dejittering buffer, Markov model, underflow probability, overflow probability.

ABSTRACT

In this paper a Markov model is developed that, given a simple description of the network delay and the packet interarrival times of a variable bit rate video stream, can be used to dimension the two important parameters of the dejittering process: the capacity Q the dejittering buffer should have and the dejittering delay T_{jit} the dejittering process should wait before starting the playout of the first packet. The correct setting of these parameters is important to keep both the packet loss due to dejittering buffer overflow and/or underflow and the introduced dejittering delay below some acceptable limits.

INTRODUCTION

In the domain of access networks there is a tendency towards increasing the capacity of the access bit pipes. This provides opportunities to offer beyond the usual data transmission the Internet was originally designed for, also bandwidth-hungry services like high-quality digital television. A major problem with the real-time transmission of audio-visual information over the Internet is however that the end-to-end delay is variable, depending on the load of the network. These delay variations (also called jitter) need to be reduced at the receiving end, or the decoder will not operate correctly (Apostolopoulos et al., 2002).

The task of reducing the jitter is typically performed by placing a dejittering buffer before the decoder. In this buffer, the first packet of a video stream is retained for some time (the *dejittering delay*) before it is offered to the decoder. From then on, packets are played out with the same cadence as with which the encoder generated the video packet stream. Packets that arrive before their playout instant are temporarily stored in the dejittering buffer, packets that arrive too late are useless and considered lost. For each packet that arrives too late, the dejittering buffer underflows (i.e., runs empty) at the moment such a packet is supposed to be played out.

For a given dejittering delay and dejittering buffer capacity, there is a maximal amount of network jitter that can be accommodated without buffer underflow or overflow, either of which would result in a poor video quality. But while a dejittering buffer can compensate for jitter, the dejittering mechanism also introduces additional delay. Specifying the dejittering delay and the size of the dejittering buffer thus always involves a trade-off between packets being lost to the video application and their delay.

The aim of this paper is to develop a mathematical model that, given a simple description of the network delay and the packet interarrival times of a variable bit rate video stream, can be used to dimension the two important parameters of the dejittering process: the capacity Q the dejittering buffer should have and the dejittering delay T_{jit} the dejittering process should wait before starting the playout of the first packet. The correct setting of these parameters is important to keep both the packet loss due to dejittering buffer overflow and/or underflow and the introduced dejittering delay below some acceptable limits.

SYSTEM DESCRIPTION AND ASSUMPTIONS

Consider a video source that sends its traffic over a packet-based network towards a video player, at a rate that is not necessarily constant (i.e., a variable bit rate source). Because of varying (queueing) delays within the routers in the network, the end-to-end delay between source and receiver can vary from packet to packet. To compensate for this delay jitter introduced by the network, the video player uses a dejittering buffer.

Assume that the time is divided into fixed-length intervals, referred to as slots, and that the variable character of the interarrival time (IAT) of successive packets at the network is characterized by the probability distribution

$$b_i = \mathrm{P}\{\mathrm{IAT} = i \text{ slots}\}, \quad 0 < iat_{\min} \le i \le iat_{\max}.$$

Remark that taking $iat_{\min}$ equal to $iat_{\max}$ results in the characterization of constant bit rate traffic.

The network delays experienced by the packets are assumed to be identically distributed random variables (not necessarily independent) with probability distribution

$$d_i = \mathrm{P}\{\text{network delay} = i \text{ slots}\}, \qquad 0 < delay_{\min} \le i \le delay_{\max}.$$

For ease of notation later on, we assume that $d_i \neq 0$, $\forall i \in \{delay_{\min}, \ldots, delay_{\max}\}$. This is however not an essential requirement.

A delay distribution could be obtained from delay measurements in the considered network, or from the service level specification of the network provider. Assuming that the delay of each packet is described by the same probability distribution is only valid if the network conditions (e.g., routes, loads, etc.) do not change drastically over the time that the video flow exists.

Independency of the delay of succeeding packets can however not be assumed, since packet delays in general exhibit a temporal dependency, particularly for high delays (Jiang and Schulzrinne, 2000). The reason for this dependency is intuitively explained as follows. A high delay for packet k indicates a non-empty router buffer in the network. Since it takes some time for such a buffer to drain, if the inter-packet gap at the sender is small, the buffer depth may not have changed much upon arrival of packet $k+1$ at that buffer, and this packet will also experience a high delay. We will take this phenomenon into account by assuming that the probability with which the delay $D(k+1)$ of a packet $k+1$ takes a certain value, given the delay $D(k)$ experienced by packet k and the interarrival time $\mathrm{IAT}(k, k+1)$ between the two packets at the network, has the form of a conditional distribution:

$$\mathrm{P}\{D(k+1) = j \,|\, D(k) = i \wedge \mathrm{IAT}(k,k+1) = l\} = \begin{cases} 0, & \text{if } j \le i - l, \\ f_j \Big/ \left(1 - \sum_{m=delay_{\min}}^{i-l} f_m\right), & \text{otherwise,} \end{cases}$$

with $delay_{\min} \le i, j \le delay_{\max}$, $iat_{\min} \le l \le iat_{\max}$, all $f_j \ge 0$ and $\sum_{j=delay_{\min}}^{delay_{\max}} f_j = 1$, where the exact values of the f_j's are still unknown at this point.

Since $\mathrm{P}\{D(k) = i\} = d_i$ and $\mathrm{P}\{D(k+1) = j\} = d_j$, the theorem on total probability gives for all $j \in \{delay_{\min}, \ldots, delay_{\max}\}$ that

$$\begin{aligned} d_j &= \mathrm{P}\{D(k+1) = j\} \\ &= \sum_{i=delay_{\min}}^{delay_{\max}} \sum_{l=iat_{\min}}^{iat_{\max}} \mathrm{P}\{D(k+1) = j \,|\, D(k) = i \wedge \mathrm{IAT}(k,k+1) = l\} \\ &\quad \mathrm{P}\{\mathrm{IAT}(k,k+1) = l \,|\, D(k) = i\}\, \mathrm{P}\{D(k) = i\} \\ &= \sum_{i=delay_{\min}}^{delay_{\max}} \sum_{l=\max\{iat_{\min}, i-j+1\}}^{iat_{\max}} f_j\, d_i \left(\frac{b_l}{1 - \sum_{m=delay_{\min}}^{i-l} f_m} \right). \end{aligned} \tag{1}$$

By solving this set of $delay_{\max} - delay_{\min} + 1$ equations, the $delay_{\max} - delay_{\min} + 1$ unknown f_j's are obtained. Note that solving this set of equations is not difficult, since $i - l < j$ in Equation (1). So the first equation corresponding to $j = delay_{\min}$ only contains the unknown $f_{delay_{\min}}$. The second equation contains $f_{delay_{\min}+1}$ and $f_{delay_{\min}}$, where $f_{delay_{\min}}$ can be substituted from the solution of the first equation, and so on.

Denote by $\mathrm{IDT}(k, k+1)$ the interdeparture time from the network between packet k and packet $k+1$ (i.e., their interarrival time at the dejittering buffer). Then

$$\mathrm{IDT}(k,k+1) = \mathrm{IAT}(k,k+1) + D(k+1) - D(k). \tag{2}$$

Define for notational convenience later on the matrices $\mathbf{U}_s$, with elements $(\mathbf{U}_s)_{i,j}$, ($delay_{\min} \le i, j \le delay_{\max}$) equal to

$$\begin{aligned} (\mathbf{U}_s)_{i,j} &= \mathrm{P}\{\mathrm{IDT}(k,k+1) = s+1 \wedge D(k+1) = j \,|\, D(k) = i\} \\ &= \mathrm{P}\{\mathrm{IAT}(k,k+1) = s+1-j+i \wedge D(k+1) = j \,|\, D(k) = i\} \\ &= \mathrm{P}\{D(k+1) = j \,|\, \mathrm{IAT}(k,k+1) = s+1-j+i \wedge D(k) = i\} \\ &\quad \mathrm{P}\{\mathrm{IAT}(k,k+1) = s+1-j+i \,|\, D(k) = i\} \\ &= \begin{cases} \frac{f_j\, b_{s+1-j+i}}{1 - \sum_{m=delay_{\min}}^{j-s-1} f_m}, & \text{if } s \ge 0 \text{ and } \\ \qquad iat_{\min} \le s+1-j+i \le iat_{\max}, & \\ 0, \ \text{otherwise.} & \end{cases} \end{aligned}$$

Remark that for $s \notin \{0, \ldots, S\}$, with $S = iat_{\max} + delay_{\max} - delay_{\min} - 1$, $\mathbf{U}_s = \mathbf{0}$.

DEJITTERING PROCESS

For every slot n, define the states (l_n, t_n, j_n) with $l_n > 0$, and (l_n, t_n, w_n, j_n) with $l_n = 0$, where the different variables have the following interpretation:

- l_n: for $l_n > 0$, the number of slots ago the next packet for playout arrived, and for $l_n = 0$, the fact that the next packet for playout did not yet arrive, or arrives in the current slot n,
- t_n: the number of slots to go until the next playout should start, where $t_n = 0$ means that the next packet should be played out in the current slot n,
- w_n (only defined if $l_n = 0$): the number of slots to go until the next packet for playout arrives, where $w_n = 0$ means that it will arrive in the current slot n,
- j_n: the network delay experienced by the next packet for playout.

Remark that states with $l_n = 0$, $t_n = 0$ and $w_n > 0$ correspond to a packet being late for playout. Note further that given the state of the dejittering process in slot n, it is possible to calculate with which probability the process will be in which state in slot $n+1$. So the states defined above describe a discrete-time Markov chain. The transition probabilities of going from one state to another state will be derived now.

First of all, transitions from a state with $t_n > 0$ are trivial, since the next packet for playout is the same packet in slot $n+1$ as in slot n. With probability one the system will evolve

- from a state $(l_n > 0, t_n > 0, j_n)$ in slot n to the state $(l_{n+1} = l_n + 1, t_{n+1} = t_n - 1, j_{n+1} = j_n)$ in slot $n+1$,
- from a state $(l_n = 0, t_n > 0, w_n > 0, j_n)$ in slot n to the state $(l_{n+1} = 0, t_{n+1} = t_n - 1, w_{n+1} = w_n - 1, j_{n+1} = j_n)$ in slot $n+1$, and
- from a state $(l_n = 0, t_n > 0, w_n = 0, j_n)$ in slot n to the state $(l_{n+1} = 1, t_{n+1} = t_n - 1, j_{n+1} = j_n)$ in slot $n+1$.

If the system is in slot n in a state with $t_n = 0$, i.e., the next packet should be played out in this slot, then with probability $(\mathbf{U}_s)_{i,j}$, $s \in \{0, \ldots, S\}$ and $delay_{\min} \leq i, j \leq delay_{\max}$, a transition is made

- from a state $(l_n > 0, t_n = 0, j_n = i)$ in slot n to the state $(l_{n+1} = l_n - s, t_{n+1} = s + i - j, j_{n+1} = j)$ in slot $n+1$ if $s - l_n < 0$, and to the state $(l_{n+1} = 0, t_{n+1} = s + i - j, w_{n+1} = s - l_n, j_{n+1} = j)$ if $s - l_n \geq 0$,
- from a state $(l_n = 0, t_n = 0, w_n, j_n = i)$ in slot n to the state $(l_{n+1} = 0, t_{n+1} = s + i - j, w_{n+1} = w_n + s, j_{n+1} = j)$.

This is because when a packet k should be played out in slot n, and packet $k+1$ arrives $s+1$ slots later than packet k at the player,

- packet $k+1$ should be played out $\mathrm{IAT}(k, k+1)$ slots later than packet k, i.e., in slot $n + s + 1 - j + i$ (by Equation (2)),
- packet k arrived in slot $n - l_n$ if $l_n > 0$, and thus packet $k+1$ arrives in slot $n - l_n + s + 1$, which is before or later than slot $n+1$ depending on if $s - l_n$ is positive or negative,
- packet k arrives in slot $n + w_n$ if $l_n = 0$, and thus packet $k+1$ arrives in slot $n + w_n + s + 1$.

Denote $\mathcal{D} = \{delay_{\min}, \ldots, delay_{\max}\}$. Because the receiver waits T_{jit} slots after the arrival of the first packet before starting the playout of the video, the evolution of the Markov chain defined above will start upon arrival of the first packet in a state $(0, T_{jit}, 0, d^*)$, where $d^* \in \mathcal{D}$ is the network delay experienced by the first packet.

Define the set of states $\mathcal{C}(T_{jit} + d^*)$ as

$$\begin{aligned}
&\mathcal{C}(T_{jit} + d^*) = \\
&\quad \big\{(0, t, w, j) | j \in \mathcal{D}, w = t + j - T_{jit} - d^* \text{ and} \\
&\quad t \in \{\max(0, T_{jit} + d^* - j), \ldots, iat_{\max} - 1\}\big\} \\
&\quad \bigcup \big\{(l, t, j) | j \in \mathcal{D}, l = T_{jit} + d^* - t - j \text{ and} \\
&\quad t \in \{0, \ldots, \min(T_{jit} + d^* - j, iat_{\max}) - 1\}\big\}. \quad (3)
\end{aligned}$$

The states of $\mathcal{C}(T_{jit} + d^*)$ form a Markov chain with exactly one irreducible closed set of states (see the appendix for details). States not in this closed set are transient. As a consequence, this Markov chain has a unique stationary distribution $\boldsymbol{\pi}$.

If $T_{jit} \leq iat_{\max} - 1$, the start state $(0, T_{jit}, 0, d^*)$ belongs to the set $\mathcal{C}(T_{jit} + d^*)$. If $T_{jit} > iat_{\max} - 1$, $(0, T_{jit}, 0, d^*)$ is not an element of $\mathcal{C}(T_{jit} + d^*)$, but after T_{jit} transitions with probability 1, the system reaches the state $(T_{jit}, 0, d^*)$, which does belong to $\mathcal{C}(T_{jit} + d^*)$. So when the dejittering process starts in the state $(0, T_{jit}, 0, d^*)$, the steady-state behavior of this process is described by the Markov chain with the elements of $\mathcal{C}(T_{jit} + d^*)$ as states.

To calculate the stationary distribution $\boldsymbol{\pi}$, the set of equations

$$\boldsymbol{\pi}\mathbf{Q} = \boldsymbol{\pi}, \quad \boldsymbol{\pi}\mathbf{e} = 1, \tag{4}$$

should be solved, where $\mathbf{Q}$ is the transition matrix describing the transition probabilities among the states of $\mathcal{C}(T_{jit} + d^*)$. By dividing $\mathbf{Q}$ into blocks corresponding to the first index of the states, i.e.,

$$\mathbf{Q} = \begin{pmatrix}
\mathbf{B}_1^{(0)} & \mathbf{B}_0^{(0)} & \mathbf{0} & \ldots & \mathbf{0} \\
\mathbf{B}_2^{(1)} & \mathbf{A}_1^{(1)} & \mathbf{A}_0^{(1)} & \ldots & \mathbf{0} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{B}_{l_{\max}}^{(l_{\max}-1)} & \mathbf{A}_{l_{\max}-1}^{(l_{\max}-1)} & \mathbf{A}_{l_{\max}-2}^{(l_{\max}-1)} & \ldots & \mathbf{A}_0^{(l_{\max}-1)} \\
\mathbf{B}_{l_{\max}+1}^{(l_{\max})} & \mathbf{A}_{l_{\max}}^{(l_{\max})} & \mathbf{A}_{l_{\max}-1}^{(l_{\max})} & \ldots & \mathbf{A}_1^{(l_{\max})}
\end{pmatrix},$$

the lower block-Hessenberg form of this matrix becomes clear, which means that equation (4) can be solved using for example the algorithm described in (Van Houdt et al., 2003), which is an adaptation of the algorithm of Latouche, Jacobs and Gaver (Latouche et al., 1984). The diagonal blocks of $\mathbf{Q}$ are square matrices, the other blocks are rectangular with appropriate dimensions. Once $\boldsymbol{\pi}$ is calculated, several performance measures of the dejittering process can be calculated.

PERFORMANCE MEASURES

This section explains how to calculate the underflow and overflow probabilities from the presented model. Remember from before that for each different value of $T_{jit} + d^*$, a Markov chain with a different set of states $\mathcal{C}(T_{jit} + d^*)$, and consequently a different stationary distribution should be considered. Despite this, the simple notation '$\boldsymbol{\pi}$' is used for such a stationary distribution, since adding a reference to the value $T_{jit} + d^*$ will just overload notations, while the stationary distributions corresponding to different values for $T_{jit} + d^*$ will never be used together in a single formula.

Underflow Probability

The probability that the dejittering buffer underflows, i.e., that a packet is not in the dejittering buffer on its playout time, given that the receiver waits T_{jit} slots after the arrival of the first packet to start the playout, and given that the first packet experiences a network delay of d^* slots, is

$$P_{late}(T_{jit}, d^*) = \frac{\sum_{w>0} \sum_{j\in\mathcal{D}} \boldsymbol{\pi}_{(0,0,w,j)}}{\sum_{w} \sum_{j\in\mathcal{D}} \boldsymbol{\pi}_{(0,0,w,j)} + \sum_{l>0} \sum_{j\in\mathcal{D}} \boldsymbol{\pi}_{(l,0,j)}}, \quad (5)$$

i.e., the fraction of all slots in which a packet should be played out, but the packet did not yet arrive, to all slots in which a packet should be played out.

Remark that when $T_{jit} + d^* \geq delay_{\max}$, all packets are always on time for playout, since there does not exist a state of the form $(0, 0, w, j) \in \mathcal{C}(T_{jit} + d^*)$ with $w > 0$ and $j \in \mathcal{D}$. As a consequence, the numerator of (5) equals zero. However, the delay d^* experienced by the first packet will not be known to the receiver at the moment it sets the parameter T_{jit}. Choosing $T_{jit} \geq delay_{\max} - delay_{\min}$ ensures that all packets will always be on time for playout, although the introduction of an extra delay of this amount will not always be acceptable if the difference between the maximum and the minimum delay becomes large.

The average underflow probability given that the receiver waits T_{jit} slots after the arrival of the first packet to start the playout is obtained by considering all the different values d^* can take:

$$P_{late}(T_{jit}) = \sum_{k\in\mathcal{D}} P_{late}(T_{jit}, d^* = k)\, d_k.$$

Overflow Probability

Packets that arrive too late for playout will not enter the dejittering buffer, since they are of no use to the application anymore. Packets that arrive on time could however still get lost when upon their arrival the dejittering buffer is full. The probability that this happens when the dejittering buffer has a dimension of Q packets, will be approximated by the probability that a timely arriving packet sees upon its arrival Q or more packets present in an infinite capacity buffer.

Consider a random timely arriving packet, and suppose that the playout instant corresponding to this packet is slot n. Then the Markov chain describing the dejittering process will be in a state of the following types in slot n: $(l_n > 0, t_n = 0, j_n \in \mathcal{D})$ or $(l_n = 0, t_n = 0, w_n = 0, j_n \in \mathcal{D})$. This means that the packet arrived in slot $n - l_n$, and the packet saw at that moment as many packets present in the dejittering buffer as will be played out in the time interval $[n - l_n; n[$. So the probability that a random timely arriving packet that arrives $l \geq 0$ slots before its playout time finds q packets in the dejittering buffer upon its arrival, equals the probability that q succeeding interarrival times of the variable bit rate stream last l or less slots, while $q + 1$ succeeding interarrival times last more than l slots.

Denote by $X(q)$ the duration of q succeeding interarrival times. Then

$$\begin{aligned}
&\mathrm{P}\{X(q) \leq l \wedge X(q+1) > l\} \\
&= \sum_{i=\max(q.iat_{min}, l-iat_{\max}+1)}^{\min(l, q.iat_{\max})} \mathrm{P}\{X(q) = i \wedge X(q+1) > l\} \\
&= \sum_{i=\max(q.iat_{min}, l-iat_{\max}+1)}^{\min(l, q.iat_{\max})} \mathrm{P}\{X(q+1) > l \,|\, X(q) = i\} \\
&\qquad\qquad\qquad\qquad \mathrm{P}\{X(q) = i\} \\
&= \sum_{i=\max(q.iat_{min}, l-iat_{\max}+1)}^{\min(l, q.iat_{\max})} \; \sum_{j=\max(l-i+1, iat_{\min})}^{iat_{\max}} b_j \;\; I_q^i,
\end{aligned}$$

where I_q^i denotes the probability that q succeeding interarrival times of the variable bit rate stream last i slots. This probability is calculated using the following recursive relation:

$$I_0^i = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$I_1^i = \begin{cases} b_i, & \text{if } iat_{\min} \le i \le iat_{\max}, \\ 0, & \text{otherwise,} \end{cases}$$

and for $q > 1, I_q^i =$

$$\begin{cases} 0, & \text{if } i < q.iat_{\min} \text{ or } i > q.iat_{\max}, \\ \sum_{j=\max(i-(q-1).iat_{\max}, iat_{\min})}^{\min(iat_{\max}, i-(q-1).iat_{\min})} I_1^j \, I_{q-1}^{i-j}, & \text{otherwise.} \end{cases}$$

So the probability that a timely arriving packet sees upon arrival q packets in the dejittering buffer, given T_{jit} and d^*, is:

$$\mathrm{P}\{Q_a = q \,|\, T_{jit}, d^*\} = \frac{\sum_{j\in\mathcal{D}} \pi_{(0,0,0,j)} 1_{\{q=0\}}}{\sum_{j\in\mathcal{D}} \pi_{(0,0,0,j)} + \sum_{l>0}\sum_{j\in\mathcal{D}} \pi_{(l,0,j)}} + \frac{\sum_{l>0}\sum_{j\in\mathcal{D}} \pi_{(l,0,j)} \sum_{i=\max(q.iat_{min}, l-iat_{\max}+1)}^{\min(l, q.iat_{\max})} \sum_{k=\max(l-i+1, iat_{\min})}^{iat_{\max}} b_k \, I_q^i}{\sum_{j\in\mathcal{D}} \pi_{(0,0,0,j)} + \sum_{l>0}\sum_{j\in\mathcal{D}} \pi_{(l,0,j)}},$$

and given only T_{jit}, this becomes

$$\mathrm{P}\{Q_a = q \,|\, T_{jit}\} = \sum_{k\in\mathcal{D}} \mathrm{P}\{Q_a = q \,|\, T_{jit}, d^* = k\} \, d_k.$$

Then the overflow probability given T_{jit} and d^* for a dejittering buffer of size Q is approximated by

$$P_Q(T_{jit}, d^*) \approx \mathrm{P}\{Q_a \ge Q \,|\, T_{jit}, d^*\},$$

and given T_{jit} only by

$$P_Q(T_{jit}) \approx \mathrm{P}\{Q_a \ge Q \,|\, T_{jit}\}.$$

Remark that the maximal value l can take in a state with $t = 0$ of $\mathcal{C}(T_{jit} + d^*)$, is $l_{\max} = \max(T_{jit} + d^* - delay_{\min}, 0)$. Because there will be at most one playout per $iat_{\min}$ slots, a timely arriving packet sees never more than $\lfloor \frac{l_{\max}}{iat_{\min}} \rfloor$ packets in the dejittering buffer upon arrival. So when the dejittering buffer size Q is chosen equal to or larger than $\lfloor \frac{T_{jit}+delay_{\max}-delay_{\min}}{iat_{\min}} \rfloor + 1$, no packets will get lost due to buffer overflow. Setting $Q = \lfloor \frac{2(delay_{\max}-delay_{\min})}{iat_{min}} \rfloor + 1$ and $T_{jit} = delay_{\max} - delay_{\min}$ corresponds to the optimal choice for T_{jit} and Q for which no packets get lost due to the dejittering process, i.e., not because of buffer underflow and not because of buffer overflow.

NUMERICAL EXAMPLE

Consider a network delay distribution with the shape of a shifted gamma distribution, as shown in Figure 1. Studies (Corlett et al., 2002; Bovy et al., 2002) into one-way end-to-end packet delays have found that the vast majority of these delays is well approximated by a shifted gamma distribution, and that excessive delays are in reality quite rare. Figure 2 shows the packet interarrival time distribution that will be considered in this example.

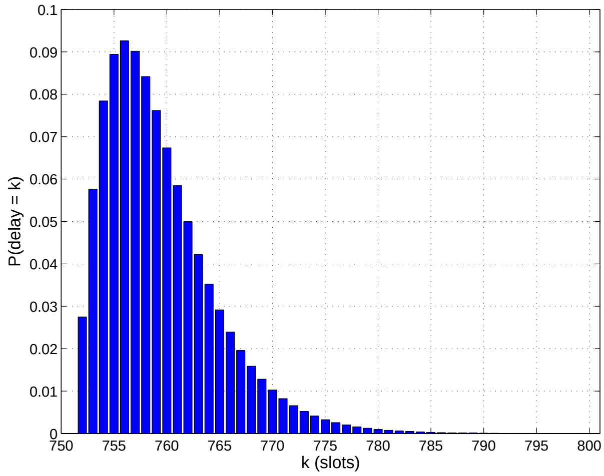

Figure 1: Network delay distribution

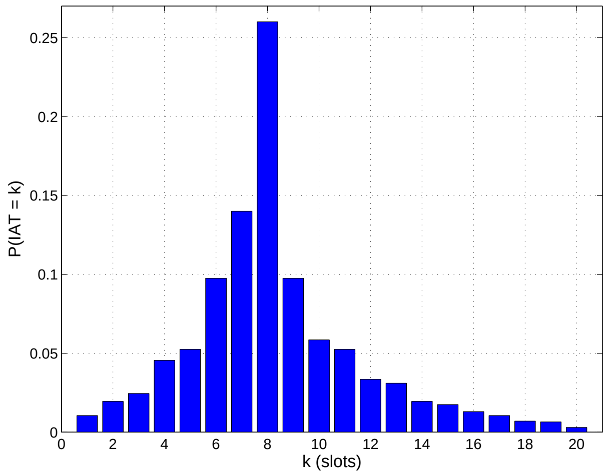

Figure 2: Packet interarrival time distribution of the variable bit rate stream

Figures 3 until 6 present performance results obtained with the mathematical model developed in this paper, when the data shown in the Figures 1 and 2 is used as input to the model.

Figure 3 shows the underflow probability for different values of $T_{jit} + d^*$. $T_{jit} + d^*$ is always larger than or equal to $delay_{\min}$, which is 751 slots in this example. For $T_{jit} + d^*$ larger or equal to $delay_{\max}$, which is 800 slots, the underflow probability is zero. A curve as shown in Figure 3 can be used to dimension the parameter T_{jit} of the dejittering process. This parameter should be set as small as possible to reduce the extra delay introduced by the dejittering process, but large enough to keep the packet loss due to underflow below some target value. For a certain target

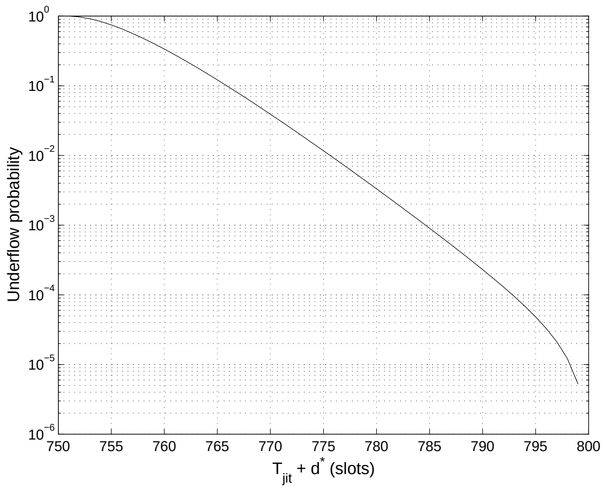

Figure 3: Underflow probabilities given T_{jit} and d^*

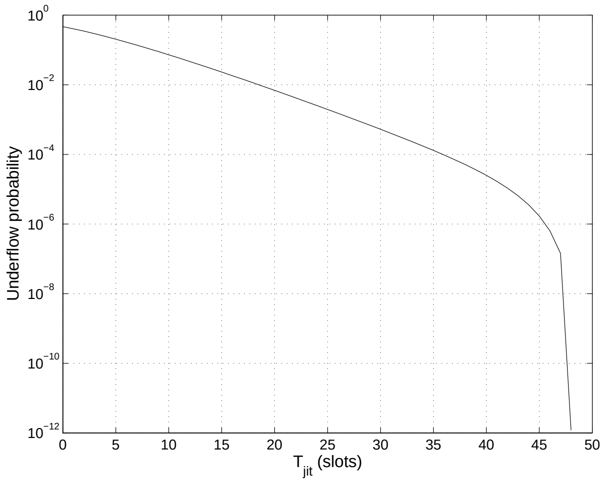

Figure 4: Average underflow probabilities given T_{jit}

loss limit, the corresponding value for $T_{jit}+d^*$ is read from the curve. Remark that in general the delay d^* experienced by the first packet of the video stream is not known at the moment a value for T_{jit} should be set. So under the worst case assumption that the first packet of the video stream experiences a delay $d^* = delay_{\min}$, the value for T_{jit} to be used is thus found.

The curve shown in Figure 4 gives the average underflow probability that would be obtained in a network with certain delay characteristics, when the same value of T_{jit} would be used in the dejittering process of all users receiving a video stream with the considered characteristics. Users for which the first packet of the video stream experiences a delay larger than $delay_{\min}$ will get a packet loss probability due to underflow smaller than the guaranteed one. As a consequence, the average underflow probability will be smaller than the guaranteed one.

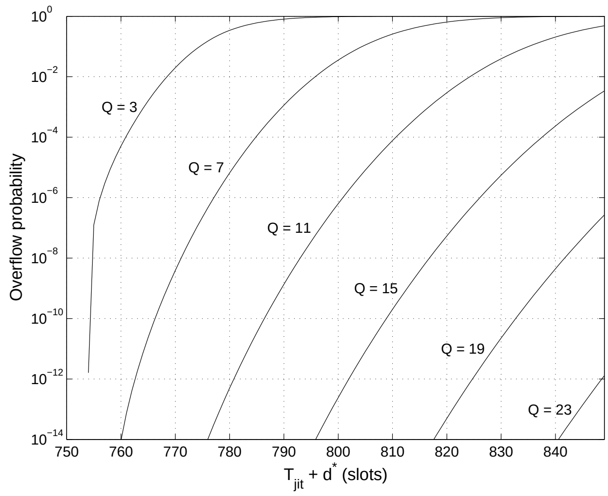

Figure 5: Overflow probabilities given T_{jit} and d^*, for different values of the dejittering buffer size Q

Figure 5 and 6 depict the overflow probabilities obtained for some values of Q, the size of the dejittering buffer. Figure 5 shows overflow probabilities given $T_{jit} + d^*$, while in Figure 6 average overflow probabilities given T_{jit} are seen.

Again the results given $T_{jit} + d^*$ can be used for dimensioning purposes, this time of the capacity Q the dejittering buffer should have to ensure that the packet loss probability due to overflow of the dejittering buffer stays below a guaranteed limit. As can be seen on Figure 5, for a target limit on the overflow probability and under the worst case assumption that d^* equals $delay_{\max}$, several valid combinations (T_{jit}, Q) are found. In practice however, one wants to guarantee a limit on both the packet loss due to overflow and due to underflow, which makes that only one combination (T_{jit}, Q) is considered as optimal, namely the one with the smallest value for T_{jit} that guarantees the limit on the underflow probability. Remark further that a guarantee ϵ for the underflow probability and a guarantee δ for the overflow probability together give a guarantee $\epsilon+\delta(1-\epsilon)$ on the general packet loss probability introduced by the dejittering process. So to guarantee a general loss probability of for example 1e-4, both ϵ and δ should be chosen smaller than 1e-4, in such a way that $\epsilon + \delta(1 - \epsilon) =$ 1e-4.

For a fixed value of Q, overflow probability results as shown in Figure 6 are the average overflow probabilities that are obtained in a network when a certain value for T_{jit} is used in the dejittering process of all users. Users for which the first packet of the video stream experiences a delay smaller than $delay_{\max}$ will get an overflow probability smaller than the guaranteed one. Remark that for typical delay distributions, the probability that a (first) packet will ex-

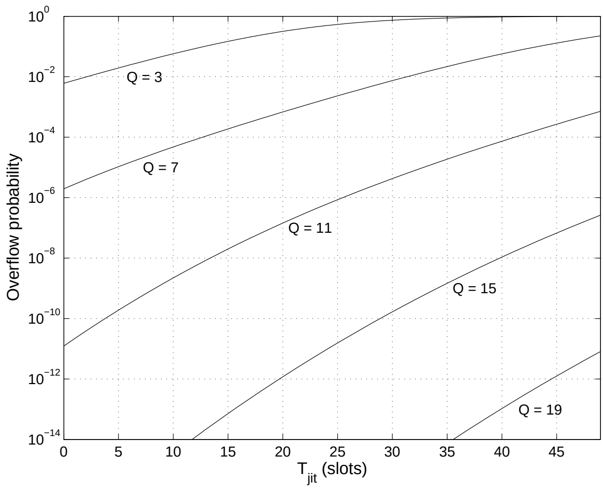

Figure 6: Average overflow probabilities given T_{jit}, for different values of the dejittering buffer size Q

perience the maximal delay is very small, so the average overflow probability will be considerably smaller than the guaranteed one which is based on the worst case delay the first packet of a video stream can experience.

CONCLUSIONS

In this paper a Markov model was developed that can be used to dimension the two important parameters of the dejittering process of variable bit rate video streams: the dejittering delay T_{jit} the dejittering process should wait before starting the playout of the first packet, and the capacity Q of the dejittering buffer. The model takes two probability distributions as input: that of the interarrival time of packets of the variable bit rate stream at the network, and that of the network delay. For every chosen value of $T_{jit} + d^*$, where d^* is the network delay that is experienced by the first packet of the video stream, a Markov chain with a different set of states $\mathcal{C}(T_{jit}+d^*)$ was defined. From the stationary distribution of such a Markov chain, the dejittering buffer underflow and overflow probability corresponding to $T_{jit}+d^*$ is then calculated. By weighting the results with the probability of all possible values d^* can take, underflow and overflow probabilities in function of T_{jit} are derived. To illustrate its calculability, the model was applied to a numerical example, where the delay distribution used in this example is rather typical.

In the video source considered in this paper, no temporal dependency among the interarrival times at the network of succeeding packets was assumed. As was shown, the overflow probability results depend on the probability of having more than x interarrival times in a time interval of a certain length. As a consequence, it is to be expected that a larger buffer capacity Q will be needed with bursty video sources, because then there will be periods with low arrival rates (large interarrival times) followed by periods with higher arrival rates (small interarrival times), and the buffer size Q should be large enough to capture a possible burst of early arrivals. The exact implications of bursty arrival processes on the dimensioning of the dejittering buffer are for further study.

Another area for further study is the use and modelling of dynamic or adaptive dejittering techiques (ETSI, 2003; Narbutt and Murphy, 2001). Dejittering considered in this paper is static dejittering, since once the decision about the value of the dejittering delay T_{jit} is taken at the establishment of the connection, it is never reviewed during the lifetime of the video stream. Dynamic dejittering techniques adjust the choice about the dejittering delay from time to time. These techniques become more common for voice streams, where the dejittering delay is typically changed at a new talkspurt, to limit the audible distortion. For video the instants at which such a change could be made are less clear. In (Laoutaris et al., 2004) dynamic dejittering for constant bit rate video traffic is considered. For variable bit rate video it is however more complicated to estimate on time which part of the variability in the arrival rate of the packets at the receiver is due to network delay variations, and which part is due to the variable character of the video stream.

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APPENDIX

The set of states $\mathcal{C}(T_{jit}+d^*)$, defined in (3), has the following properties:

1. $\mathcal{C}(T_{jit}+d^*)$ contains $\#\mathcal{D}.iat_{\max}$ elements.
2. From a state in $\mathcal{C}(T_{jit}+d^*)$ it is impossible to reach a state outside $\mathcal{C}(T_{jit}+d^*)$.
3. $\mathcal{C}(T_{jit}+d^*)$ always contains at least one state with $j = delay_{\max}$, namely the state $(0, iat_{\max} - 1, iat_{\max} - 1 + delay_{\max} - T_{jit} - d^*, delay_{\max})$ if $T_{jit} + d^* \leq iat_{\max} - 1 + delay_{\max}$, and the state $(T_{jit} + d^* - iat_{\max} + 1 - delay_{\max}, iat_{\max} - 1, delay_{\max})$ otherwise.
4. From all states in $\mathcal{C}(T_{jit}+d^*)$ it is possible to reach the state mentioned in property 3.

Namely,

1. $\mathcal{C}(T_{jit}+d^*)$ contains exactly one element for every combination (j,t), with $j \in \mathcal{D}$ and $t \in \{0,\ldots,iat_{\max}-1\}$.
2. This is because all states reachable with one transition from a random state in $\mathcal{C}(T_{jit}+d^*)$ are again elements of $\mathcal{C}(T_{jit}+d^*)$, as is easily seen by considering all one-step transitions described in the dejittering process Section.
3. Trivial.
4. Consider a random state of $\mathcal{C}(T_{jit}+d^*)$. Such a state has one of the following forms: (a) $(0, t, t + j - T_{jit} - d^*, j)$ with $T_{jit} + d^* - j > 0$, (b) $(0, t, t+j-T_{jit}-d^*, j)$ with $T_{jit}+d^*-j \leq 0$, or (c) $(T_{jit} + d^* - t - j, t, j)$. After a finite number of transitions with probability 1, the following states are reached: $(T_{jit} + d^* - j, 0, j)$ in cases (a) and (c), and $(0, 0, j - T_{jit} - d^*, j)$ in case (b). Then with probability $(\mathbf{U}_s)_{j,delay_{\max}} > 0$, where $s = iat_{\max} - 1 - j + delay_{\max}$, the state mentioned in property 3 is reached.

Combining all these properties leads to the property that the $\#\mathcal{D}.iat_{\max}$ states of $\mathcal{C}(T_{jit}+d^*)$ form a Markov chain with exactly one irreducible closed set of states. This set contains all states of $\mathcal{C}(T_{jit}+d^*)$ that are reachable from the state mentioned in property 3. States not in this closed set are transient.

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RETRIAL QUEUEING MODEL FOR MULTIMEDIA OVER DOWNLINK IN 3.5G WIRELESS NETWORK

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KEYWORDS

3.5G Wireless Networks, Multimedia Traffic, Stochastic Models, Performance Modelling, Retrials

ABSTRACT

In this paper, a model for multimedia transmission over downlink shared channel in 3.5G wireless network with possible retrials is presented. The multimedia stream consists of multiple substreams that are aggregated into one real-time and one nonreal-time flows. Correlation with each flow and between flows is assumed. Additionally, we propose a combined time-space priority buffer management scheme to optimize quality of service requirements for each flow. The problem is formulated in terms of a queue with two priority classes, one of which has time priority while the another has space priority. The input is described by the Batch Marked Markovian Arrival Process ($BMMAP$). Service time distributions are of PH (phase) type dependent on the class of a customer. The buffer is finite, but the customers of a class having higher priority for taking into the service from a buffer (time priority) can occupy only a part of this buffer. The customers of a class having lower priority, which meet full buffer upon arrival, become repeated customers and try to enter the system again in random time intervals. Queueing system's behavior is described in terms of multi-dimensional continuous time space inhomogeneous skip-free to the left Markov chain. It allows to exploit an effective algorithm for calculation of the stationary distribution of the queueing system. Loss probability for customers of both classes is calculated. Waiting time distribution for priority customers is calculated.

1 Introduction

The model investigated in this paper is generalization of the model considered in (Al-Begain et al, 2005). In contrast to (Al-Begain et al, 2005), where the lower priority customers are lost in case when the buffer is full at an arrival epoch, here we assume that these customers have a chance to try their luck to reach the system after some random time interval. This paper contains the formulation of the model and its analysis via the tool of the multi-dimensional continuous time asymptotically quasi-Toeplitz Markov chains (Klimenok and Dudin, 2005). This will then be the base for our optimization goal in which we use the threshold value to obtain desirable QoS for the multimedia traffic.

2 Mathematical model

We have a single-server queue with a finite buffer of a capacity R, $R > 0$. The input flow is described by the $BMMAP$ (He, 1996). Customers arrival in the $BMMAP$ is directed by the irreducible continuous time Markov chain ν_t, $t \geq 0$, with a finite state space $\{0, 1, \dots, W\}$. Sojourn time of the chain ν_t, $t \geq 0$, in the state ν has exponential distribution with a parameter λ_ν. After this time expires, with probability $p_0(\nu, \nu')$ the chain jumps into the state ν' without generation of customers and with probability $p_k^{(l)}(\nu, \nu')$ the chain jumps into the state ν' and a batch consisting of k customers of type l (type-l customers) is generated, $k \geq 1$. Here we will assume that only two types of customers are served in the system, so $l = 1, 2$. The introduced probabilities satisfy conditions:

$$p_0(\nu, \nu) = 0,$$

$$\sum_{l=1}^{2}\sum_{k=1}^{\infty}\sum_{\nu'=0}^{W} p_k^{(l)}(\nu,\nu') + \sum_{\nu'=0}^{W} p_0(\nu,\nu') = 1,\ \nu = \overline{0,W}.$$

The parameters defining the $BMMAP$ can be kept in the square matrices D_0, $D_k^{(1)}$, $D_k^{(2)}$, $k \geq 1$, of size $\bar{W} = W + 1$ defined by their entries:

$$(D_0)_{\nu,\nu} = -\lambda_\nu,\ (D_0)_{\nu,\nu'} = \lambda_\nu p_0(\nu,\nu'),$$

$$(D_k^{(l)})_{\nu,\nu'} = \lambda_\nu p_k^{(l)}(\nu,\nu'),\ \nu,\nu' = \overline{0,W},\ k \geq 1,\ l = 1,2.$$

Denote $D(1) = D_0 + \sum\limits_{l=1}^{2}\sum\limits_{k=1}^{\infty} D_k^{(l)}$, $\hat{D}_k^{(l)} = \sum\limits_{i=k}^{\infty} D_i^{(l)}$, $l = 1,2$, $k \geq 1$, $\boldsymbol{\theta}$ is the stationary probability vector of the states of the Markov chain ν_t, $t\ \geq 0$.

Vector $\boldsymbol{\theta}$ is the unique solution to the system

$$\boldsymbol{\theta} D(1) = \mathbf{0}, \boldsymbol{\theta}\mathbf{e} = 1.$$

Here and below $\mathbf{0}$ is the row vector of dimension which should be clear from context, $\mathbf{e}$ is the column vector of appropriate dimension consisting of all 1's.

Intensity λ_l of type-l customers arrival is calculated by $\lambda_l = \boldsymbol{\theta} \sum\limits_{k=1}^{\infty} k D_k^{(l)}\mathbf{e}, l = 1,2.$

Intensity $\lambda_l^{(b)}$ of batches of type-l customers arrival is calculated by $\lambda_l^{(b)} = \boldsymbol{\theta}\hat{D}_1^{(l)}\mathbf{e},\ \ l = 1,2.$

Variation v_l of inter-arrival times of batches of type-l customers arrival is calculated by

$$v_l = \frac{2\boldsymbol{\theta}(-D_0 - \hat{D}_1^{(\bar{l})})^{-1}\mathbf{e}}{\lambda_l^{(b)}} - \left(\frac{1}{\lambda_l^{(b)}}\right)^2,\ \ \bar{l} \neq l, \bar{l}, l = 1,2.$$

Coefficient of correlation $C_{cor}^{(l)}$ of two successive intervals between type-l customers arrival is computed by

$$C_{cor}^{(l)} = \left[\frac{\boldsymbol{\theta}(-D_0 - \hat{D}_1^{(\bar{l})})^{-1}}{\lambda_l^{(b)}}\hat{D}_1^{(l)}(-D_0 - \hat{D}_1^{(\bar{l})})^{-1}\mathbf{e} - \left(\frac{1}{\lambda_l^{(b)}}\right)^2\right] v_l^{-1},\ \bar{l} \neq l, \bar{l}, l = 1,2.$$

Coefficient of correlation $C_{cor}^{(l,\bar{l})}$ of two successive intervals between type-l and type-$\bar{l}$ customers arrival (coefficient of cross correlation) is computed by

$$C_{cor}^{(l,\bar{l})} = \left[\frac{\boldsymbol{\theta}(-D_0 - \hat{D}_1^{(\bar{l})})^{-1}}{\lambda_l^{(b)}}\hat{D}_1^{(l)}(-D_0 - \hat{D}_1^{(l)})^{-1}\mathbf{e} - \frac{1}{\lambda_l^{(b)}\lambda_{\bar{l}}^{(b)}}\right](\sqrt{v_l}\sqrt{v_{\bar{l}}})^{-1},\ \bar{l} \neq l, \bar{l}, l = 1,2.$$

Type-1 customers are accepted into the system if the buffer is not full. If the buffer is full or the size of arriving batch exceeds the available space in a buffer we assume that the corresponding part of the batch (if any) is accepted into the buffer while the rest moves to some virtual pool called the orbit and tries to get the service later on. Each customer staying in the orbit makes the repeated attempts in random intervals having length exponentially distributed with parameter α ($\alpha > 0$), independently of the other customers. If the server is idle or the buffer is not full at the retrial epoch, the customer occupies the server or takes a place in the buffer. If the server is busy at the retrial epoch and the buffer is full, the customer returns to the orbit. Every customer tries to get the service until it succeeds to occupy the server or a place in the buffer.

Type-2 customers have a priority with respect to type-1 customers. If at least one type-2 customers presents in the system at the service completion epoch, then type-2 customer will get the service. Type-1 customers have a chance to get service only if no one type-2 customer presents in the system. Interruption of the service is not allowed.

However, type-2 customers have more restricted access into the system. No more then N, $1 \leq N < R$, type-2 customers can be accepted into the buffer. Discipline of partial admission is applied as well.

The service time of customers has PH (phase) type distribution. It means the following. Service of type-l customer is defined as a time until the continuous-time Markov chain $\eta_t^{(l)}$, $t \geq 0$, which has the states $(1,\ldots,M_l)$ as the transient and state 0 as the absorbing one, reaches the absorbing state. The initial state of the chain is selected in a random way, according to the probability distribution defined by the row vector $(\boldsymbol{\beta}_l, 0)$, where $\boldsymbol{\beta}_l$ is the stochastic row vector of dimension M_l. Transitions of the Markov chain $\eta_t^{(l)}$, $t \geq 0$, are described by the generator $\begin{pmatrix} S^{(l)} & S_0^{(l)} \\ 0 & 0 \end{pmatrix}$ where the matrix $S^{(l)}$ is sub-generator and the column vector $S_0^{(l)}$ is defined by $S_0^{(l)} = -S^{(l)}\mathbf{e}$. The average service time is given by $\boldsymbol{\beta}_1^{(l)} = \boldsymbol{\beta}_l(-S^{(l)})^{-1}\mathbf{e}$. For more details about the PH type distribution see (Neuts, 1981).

3 Markovian process of the system states

Let j_t be the number of customers presenting in the orbit at the epoch t; $i_t^{(l)}$ be the number of type-l customers presenting in a buffer at the epoch t, $l = 1,2$; ξ_t be the type of the customer in service at epoch t; ν_t be the state of the directing process of the $BMMAP$; $\eta_t^{(\xi_t)}$ be the state of the process which defines the service at the epoch $t,\ t \geq 0$. It is clear that the process

$$\zeta_t = \{j_t, i_t^{(2)},\ i_t^{(1)},\ \xi_t,\ \nu_t,\ \eta_t^{(\xi_t)}\},\ t \geq 0,\ j_t \geq 0,$$

$$i_t^{(1)} = \overline{0, R - i_t^{(2)}}, i_t^{(2)} = \overline{0,N}, \nu_t = \overline{0,W},\ \eta_t^{(l)} = \overline{1,M_l},$$

$$\xi_t = \begin{cases} l, & \text{if type-}l\text{ customer is in the service, } l = 1,2, \\ 0, & \text{if the server is idle.} \end{cases}$$

is the Markov chain.

Denote the stationary state probabilities of this Markov chain by:

$$p(j,0,0,0,\nu) =$$

$$\lim_{t\to\infty} P\{j_t = j, i_t^{(2)} = 0, i_t^{(1)} = 0, \xi_t = 0, \nu_t = \nu\},$$

$$p(j,i_2,i_1,r,\nu,\eta) = \lim_{t\to\infty} P\{j_t = j, i_t^{(2)} = i_2, i_t^{(1)} = i_1,$$

$$\xi_t = r, \nu_t = \nu, \eta_t = \eta\}, j \geq 0, i_1 = \overline{0, R - i_2},$$

$$i_2 = \overline{0,N},\ \nu = \overline{0,W}, \eta = \overline{1,M_r}, r = 1,2.$$

Condition of these limits existence will be given below.

To simplify operation with the probabilities, we enumerate the states of the processes in the lexicographic order and introduce the vectors of stationary probabilities:

$$\mathbf{p}(j,0,0,0) = (p(j,0,0,0,0), p(j,0,0,0,1), \ldots,$$

$$p(j,0,0,0,W)), \mathbf{p}(j,i_2,i_1,r) = (p(j,i_2,i_1,r,0,1), \ldots,$$

$$p(j,i_2,i_1,r,0,M_r), p(j,i_2,i_1,r,1,0), \ldots,$$

$$p(j,i_2,i_1,r,1,M_r), \ldots, p(j,i_2,i_1,r,W,M_r)), r = 1,2.$$

So, the row vector $\mathbf{p}(j,0,0,0)$ has dimension $\bar{W}$, the row vector $\mathbf{p}(j,i_2,i_1,r)$ has dimension $\bar{W}M_r$, $r = 1,2$.

In what follows, we use the following denotations: I is the identity matrix of dimension defined by the suffix, $\otimes$ and $\oplus$ are the symbols of Kronecker product and sum of the matrices correspondingly.

Combine now the states corresponding to a fixed value of components $j,\ i_2$ and r and introduce macro-vectors of stationary probabilities:

$$\tilde{\mathbf{p}}(j,0) = \mathbf{p}(j,0,0,0),\ \mathbf{p}(j,i_2,r) = (\mathbf{p}(j,i_2,0,r),$$

$$\mathbf{p}(j,i_2,1,r), \ldots, \mathbf{p}(j,i_2,R{-}i_2,r)),\ i_2 = \overline{0,N},\ r = 1,2,$$

macro-vectors $\mathbf{p}(j)$ consisting of macro-vectors $\tilde{\mathbf{p}}(j,0)$, $\mathbf{p}(j,i_2,r)$, $i_2 = \overline{0,N}$, $r = 1,2$, $j \geq 0$, listed in lexicographic order and macro-vector

$$\mathbf{p} = (\mathbf{p}(0),\ \mathbf{p}(1), \mathbf{p}(2), \ldots).$$

Introduce the following notations of matrices:

$\mathcal{Z}^{(1)}$ has dimension $\bar{W} \times (R{+}1)\bar{W}M_1$ and is defined by $\mathcal{Z}^{(1)} = (D_1^{(1)} \otimes \boldsymbol{\beta}_1, \ldots, D_R^{(1)} \otimes \boldsymbol{\beta}_1, D_{R+1}^{(1)} \otimes \boldsymbol{\beta}_1)$;

$\mathcal{Z}_i^{(2)}$ has dimension $\bar{W} \times (R+2-i)\bar{W}M_2$ and is calculated by $\mathcal{Z}_i^{(2)} = (D_i^{(2)} \otimes \boldsymbol{\beta}_2, O, \ldots, O)$, $i = \overline{1,N}$, $\mathcal{Z}_{N+1}^{(2)} = (\hat{D}_{N+1}^{(2)} \otimes \boldsymbol{\beta}_2, O, \ldots, O)$;

$\mathcal{G}_i^{(l)}$ is the matrix of dimension $(R-i+1)\bar{W}M_l \times (R-i+1)\bar{W}M_l$, $l = 1,2$. The matrix represents the block diagonal matrix with $R-i$ diagonal blocks $I_{\bar{W}M_l}$ supplemented from the right side by the zero block column of dimension $(R-i)\bar{W}M_l \times \bar{W}M_l$ and from below by the zero block row of dimension $\bar{W}M_l \times (R-i+1)\bar{W}M_l$, $\tilde{\mathcal{G}}_0 = I_{\bar{W}}$;

$\mathcal{S}_{i,k}^{(l)}$ is the square matrix of dimension $(R-i+1)\bar{W}M_l$, $i = \overline{0,N}$, $l = 1,2$, defined as follows:

$$\mathcal{S}_{i,k}^{(l)} = \begin{pmatrix} O & O & \ldots & O & D_{k+R-i}^{(1)} \otimes I_{M_l} \\ O & O & \ldots & O & D_{k+R-i-1}^{(1)} \otimes I_{M_l} \\ O & O & \ldots & O & D_{k+R-i-2}^{(1)} \otimes I_{M_l} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \ldots & O & D_{k+1}^{(1)} \otimes I_{M_l} \\ O & O & \ldots & O & D_k^{(1)} \otimes I_{M_l} \end{pmatrix},$$

$\tilde{\mathcal{S}}_{0,k} = (O, \ldots, O, D_{k+R}^{(1)}, O, \ldots, O)$;

$\mathcal{D}_{i,j}^{(l)}$ is the matrix of dimension $(R+1-i)\bar{W}M_l \times (R+1-i-j)\bar{W}M_l$, $j = \overline{1, N-1-i}$, $i = \overline{0, N-1}$, $l = 1,2$. It consists of the block diagonal matrix with the diagonal blocks $\{D_j^{(2)} \otimes I_{M_l}, \ldots, D_j^{(2)} \otimes I_{M_l}, \hat{D}_j^{(2)} \otimes I_{M_l}\}$ supplemented from below by the zero matrix;

$\mathcal{D}_{i,N-i}^{(l)}$ has the analogous structure, but the blocks of the diagonal matrix are equal to $\hat{D}_{N-i}^{(2)} \otimes I_{M_l}$, $l = 1,2$;

$\mathcal{B}_i^{(l,2)}$ is the matrix of dimension $(R{+}1{-}i)\bar{W}M_l \times (R+2-i)\bar{W}M_2$, $l = 1,2$. For $i = \overline{1,N}$, this matrix represents the block diagonal matrix with diagonal blocks $I_{\bar{W}} \otimes S_0^{(l)}\boldsymbol{\beta}_2$ supplemented from the right side by the zero block column of dimension $(R+1-i)\bar{W}M_l \times \bar{W}M_2$;

$\mathcal{B}_0^{(l,1)}$ is the matrix of dimension $(R{+}1)\bar{W}M_l{\times}(R{+}2)\bar{W}M_1$, $l = 1,2$. The matrix represents the block diagonal matrix with R diagonal blocks $I_{\bar{W}} \otimes S_0^{(l)}\boldsymbol{\beta}_1$ supplemented from the right side by the zero block column of dimension $(R+1)\bar{W}M_l \times \bar{W}M_1$ and from above by the zero block row of dimension $\bar{W}M_l \times (R+1)\bar{W}M_1$;

$\mathcal{T}^{(l)}$ has dimension $(R{+}1)\bar{W}M_l \times \bar{W}$, $l = 1,2$, and is defined as the block column having zero blocks except the first block which is equal to $I_{\bar{W}} \otimes S_0^{(l)}$;

$\mathcal{A}_i^{(l)}$ is the square matrix of dimension $(R-i+1)\bar{W}M_l$, $i = \overline{0,N-1}$, $l = 1,2$, defined as follows:

$$
\mathcal{A}_i^{(l)} = \begin{pmatrix} D_0 \oplus S^{(l)} & D_1^{(1)} \otimes I_{M_l} & D_2^{(1)} \otimes I_{M_l} & \dots & D_{R-i-1}^{(1)} \otimes I_{M_l} & D_{R-i}^{(1)} \otimes I_{M_l} \\ O & D_0 \oplus S^{(l)} & D_1^{(1)} \otimes I_{M_l} & \dots & D_{R-i-2}^{(1)} \otimes I_{M_l} & D_{R-i-1}^{(1)} \otimes I_{M_l} \\ O & O & D_0 \oplus S^{(l)} & \dots & D_{R-i-3}^{(1)} \otimes I_{M_l} & D_{R-i-2}^{(1)} \otimes I_{M_l} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \dots & D_0 \oplus S^{(l)} & D_1^{(1)} \otimes I_{M_l} \\ O & O & O & \dots & O & (D_0 + \hat{D}_1^{(2)}) \oplus S^{(l)} \end{pmatrix}; \tag{1}
$$

$\mathcal{A}_N^{(l)}$ is the square matrix of dimension $(R-N+1)\bar{W}M_l$, $l=1,2$, having the same structure, as the matrix (1), but the matrix D_0 on diagonal blocks is replaced with $D_0 + \hat{D}_1^{(2)}$;

$\mathcal{R}_i^{(l)}$ is the matrix of dimension $(R-i+1)\bar{W}M_l \times (R-i+1)\bar{W}M_l$, $l=1,2$. The matrix represents the block diagonal matrix with $R-i$ diagonal blocks $I_{\bar{W}M_l}$ supplemented from the left side by the zero block column of dimension $(R-i)\bar{W}M_l \times \bar{W}M_l$ and from below by the zero block row of dimension $\bar{W}M_l \times (R-i+1)\bar{W}M_l$, $\widetilde{\mathcal{R}}_0 = (I_{\bar{W}} \otimes \boldsymbol{\beta}_1, O, \dots, O)$.

Theorem 1. The macro-vector $\mathbf{p}$ satisfies the following equation:

$$
\mathbf{p}Q = \mathbf{0}, \tag{2}
$$

where Q is generator of the Markov chain $\zeta_t, t \geq 0$, having the following structure:

$$
Q = \begin{pmatrix} Q_{0,0} & Q_{0,1} & Q_{0,2} & Q_{0,3} & \cdots \\ Q_{1,0} & Q_{1,1} & Q_{1,2} & Q_{1,3} & \cdots \\ O & Q_{2,1} & Q_{2,2} & Q_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3}
$$

where the blocks $Q_{j,h}$ are calculated as follows:

$$
Q_{j,j} = \mathbf{Z} - j\alpha\mathbf{G},\ j \geq 0,
$$

$$
Q_{j,j-1} = j\alpha\mathbf{R},\ j \geq 1, Q_{j,j+k} = \mathbf{S}_k,\ j \geq 0, k \geq 1,
$$

where

$$
\mathbf{Z} = \begin{pmatrix} D_0 & \mathcal{Z}_0 & \mathcal{Z}_1 & \mathcal{Z}_2 & \dots & \mathcal{Z}_N \\ \mathcal{T} & \mathcal{A}_0 & \mathcal{D}_{0,1} & \mathcal{D}_{0,2} & \dots & \mathcal{D}_{0,N} \\ O & \mathcal{B}_1 & \mathcal{A}_1 & \mathcal{D}_{1,1} & \dots & \mathcal{D}_{1,N-1} \\ O & O & \mathcal{B}_2 & \mathcal{A}_2 & \dots & \mathcal{D}_{2,N-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & O & \dots & \mathcal{A}_N \end{pmatrix},
$$

$$
\mathcal{T} = \begin{pmatrix} \mathcal{T}^{(1)} \\ \mathcal{T}^{(2)} \end{pmatrix}, \mathcal{A}_0 = \begin{pmatrix} \mathcal{A}_0^{(1)} + \mathcal{B}_0^{(1,1)} & O \\ \mathcal{B}_0^{(2,1)} & \mathcal{A}_0^{(2)} \end{pmatrix},
$$

$$
\mathcal{A}_i = \mathrm{diag}\{\mathcal{A}_i^{(1)}, \mathcal{A}_i^{(2)}\}, \mathcal{B}_i = \begin{pmatrix} O & \mathcal{B}_i^{(1,2)} \\ O & \mathcal{B}_i^{(2,2)} \end{pmatrix}, i = \overline{1,N},
$$

$$
\mathcal{D}_{i,j} = \mathrm{diag}\{\mathcal{D}_{i,j}^{(1)}, \mathcal{D}_{i,j}^{(2)}\},\ j = \overline{1, N-i},\ i = \overline{0, N-1},
$$

$$
\mathcal{Z}_0 = (\mathcal{Z}^{(1)}, \mathcal{Z}_1^{(2)}), \mathcal{Z}_i = (O, \mathcal{Z}_{i+1}^{(2)}),\ i = \overline{1,N},
$$

$$
\mathbf{G} = \begin{pmatrix} \widetilde{\mathcal{G}}_0 & O & O & \dots & O \\ O & \mathcal{G}_0 & O & \dots & O \\ O & O & \mathcal{G}_1 & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & \mathcal{G}_N \end{pmatrix},
$$

$$
\mathbf{R} = \begin{pmatrix} O & \widetilde{\mathcal{R}}_0 & O & \dots & O \\ O & \mathcal{R}_0 & O & \dots & O \\ O & O & \mathcal{R}_1 & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & \mathcal{R}_N \end{pmatrix},
$$

$$
\mathbf{S}_k = \begin{pmatrix} O & \widetilde{\mathcal{S}}_{0,k} & O & \dots & O \\ O & \mathcal{S}_{0,k} & O & \dots & O \\ O & O & \mathcal{S}_{1,k} & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & \mathcal{S}_{N,k} \end{pmatrix},
$$

$$
\mathcal{G}_i = diag\{\mathcal{G}_i^{(1)}, \mathcal{G}_i^{(2)}\}, \mathcal{R}_i = diag\{\mathcal{R}_i^{(1)}, \mathcal{R}_i^{(2)}\}, i = \overline{1,N},
$$

$$
\mathcal{S}_{i,k} = diag\{\mathcal{S}_{i,k}^{(1)}, \mathcal{S}_{i,k}^{(2)}\}, i = \overline{1,N},
$$

$\mathrm{diag}\{C_1, \dots, C_N\}$ denotes the block diagonal matrix with the diagonal entries $\{C_1, \dots, C_N\}$.

Theorem 2. Stationary distribution of the Markov chain $\zeta_t, t \geq 0$, exists if the following inequality holds good:

$$
\lambda_1 \boldsymbol{\beta}_1^{(1)} < 1. \tag{4}
$$

Note that stability of the system is defined only by the intensities of arrival and service of type-1 customers. It is easy explained intuitively. When the number of customers in the orbit is very large, type-2 customers have no chance to enter the system because type-1 customer from the orbit immediately occupies a free place in the buffer upon any service completion epoch.

The continuous-time multi-dimensional Markov chain describing the behavior of the system belongs to the class of multi-dimensional asymptotically quasi-Toeplitz Markov chains. So, the effective and stable algorithm for solving system (2), (3), which is presented in (Klimenok and Dudin, 2005), can be applied for solving this system.

Having the stationary distribution been computed we can easy calculate different performance measures of the system can be calculated and optimization problems can be solved.

Corollary 1. Average number $L_1^{(l)}$ of type-l customers in the buffer is calculated by

$$
L_1^{(l)} = \sum_{r=1}^{2} \sum_{j=0}^{\infty} \sum_{i_2=0}^{N} \sum_{i_1=0}^{R-i_2} i_l \mathbf{p}(j, i_2, i_1, r)\mathbf{e}, l = 1, 2.
$$

Average number M_1 of type-1 customers in the orbit is calculated by

$$M_1 = \sum_{j=0}^{\infty} j \Big[\sum_{r=1}^{2} \sum_{i_2=0}^{N} \sum_{i_1=0}^{R-i_2} \mathbf{p}(j, i_2, i_1, r)\mathbf{e} + \tilde{\mathbf{p}}(j, 0)\mathbf{e} \Big].$$

Variation $V^{(l)}$ of the number of type-l customers in the system is calculated by

$$V^{(l)} = \sum_{r=1}^{2} \sum_{i_2=0}^{N} \sum_{i_1=0}^{R-i_2} i_l^2 \mathbf{p}(i_2, i_1, r) - (L_1^{(l)})^2, l = 1, 2.$$

Average number $\hat{L}_1^{(1)}$ of type-1 customers in the system is calculated by $\hat{L}_1^{(1)} = L_1^{(1)} + M_1$.

Fraction F_r of time when the system processes the type-r customers is computed by

$$F_r = \sum_{j=0}^{\infty} \sum_{i_2=0}^{N} \sum_{i_1=0}^{R-i_2} \mathbf{p}(j, i_2, i_1, r)\mathbf{e}, r = 1, 2.$$

Fraction F_0 of time when the server is idle is computed by $F_0 = \sum\limits_{j=0}^{\infty} \tilde{\mathbf{p}}(j, 0)\mathbf{e}$.

4 Loss probability

We will assume below that customers of the same priority class are served according to $FIFO$ (first in - first out) discipline and the customers in an arriving batch of size k are numbered in such a way as an arbitrary customer gets the number $m, m = \overline{1, k}$, $k \geq 1$, with equal probability $\frac{1}{k}$.

Theorem 3. *The loss probability $P_{loss}^{(2)}$ is calculated by*

$$P_{loss}^{(2)} = \lambda_2^{-1} \Big(\sum_{j=0}^{\infty} \mathbf{p}(j, 0, 0, 0) \sum_{k=N+2}^{\infty} (k-N-1) D_k^{(2)} \mathbf{e} +$$

$$\sum_{l=1}^{2} \sum_{j=0}^{\infty} \sum_{i_2=0}^{N} \sum_{i_1=0}^{R-i_2} \mathbf{p}(j, i_2, i_1, l) \times \sum_{k=\min\{N-i_2, R-i_2-i_1\}-1}^{\infty}$$

$$(k - \min\{N - i_2, R - i_2 - i_1\}) D_k^{(2)} \otimes I_{M_l} \mathbf{e} \Big). \quad (5)$$

5 Waiting time distribution

Denote $W_2(x)$ waiting time distribution function for type-2 (real-time) flow. Let $\omega_2(v)$ be its Laplace-Stieltjes transform $\omega_2(v) = \int\limits_0^{\infty} e^{-vx} dW_2(x)$, $Re\ v > 0$, $W_1^{(2)}$ be the mean waiting time, $W_2^{(2)}$ be the second initial moment of distribution function and J_2 be a jitter for type-2 customers.

Theorem 3. Laplace-Stieltjes transform $\omega_2(v)$ of waiting time distribution function for type-2 customers is calculated as follows:

$$\omega_2(v) = \lambda_2^{-1} \Big(P_{loss}^{(2)} + \sum_{j=0}^{\infty} \mathbf{p}(j, 0, 0, 0) \sum_{k=1}^{\infty} D_k^{(2)} \mathbf{e} \times$$

$$\sum_{i=0}^{\min\{k-1,N\}} \big(\boldsymbol{\beta}_2 (vI - S^{(2)})^{-1} S_0^{(2)} \big)^i + \quad (6)$$

$$\sum_{l=1}^{2} \sum_{j=0}^{\infty} \sum_{i_2=0}^{N-1} \sum_{i_1=0}^{R-i_2-1} \mathbf{p}(j, i_2, i_1, l) \sum_{k=1}^{\infty} D_k^{(2)} \mathbf{e} \otimes \boldsymbol{\gamma}^{(l)}(v) \times$$

$$\sum_{m=0}^{\min\{k, N-i_2, R-i_2-i_1\}-1} \big(\boldsymbol{\beta}_2 (vI - S^{(2)})^{-1} S_0^{(2)} \big)^{i_2+m} \Big),$$

where $\boldsymbol{\gamma}^{(l)}(v) = (vI - S^{(l)})^{-1} S_0^{(l)}$, $l = 1, 2$.

Corollary 2. The mean waiting time $W_1^{(2)}$, the second initial moment $W_2^{(2)}$ of distribution function and a jitter J_2 for type-2 customers are calculated by

$$W_1^{(2)} = -\omega_2'(0) = \lambda_2^{-1} \Big(\sum_{j=0}^{\infty} \mathbf{p}(j, 0, 0, 0) \sum_{k=1}^{\infty} D_k^{(2)} \mathbf{e} \boldsymbol{\beta}_1^{(2)} \times$$

$$\frac{\min\{k, N+1\}}{2} \min\{k-1, N\} +$$

$$\sum_{l=1}^{2} \sum_{j=0}^{\infty} \sum_{i_2=0}^{N-1} \sum_{i_1=0}^{R-i_2-1} \mathbf{p}(j, i_2, i_1, l) \sum_{k=1}^{\infty} D_k^{(2)} \otimes I_{M_l} \times$$

$$\Big(\mathbf{e} \otimes \bar{\boldsymbol{\gamma}}^{(l)} + \mathbf{e} \boldsymbol{\beta}_1^{(2)} \min\{k, N - i_2, R - i_1 - i_2\} \times$$

$$[i_2 + \frac{\min\{k, N - i_2, R - i_1 - i_2\} - 1}{2}] \Big) \Big),$$

$$W_2^{(2)} = -\omega_2''(0) = \lambda_2^{-1} \Big(\sum_{j=0}^{\infty} \mathbf{p}(j, 0, 0, 0) \sum_{k=1}^{\infty} D_k^{(2)} \mathbf{e} \times$$

$$\sum_{i=0}^{\min\{k-1,N\}} \Big(i \boldsymbol{\beta}_2^{(2)} + \frac{i(i-1)}{2} (\boldsymbol{\beta}_1^{(2)})^2 \Big) +$$

$$\sum_{l=1}^{2} \sum_{j=0}^{\infty} \sum_{i_2=0}^{N-1} \sum_{i_1=0}^{R-i_2-1} \mathbf{p}(j, i_2, i_1, l) \sum_{k=1}^{\infty} D_k^{(2)} \otimes I_{M_l} \Big(\mathbf{e} \otimes \bar{\bar{\boldsymbol{\gamma}}}^{(l)} +$$

$$\sum_{m=i_2}^{\min\{k+i_2, N, R-i_1\}-1} \Big(2\mathbf{e} \otimes \bar{\boldsymbol{\gamma}}^{(l)} \boldsymbol{\beta}_1^{(2)} m + m \boldsymbol{\beta}_2^{(2)} \mathbf{e} +$$

$$m(m-1) \big(\boldsymbol{\beta}_1^{(2)} \big)^2 \mathbf{e} \Big) \Big) \Big), J_2 = W_2^{(2)} - \big(W_1^{(2)} \big)^2,$$

where $\boldsymbol{\beta}_2^{(l)} = 2\boldsymbol{\beta}_l (-S^{(l)})^{-2} \mathbf{e}$, $\bar{\boldsymbol{\gamma}}^{(l)} = (-S^{(l)})^{-1} \mathbf{e}$, $\bar{\bar{\boldsymbol{\gamma}}}^{(l)} = 2(-S^{(l)})^{-2} \mathbf{e}$.

Waiting time distribution $W_2(x)$ defines probability that waiting time of an arbitrary type-2 customer is less or equal to x. Denote by $\tilde{W}_2(x)$ the waiting time distribution for an arbitrary type-2 customer which was not lost due to the buffer overfull. It is easy to see that the distribution functions $W_2(x)$ and $\tilde{W}_2(x)$ are related as follows:

$$\tilde{W}_2(x) = \frac{W_2(x) - P_{loss}^{(2)}}{1 - P_{loss}^{(2)}}.$$

So, the following corollary is obvious.

Corollary 3. The mean waiting time $\tilde{W}_1^{(2)}$, the second initial moment $\tilde{W}_2^{(2)}$ of distribution function and a jitter $\tilde{J}_2$ for type-2 customers are calculated by

$$\tilde{W}_n^{(2)} = \frac{W_n^{(2)}}{1 - P_{loss}^{(2)}}, \, n = 1, 2, \tilde{J}_2 = \tilde{W}_2^{(2)} - \big(\tilde{W}_1^{(2)}\big)^2.$$

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SESSION 4

QUEUEING SYSTEMS II

A NEW DYNAMIC PRIORITY SCHEME: PERFORMANCE ANALYSIS

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ABSTRACT

In this paper, we introduce a new dynamic priority scheduling discipline. We consider a discrete-time, two-class queueing system with two priority queues of infinite capacity, with one server, and with the HOL-JIA (head-of-line, jump-if-arrival) priority scheme. We show that the use of probability generating functions is suitable for analysing the system contents and the packet delay. By means of some numerical examples, we draw attention to the impact of HOL-JIA priority scheduling on performance measures, and compare the HOL-JIA scheduling with other scheduling types.

INTRODUCTION

A common feature encountered in modern communication networks is competition for limited capacity, by types of traffic with different QoS (Quality of Service) characteristics and requirements. *Real-time* traffic e.g., has strict delay requirements, i.e., mean delay and delay jitter have to be small. Voice and video are the prime examples of real-time traffic. For *non-real-time* traffic on the other hand, the loss ratio is the restrictive quantity. The ability to differentiate real-time, *delay-sensitive* traffic, and non-real-time, *delay-tolerant* traffic, is one of the main keys to a succesful communication network. Priority scheduling disciplines, in which multiple priority levels are provided, and in which different priority levels are given to different types of traffic, may help achieve differentation in QoS.

In the static, head-of-line (HOL) priority scheduling discipline for instance, priority is always given to the delay-sensitive traffic. So, as long as there is delay-sensitive traffic present in the system, this traffic has service priority over the delay-tolerant traffic. The HOL priority scheme does indeed provide low delays for the delay-sensitive, high-priority traffic (Bae, 1994; Walraevens, 2003). When the network is highly loaded however, HOL priority scheduling causes excessive delays for the delay-tolerant, low-priority traffic. The Transmission Control Protocol (TCP) e.g., considers this low-priority traffic as lost, and decreases its transmission rate (unneccessarily). The impact of the HOL priority scheme on the low-priority traffic may thus be too disadvantageous in some cases. In order to find a solution for this so-called *starvation problem*, several dynamic priority schemes have been proposed in the past.

These schemes are mostly obtained by either alternately serving high- and low-priority traffic (Choi, 1998; Ganos, 1996), or by allowing priority jumps (Bae, 1994; Lin, 1990; Maertens, 2004, 2005). In the latter type, referred to as head-of-line with priority jumps (HOL-PJ), high- and low-priority packets arrive in separate, logical queues, i.e., the high- and low-priority queue, but packets of the low-priority queue can now jump to the high-priority queue. Within this context, we analyse in this paper a discrete-time, two-class priority queueing system with a logical high- and low-priority queue, both having an infinite capacity, and with one server. Packets of the high-priority queue thus have service priority over packets of the low-priority queue. During a slot the high-priority queue is served, the packet at the HOL-position (or shorter, the *HOL-packet*) of the low-priority queue jumps to the high-priority queue, *if* during the slot, low-priority packets arrive at the system. With this last condition, we let, in a controlled manner (i.e., depending on the input) low-priority traffic *flow* into the high-priority queue. This new dynamic priority scheme, denoted by HOL-JIA (head-of-line, jump-if-arrival), thus tries to hold the advantage of the HOL priority scheme, i.e., keeping the delay of high-priority packets small, while trying to alleviate the starvation problem.

For assessing the performance of queues with the HOL-JIA scheduling discipline, we use an analysis based on probability generating functions. We obtain the probability generating functions of the contents of the high- and low-priority queue, and the probability generating function of the delay of a high-priority packet. From these probability generating functions, we can easily calculate expressions for some interesting performance measures, such as mean values and variances. Furthermore, we can calculate the mean delay of a low-priority packet, although we have not obtained the corresponding probability generating function. This paper thus demonstrates that an analysis based on probability generating functions is extremely suitable for analysing this type of queues with a dynamic priority scheduling discipline.

The outline of the paper is as follows. In the next section, we give the mathematical model. In the third and fourth section, we derive the steady-state system contents, and study the packet delay of both classes respectively. We present some numerical examples in the fifth section. Finally, in the last section, we formulate some conclusions.

MATHEMATICAL MODEL

We consider a discrete-time queueing system, i.e., time is assumed to be divided into fixed-length intervals called *slots*. The system contains two queues, both with infinite capacity, and one server. It can furthermore be characterized by three processes: the arrival process, the service process, and the jumping process.

We first describe the arrival process. Two types of traffic arrive at the system: packets of class 1, which are stored in the first queue, and packets of class 2, which are stored in the second. The number of class-j arrivals during slot k is denoted by $a_{j,k}\,(j=1,2)$, and the $a_{1,k}$s and $a_{2,k}$s are assumed to be independent and identically distributed (i.i.d.) from slot-to-slot. Within one slot, the numbers of arrivals of both classes can be correlated, and this possible correlation is described by their joint probability generating function (pgf) $A(z_1,z_2) \triangleq \lim_{k\to\infty} \mathrm{E}\left[z_1^{a_{1,k}} z_2^{a_{2,k}}\right]$. The number of per-slot arrivals of class 1 and class 2 *separately*, are characterized by the marginal pgfs $A_1(z)=A(z,1)$ and $A_2(z)=A(1,z)$ respectively (with $\lambda_j=A'_j(1)$ the arrival rate of class j). The *total* number of per-slot arrivals, i.e., $a_{T,k} \triangleq a_{1,k}+a_{2,k}$, is characterized by the marginal pgf $A_T(z)=A(z,z)$ (with $\lambda_T=A'_T(1)=\lambda_1+\lambda_2$ the total arrival rate). Since there is only one server, the stability condition for this system is given by $\lambda_T<1$.

Secondly, the service times of the packets are deterministically equal to *one* slot, and packets of the first queue (the high-priority queue) have a higher priority than those of the second queue (the low-priority queue). So, whenever there are packets present in the high-priority queue, they have service priority, and only when the high-priority queue is empty, packets of the low-priority queue are served. Within both queues separately, packets are served according to a first-in, first-out (FIFO) scheduling.

Finally, the system is influenced by a jumping process. This jumping process depends on the number of class-2 arrivals: when *both* queues are non-empty at the beginning of a slot, and when class-2 packets arrive during that slot, the packet at the HOL-position of the low-priority queue jumps at the end of the slot to the high-priority queue. When the number of class-2 arrivals *equals* zero, the HOL-packet of the low-priority queue does not jump. When one of the queues is empty, the packet at the HOL-position of the non-empty queue is served. Note that, since the possible jump occurs at the end of the slot, the jumping packet is queued after the class-1 arrivals during the same slot.

SYSTEM CONTENTS

We will now derive an expression for the joint pgf of the system contents of both queues at the beginning of a random slot in *steady-state*. In the assumption that the packet in service (if any) is part of the queue that is served in that slot, we define $u_{1,k}$ and $u_{2,k}$ as the system content of the high-priority queue and the low-priority queue at the beginning of slot k respectively, and $u_{T,k}$ as the total system content at the beginning of slot k. The joint pgf of $u_{1,k}$ and $u_{2,k}$ is then denoted by $U_k(z_1,z_2) \triangleq \mathrm{E}\left[z_1^{u_{1,k}} z_2^{u_{2,k}}\right]$. The system under consideration satisfies the following *system equations*:

- if $u_{1,k}=0$:

$$\begin{cases} u_{1,k+1}=a_{1,k} \\ u_{2,k+1}=[u_{2,k}-1]^+ + a_{2,k} \end{cases}, \qquad (1)$$

- if $u_{1,k}>0, u_{2,k}=0$:

$$\begin{cases} u_{1,k+1}=u_{1,k}-1+a_{1,k} \\ u_{2,k+1}=a_{2,k} \end{cases}, \qquad (2)$$

- if $u_{1,k},u_{2,k}>0$:
  - if $a_{2,k}=0$:

$$\begin{cases} u_{1,k+1}=u_{1,k}-1+a_{1,k} \\ u_{2,k+1}=u_{2,k}+a_{2,k} \end{cases}, \qquad (3)$$

  - if $a_{2,k}>0$:

$$\begin{cases} u_{1,k+1}=u_{1,k}+a_{1,k} \\ u_{2,k+1}=u_{2,k}-1+a_{2,k} \end{cases}, \qquad (4)$$

where $[\cdots]^+$ denotes the maximum of the argument and zero. Eqs. (1) and (2) are self-explanatory: if one of the queues is empty at the beginning of slot k, a packet of the other queue (if non-empty) is served during slot k. If both queues are non-empty, it depends on the number of class-2 arrivals in slot k whether a packet of the low-priority queue jumps to the high-priority queue (Eqs. (3) and (4)). Introducing pgfs in the system equations yields the following relation between $U_{k+1}(z_1, z_2)$ and $U_k(z_1, z_2)$:

$$\begin{aligned}
&U_{k+1}(z_1, z_2) = \\
&\frac{A(z_1, z_2)}{z_2}\{(z_2-1)U_k(0,0)+U_k(0,z_2)\} \\
&+\frac{A(z_1, z_2)}{z_1}\{U_k(z_1,z_2)-U_k(z_1,0)\} \\
&+\left\{\frac{A(z_1,0)}{z_1}+\frac{(A(z_1,z_2)-A(z_1,0))}{z_2}\right\} \\
&\times\{U_k(z_1,z_2)-U_k(z_1,0)-U_k(0,z_2)+U_k(0,0)\}
\end{aligned}$$

This can be arranged as

$$\begin{aligned}
&U_{k+1}(z_1, z_2) = \\
&\frac{z_2(z_1-1)A(z_1,z_2)+(z_2-z_1)A(z_1,0)}{z_1 z_2}U_k(0,0) \\
&+\frac{(z_2-z_1)(A(z_1,z_2)-A(z_1,0))}{z_1 z_2}U_k(z_1,0) \\
&+\frac{(z_1-z_2)A(z_1,0)}{z_1 z_2}U_k(0,z_2) \\
&+\frac{z_1 A(z_1,z_2)+(z_2-z_1)A(z_1,0)}{z_1 z_2}U_k(z_1,z_2). \quad (5)
\end{aligned}$$

Furthermore, since we are interested in the steady-state distribution of the system contents, we define $U(z_1, z_2)$ as $\lim_{k\to\infty} U_k(z_1,z_2) = \lim_{k\to\infty} U_{k+1}(z_1,z_2)$. Letting $k \to \infty$ in (5), and solving the obtained equation for $U(z_1, z_2)$ then yields

$$\begin{aligned}
U(z_1, z_2) &= \frac{1}{z_1 z_2 - z_1 A(z_1,z_2)-(z_2-z_1)A(z_1,0)} \\
&\times\Big\{\{z_2(z_1-1)A(z_1,z_2)+(z_2-z_1)A(z_1,0)\}U(0,0) \\
&+(z_2-z_1)(A(z_1,z_2)-A(z_1,0))U(z_1,0) \\
&+(z_1-z_2)A(z_1,0)U(0,z_2)\Big\}. \quad (6)
\end{aligned}$$

In the right hand side of (6), there are three quantities yet to be determined: the functions $U(z_1, 0)$ and $U(0, z_2)$, and the constant $U(0,0)$. First, we compute the constant $U(0,0)$. Substitution of z_2 by z_1, in (6), returns

$$U(z_1,z_1) = U_T(z_1) = U(0,0)\frac{A_T(z_1)(z_1-1)}{z_1-A_T(z_1)}, \quad (7)$$

which thus is the expression for the pgf of the total system content. Substituting z_1 by 1 in (7), applying the normalization condition $U(1,1) = 1$, and using l'Hopital's rule results in the probability of having an empty system: $U(0,0) = 1-\lambda_T$. Notice that expression (7) is identical to the pgf of the system contents of a queue with a FIFO-discipline and with one class of arrivals (determined by $A_T(z)$). This is expected, because the total system content is independent of the order the packets are being served in.

Secondly, we determine $U(z_1, 0)$. Letting $z_2 \to 0$ in (6) and using l'Hopital's rule, we find

$$U(z_1,0) = A(z_1,0)\frac{(z_1-1)(1-\lambda_T)+z_1U^{(2)}(0,0)}{z_1-A(z_1,0)}, \quad (8)$$

where $U^{(2)}(0,0) \triangleq \left.\frac{\partial U(z_1,z_2)}{\partial z_2}\right|_{z_1=z_2=0}$ denotes the probability of having an empty high-priority queue and one packet in the low-priority queue. By applying Rouché's theorem, it can be proven that the equation $z_1 = A(z_1, 0)$ has a unique solution in the unit circle ($|z_1| < 1$). Let this solution be denoted by s. Then, since $U(z_1,0)$ is analytic for $|z_1| < 1$, we obtain from (8) that $sU^{(2)}(0,0) = (1-s)(1-\lambda_T)$. As a consequence,

$$U(z_1,0) = A(z_1,0)\frac{(z_1-s)(1-\lambda_T)}{s(z_1-A(z_1,0))}. \quad (9)$$

Thirdly, we derive an expression for the function $U(0,z_2)$. By again applying Rouché's theorem, we can shown that for a given z_2 ($|z_2| < 1$), the equation (in z_1) $z_1z_2 - z_1A(z_1,z_2)-(z_2-z_1)A(z_1,0) = 0$ has a unique solution in the unit circle $|z_1| < 1$. This solution — denoted by $Y(z_2)$ — is implicitly defined by

$$Y(z) \triangleq \frac{Y(z)}{z}A(Y(z),z)+\frac{(z-Y(z))}{z}A(Y(z),0). \quad (10)$$

Since $Y(z_2)$ is a zero of the denominator of the right hand side of (6), and since the pgf $U(z_1, z_2)$ is analytic for $|z_1|, |z_2| < 1$, $Y(z_2)$ must also be a zero of the numerator, we immediately obtain from (6) that

$$\begin{aligned}
U(0,z_2) &= \frac{1}{(z_2-Y(z_2))A(Y(z_2),0)} \\
&\times\Big\{(1-\lambda_T)\{z_2(Y(z_2)-1)A(Y(z_2),z_2) \\
&+(z_2-Y(z_2))A(Y(z_2),0)\}+(z_2-Y(z_2)) \\
&(A(Y(z_2),z_2)-A(Y(z_2),0))U(Y(z_2),0)\Big\}. \quad (11)
\end{aligned}$$

Using Eqs. (8) and (11), finally yields

$$\begin{aligned}
U(z_1,z_2) &= \frac{(1-\lambda_T)z_1}{s(z_1-A(z_1,0))} \\
&\times\left\{\frac{(z_2-z_1)A(z_1,0)(s+A(z_1,z_2)-A(z_1,0))}{z_1z_2-z_1A(z_1,z_2)-(z_2-z_1)A(z_1,0)}\right. \\
&+\left.\frac{sA(z_1,z_2)\{(1-z_2)A(z_1,0)+z_2(z_1-1)\}}{z_1z_2-z_1A(z_1,z_2)-(z_2-z_1)A(z_1,0)}\right\} \\
&+\frac{(1-\lambda_T)(z_1-z_2)A(z_1,0)}{s(z_1z_2-z_1A(z_1,z_2)-(z_2-z_1)A(z_1,0))}
\end{aligned}$$

$$\times \left\{ \frac{Y(z_2)(s + A(Y(z_2), z_2) - A(Y(z_2), 0))}{Y(z_2) - A(Y(z_2), 0)} + \frac{sz_2 A(Y(z_2), z_2)}{z_2 - A(Y(z_2), z_2)} - \frac{sA(Y(z_2), z_2)(z_2 - A(Y(z_2), 0))}{(z_2 - A(Y(z_2), z_2))(Y(z_2) - A(Y(z_2), 0))} \right\}.$$

From this joint pgf, we can calculate the marginal pgfs $U_1(z)$ and $U_2(z)$ of the system contents of the high- and low-priority queue by substituting z_1 and z_2 by the appropriate values:

$$\begin{aligned} U_1(z) &\triangleq \lim_{k\to\infty} \mathrm{E}\left[z^{u_{1,k}}\right] = U(z,1) \\ &= \frac{(1-\lambda_T)(s - A(z,0))}{s(z - A(z,0))} \\ &\quad \times \frac{z(z-1)(A_1(z) - A(z,0))}{z - A(z,0) - z(A_1(z) - A(z,0))} \\ &\quad - \frac{(z-1)A(z,0)}{sA_2(0)} \\ &\quad \times \frac{(1-\lambda_T)(s - A_2(0)) + s(\lambda_1 - A_2(0))}{z - A(z,0) - z(A_1(z) - A(z,0))}, \end{aligned} \tag{12}$$

$$\begin{aligned} U_2(z) &\triangleq \lim_{k\to\infty} \mathrm{E}\left[z^{u_{2,k}}\right] = U(1,z) \\ &= \frac{(1-\lambda_T)A_2(0)(z-1)}{s(z - A_2(z) - (z-1)A_2(0))} \\ &\quad \times \left\{ \frac{s - A_2(0) + (1-s)A_2(z)}{1 - A_2(0)} - \frac{Y(z)(s + A(Y(z),z) - A(Y(z),0))}{Y(z) - A(Y(z),0)} - \frac{szA(Y(z),z)}{z - A(Y(z),z)} + \frac{sA(Y(z),z)(z - A(Y(z),0))}{(z - A(Y(z),z))(Y(z) - A(Y(z),0))} \right\}, \end{aligned} \tag{13}$$

with $s = A(s,0)$ and $Y(z)$ implicitly defined by (10).

Furthermore, from the marginal pgfs (7), (12), and (13), expressions for the moments can be easily calculated. First note that the function $Y(z)$, defined by (10), can only be explicitly found in case of some simple arrival processes. However, its derivatives for $z = 1$, necessary to calculate the moments of the system contents of the low-priority queue, can be calculated in closed-form ($Y'(1)$, e.g., is given by $\frac{1 - \lambda_2 - A(1,0)}{\lambda_1 - A(1,0)}$). By then taking the first derivative of (7), (12), and (13), for $z = 1$ (and by making extensive use of l'Hopital's rule), we get expressions for $\mathrm{E}[u_T]$, $\mathrm{E}[u_1]$, and $\mathrm{E}[u_2]$, i.e., the mean values of the total system contents, and of the system contents of the high- and low-priority queue respectively. By taking higher order derivatives of the respective pgfs for $z = 1$, expressions of higher moments can also be obtained.

PACKET DELAY

In this section, we analyse the delay of a packet, which is defined as the total amount of time the packet spends in the system, and which thus renders the number of slots between the end of the packet's arrival slot and the end of its departure slot. We first derive an expression for the pgf of the delay of a class-1 packet.

Therefore, we consider a "tagged" class-1 packet, where slot k is assumed to be the arrival slot of the tagged packet. Remember that the possible jump of the HOL-packet of the low-priority queue takes place *at the end* of slot k (see second section). This has the effect that *all* class-1 packets that arrive during slot k, including the tagged class-1 packet, are queued in front of the possibly jumping packet. As a consequence, the delay of the tagged class-1 packet only depends on the system contents of the high-priority queue at the beginning of slot k ($u_{1,k}$). If $f_{1,k}$ then represents the number of class-1 packets that arrive during slot k, but which have to be served before the tagged packet, it is obvious that the total amount of time the tagged class-1 packet spends in the system equals

$$d_1 = [u_{1,k} - 1]^+ + f_{1,k} + 1. \tag{14}$$

Indeed, the tagged packet has to wait in the high-priority queue until *all* packets that were already in this queue at the moment of its arrival are effectively served. The *waiting time* of the tagged packet is thus determined by $u_{1,k}$ less the (possibly) in slot k served HOL-packet of the high-priority queue, and by $f_{1,k}$. The delay of the tagged packet then equals the waiting time augmented with the service time of the tagged packet, which equals one. Introducing pgfs in (14) produces

$$D_1(z) = F_1(z)\left[U_1(z) + (z-1)U_1(0)\right],$$

where we have used the fact that $u_{1,k}$ and $f_{1,k}$ are uncorrelated (due to the i.i.d. arrivals from slot-to-slot) and that the pgf $F_1(z) \triangleq \mathrm{E}\left[z^{f_{1,k}}\right]$ can be calculated by taking into account that an arbitrary packet is more likely to arrive in a larger bulk (Bruneel, 1993), yielding

$$F_1(z) = \frac{A_1(z) - 1}{\lambda_1(z-1)}.$$

Finally, using (12) and the latter equation, we find

$$\begin{aligned} D_1(z) &= \frac{(1-\lambda_T)(s - A(z,0))}{\lambda_1 s(z - A(z,0))} \\ &\quad \times \frac{z(A(z)-1)(A_1(z) - A(z,0))}{z - A(z,0) - z(A_1(z) - A(z,0))} \\ &\quad - \frac{(1-\lambda_T)(s - A_2(0)) + s(\lambda_1 - A_2(0))}{\lambda_1 s A_2(0)} \end{aligned}$$

$$\times \frac{z(A_1(z)-1)(1-A_1(z)+A(z,0))}{z-A(z,0)-z(A_1(z)-A(z,0))}. \tag{15}$$

Note that by taking the first derivative of (15) for $z = 1$, we get an expression for $\mathrm{E}[d_1]$, i.e., the mean class-1 packet delay (by taking higher order derivatives for $z = 1$, expressions of higher moments can also be obtained).

The analysis of the delay of a tagged class-2 packet is much more complicated. Indeed, consider a random slot I, and suppose the tagged packet is still in the low-priority queue at the beginning of slot I. When the high-priority queue is empty at the beginning of slot I, the tagged packet comes one position closer to the HOL-position of the low-priority queue. This is however not necessarily the case when the high-priority queue is non-empty at the beginning of slot I. It is thus necessary to keep track of the system content of the high-priority queue during the time the tagged packet is waiting in the low-priority queue, since this quantity, together with the arrival process, determines what will happen to the packet during its delay. This would require a time-dependent or *transient* analysis, which is a major challenge for priority systems. However, although it seems very difficult to derive an explicit expression for the pgf of the class-2 packet delay, it is possible to easily calculate the mean class-2 packet delay. This is shown next.

It should first be mentioned that $\mathrm{E}[u_j] = \lambda_j \mathrm{E}[d_j]$ $(j = 1, 2)$ does not hold, as one would — at first — expect according to Little's law. The reason for this is that in the calculation of the system contents, packets of the low-priority queue jump to the high-priority queue and from that moment on, they are treated as part of the system contents of the high-priority queue. This is not the case in the calculation of the packet delay. Specifically, Little's law does not hold with respect to each queue separately, because the system contents is defined on a "queue"-basis, while the packet delay is defined on a "packet"-basis. For the *complete* system on the contrary, Little's law does hold. As a consequence, $\mathrm{E}[u_T] = \lambda_T \mathrm{E}[d]$, with $\mathrm{E}[d]$ being the mean delay of an arbitrary (class-1 or class-2) packet. Since the probability that a random arriving packet is of class j equals $\frac{\lambda_j}{\lambda_T}$ $(j = 1, 2)$, we know that $\mathrm{E}[d] = \frac{\lambda_1}{\lambda_T}\mathrm{E}[d_1] + \frac{\lambda_2}{\lambda_T}\mathrm{E}[d_2]$, and thus that $\frac{\mathrm{E}[u_T]}{\lambda_T} = \frac{\lambda_1}{\lambda_T}\mathrm{E}[d_1] + \frac{\lambda_2}{\lambda_T}\mathrm{E}[d_2]$. This can finally be arranged as

$$\mathrm{E}[d_2] = \frac{\mathrm{E}[u_T] - \lambda_1 \mathrm{E}[d_1]}{\lambda_2}. \tag{16}$$

Since $\mathrm{E}[u_T]$ as well as $\mathrm{E}[d_1]$ can be calculated, we are thus able to derive an expression for the mean value of the delay of a class-2 packet.

NUMERICAL EXAMPLE

In the previous sections, we have briefly described the procedure to obtain expressions for the mean values of the studied stochastic variables from their corresponding pgfs. In this section, we will illustrate these expressions in some figures, and use these figures to show the effect of the HOL-JIA priority scheme on the mean packet delay, in comparison with the original HOL priority scheme (Walraevens, 2003), the modified HOL priority scheme (in which the HOL-packet of the low-priority queue *always* jumps to the high-priority queue when both queues are non-empty (Maertens, 2005)), and the FIFO scheme. Therefore, we consider the following arrival process: the traffic of the two classes, where class 1 has priority over class 2, is assumed to arrive according to a two-dimensional binomial process, which is fully characterized by the joint pgf

$$A(z_1, z_2) = \left(1 - \frac{\lambda_1}{N}(1 - z_1) - \frac{\lambda_2}{N}(1 - z_2)\right)^N$$

and with $N = 16$ in the figures. The arrival rate of class-j traffic is then given by λ_j, and α is defined as the fraction of (high-priority) class-1 traffic in the overall traffic mix (i.e., $\alpha = \lambda_1/\lambda_T$). This is the arrival process to a queue in an output-queueing switch with Bernoulli arrivals at its inlets, and with uniform routing.

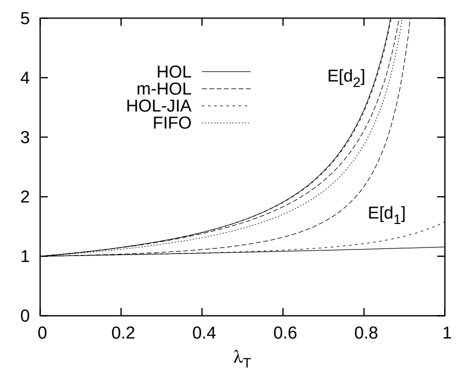

Figure 1: Mean values of packet delays versus the total arrival rate for $\alpha = 0.25$

In Figures 1 and 2, we can see the mean values of the packet delays of both classes for $\alpha = 0.25$ and $\alpha = 0.75$ respectively, as functions of the total arrival rate, for the HOL scheme, the m-HOL scheme, the FIFO scheme, and the HOL-JIA scheme. Obviously, the higher the total arrival rate, the higher E[d_1] (i.e., the mean class-1 packet delay) and E[d_2] (i.e., the mean class-2 packet delay). Secondly, it is seen that the curves of E[d_1] and E[d_2] for the HOL-JIA scheme lie between those for the HOL scheme and the

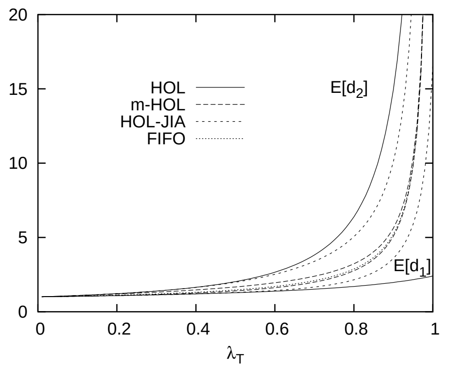

Figure 2: Mean values of packet delays versus the total arrival rate for $\alpha = 0.75$

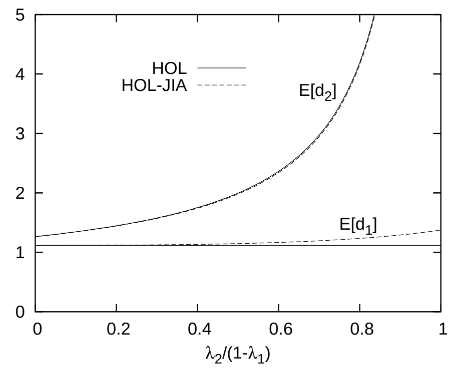

Figure 3: Mean values of packet delays versus λ_2 for $\lambda_1 = 0.2$

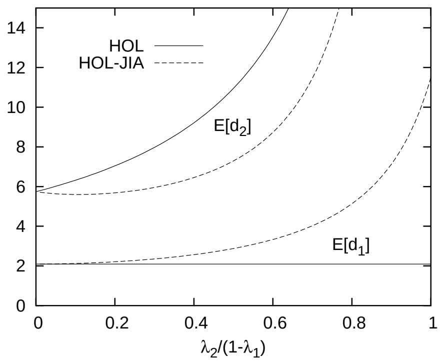

Figure 4: Mean values of packet delays versus λ_2 for $\lambda_1 = 0.7$

m-HOL scheme. This was expected, due to the jumping behavior of low-priority (class-2) packets in the stated schemes. Indeed, in the HOL scheme, there are no jumps at all, while in the m-HOL scheme, the HOL-packet of the low-priority queue jumps *in every slot* that the high-priority queue is non-empty. In the HOL-JIA scheme, jumps also depend on the number of low-priority arrivals in a slot. By introducing an additional condition on when low-priority packets jump, the number of jumps is thus smaller than in the m-HOL scheme. As to the HOL-JIA scheme, the benefit (loss) for E$[d_2]$ (E$[d_1]$), compared to the HOL scheme, is limited when $\alpha = 0.25$ (see Figure 1).

When $\alpha = 0.75$ on the contrary (see Figure 2), the decrease of E$[d_2]$ can be considerable, especially for high loads. E.g. when $\lambda_T = 0.9$, E$[d_2]$ decreases from about 15 for HOL to about 10 for HOL-JIA, with only a small increase for E$[d_1]$ (from about 2 to about 3.6). When α (i.e., the fraction of high-priority traffic in the overall traffic mix) is low, the low-priority packets are often served immediately (without jumping), and the high-priority queue is thus often empty. So, an arriving high-priority packet has a relatively high probability of arriving in an empty high-priority queue, and is thus served within a short time. When α increases, more high-priority packets arrive at the system, and, as to the HOL-JIA scheme, more low-priority packets thus jump to the high-priority queue before being served. This results in a higher mean class-1 packet delay — and a lower mean class-2 packet delay — in comparison with the HOL-scheme. Note though that for $\lambda_T \to 1$, E$[d_1]$ remains finite, as opposed to the m-HOL scheme.

Figures 3 and 4 show the mean values of the packet delays of both classes for $\lambda_1 = 0.2$ and $\lambda_1 = 0.7$ respectively, as functions of (the normalised version of) λ_2, for the HOL scheme and the HOL-JIA scheme. The figures thus illustrate the influence of the low-priority load on E$[d_1]$ and E$[d_2]$. Since there are no jumps in the HOL scheme, the amount of low-priority packets has no influence on E$[d_1]$ in this case. As to the schemes *with* priority jumps, E$[d_1]$ is obviously effected by the low-priority load. When $\lambda_1 = 0.2$ in the HOL-JIA scheme, low-priority packets are often served without jumping (see previous paragraph). The influence of low-priority packets on E$[d_1]$ is thus limited (see Figure 3). When $\lambda_1 = 0.7$ on the contrary, low-priority packets have a relatively high-probability of jumping to the high-priority queue before being served, resulting in a larger influence on E$[d_1]$ (see Figure 4). However, the small increase of E$[d_1]$, e.g., from about 2 for HOL to about 3.4 for HOL-JIA when $\lambda_2 = 0.54$, is made up largely by the decrease of E$[d_2]$ (from about 14 to about 9).

Finally, in Figures 5 and 6, the mean values of the packet delays of both classes are shown for $\lambda_T = 0.7$ and $\lambda_T = 0.9$ respectively, as functions of α, for the HOL scheme, the FIFO scheme, and the HOL-JIA scheme. Higher α means that more high-priority traffic arrives at the system in proportion to low-priority traffic. Hence, E$[d_1]$ and E$[d_2]$ increase with α in the priority schemes. However, as we have seen before, higher α also causes more jumps. Jumping low-priority packets have a negative influence on E$[d_1]$, compared to the HOL-scheme. The positive effect of

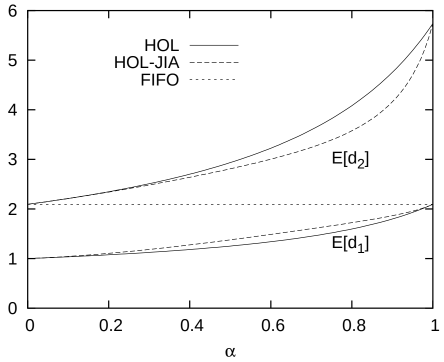

Figure 5: Mean values of packet delays versus α for $\lambda_T = 0.7$

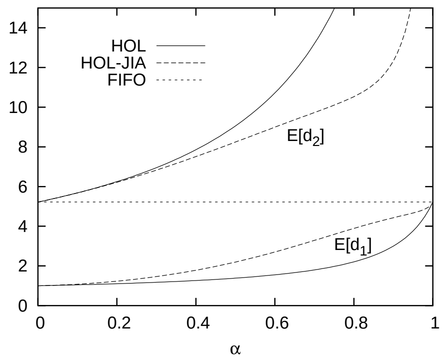

Figure 6: Mean values of packet delays versus α for $\lambda_T = 0.9$

jumps on E$[d_2]$ on the other hand, is obvious. Remark also the behavior of E$[d_1]$ and E$[d_2]$ for the HOL-JIA scheme, when α approaches 0 and 1 (see Figure 5). When $\alpha \to 0$ (i.e., the overall traffic mix only exists of low-priority traffic), the probability that an arriving high-priority packet arrives in an empty high-priority queue equals 1. Its delay is thus not influenced by low-priority packets. As a consequence, E$[d_1]$ for the HOL-JIA scheme and the HOL scheme both equal the service time of the packet, i.e., one slot. As to E$[d_2]$, we have a FIFO scheme, since basically only class-2 packets arrive at the system. When $\alpha \to 1$ (i.e., the overall traffic mix only exists of high-priority traffic), an arriving low-priority packet has a large probability of arriving in an empty low-priority queue. However, to be served, this low-priority packet has to wait in the low-priority queue until the high-priority queue becomes empty. Hence, E$[d_2]$ for the HOL-JIA scheme and the HOL scheme converge. For E$[d_1]$, we get the same expression as the mean delay in a FIFO queue.

We can thus conclude that the dynamic HOL-JIA priority scheduling discipline does what it is designed for: if the network is highly loaded (and if the overall traffic mix consists mainly of high-priority traffic), the mean delay of the low-priority packets can be decreased considerably, in comparison with the HOL scheme. The price to pay, i.e., a higher mean delay for high-priority packets, is limited (see Figure 6).

CONCLUSIONS

In this paper, we have analysed a discrete-time single-server queueing system, with two priority classes arriving at the system. Packets are orginally stored in separate queues of infinite capacity, i.e., a high- and low-priority queue, but packets of the low-priority queue can in process of time jump to the high-priority queue, according to a newly introduced dynamic priority scheduling discipline, i.e., the HOL-JIA (head-of-line, jump-if-arrival) scheme. The model further included possible correlation between the numbers of arrivals of the two priority classes during a slot. Adopting a probability generating function approach, we have derived the probability generating functions of the system contents of the high- and low-priority queue, and the probability generating function of the delay of a high-priority packet. Moments can be easily calculated from the obtained pgfs. We have also shown a procedure to calculate the mean delay of a low-priority packet. Finally, the effect of the HOL-JIA scheme, compared to other priority schemes, is illustrated by some numerical examples.

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ANALYSIS OF A TWO-NODE QUEUEING NETWORK WITH FLOW CONTROL AND NEGATIVE ARRIVALS

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KEYWORDS

Two-node queueing network, flow control, negative arrivals.

ABSTRACT

A queueing network model of a virtual circuit in a packet computer network having end–to–end window flow control is considered. In the queueing model a simplified protocol is studied, including two single server queueing systems in tandem, having buffers B_1 and B_2 of capacity $L-1$ and with an external buffer BE in front. For the flow control is considered that in the two tandem systems in each moment there are at most L packets. To the system in addition to usual (positive) customers arriving to the outer buffer, there are negative customers arriving to each of the inner nodes. Modeling with a Markov process a necessary and sufficient ergodicity condition is obtained and a matrix algebraic algorithm is developed for the computation of stationary probabilities.

INTRODUCTION

The purpose of the present article is to investigate performance of a virtual circuit (VC) in a packet computer network having end–to–end window flow control. A VC with end–to–end window flow control operates by restricting the maximum number of packets from de VC in transit within the network. This maximum number of packets which a VC can hold is called the window size and it will be denoted bellow by L.

In this paper a simplified protocol is examined. It is supposed that a packet, arriving when the number of packets in the VC have reached the window size, will be placed in an external queue, which is also called the window queue. The first moment of time when the number of packets in the VC gets less that the window size a packet will be transmitted over the circuit. In the case, when the arriving packet finds that the number of packets in the VC is less than window size, it is transmitted immediately. In addition, we consider that the number of customers inside the window may be decremented at random moments.

The packet whose transmission has ended with a positive probability $1-\theta$ leaves the VC and with a supplementary probability θ again repeats its transmission, i.e. the feedback is possible.

DESCRIPTION OF THE QUEUEING MODEL

We consider a single VC modeled as a series of two single server queueing systems, having buffers B_1 and B_2 of capacity $L-1$ and with an external buffer BE in front. The customer (packet) arriving when the window is full, i.e. if the queueing systems (servers 1 and 2 and their buffers) without external buffer have L customers, is placed in the external buffer which is supposed to be unlimited. Therefor in this case the first node is blocked for the arriving customers. When the number of customers in the system is reduced to $L-1$ a customer from the external queue (if there is any customer in it) proceeds to the first buffer. The customer arriving to the system, having less than L customers, enters in it and is placed in the first buffer or in his server if there are no customers in the first node. After completion of the service in the first server the customer proceeds to the second buffer or directly to the second server if the second node is empty. The customer whose service has ended in the second server with positive probability $1-\theta$ leaves the system and with probability θ joins the first queue. Finally we consider that nega-

tive customers arriving to the first or second node can eliminate existing customers (Gelenbe et al. 1991).
It is supposed that the positive customers flow is a Poisson one with parameter λ, that negative flows are Poisson too with parameters λ_1^- for eliminating customers in node 1 and λ_2^- directed to the second stage, the service times are independent and have exponential distributions with parameters μ_1 and μ_2 for the first and second server, respectively.
A similar queueing network without negative customers was considered in (Bocharov et al. 1992; Osterbo et al. 1985). The paper (Bocharov et al. 1992) generalizes for the case of arbitrary L and θ the results of (Osterbo et al. 1985), where the solution of the SEE for $L=2$ and $\theta=0$ was given. Additionally, in (Bocharov et al. 1992) a necessary and sufficient ergodicity condition was derived.
This paper extends the results of (Gelenbe et al. 1991) for the case when negative customers are taken into consideration.

MARKOV PROCESS AND EQUILIBRIUM EQUATIONS

The operation of the system under consideration could be described by a homogeneous Markov process $X(t)$, $t \geq 0$, on the state space

$$\mathcal{X} = \{(n,k,m) | n = 0,1,\ldots, \quad k+m = \overline{u(n)L, L}\},$$

where

$$u(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Here, $X(t) = (0,k,m)$ means that at time t, the external buffer is empty, and nodes one and two have k and m customer, respectively; $X(t) = (n, L-m, m)$, with $n \geq 1$, means that at the external buffer there are n customers waiting and their total number in the two internal nodes is equal to L, here $L-m$ and m are the number of customers in the first and second node, respectively.
Let us suppose that the stationary distribution of $X(t)$ exists, and that it is

$$p_x = \lim_{t\to\infty} P\{X(t) = x\}, \quad x \in \mathcal{X}.$$

The stationary probabilities $q_{k,m}$ of the state $(0,k,m)$ and $p_{k,m}$ of the state (n,k,m) satisfy the following system of equilibrium equations (SEE):

$$\begin{aligned}\big(\lambda + \mu_1 u(k) + \mu_2 u(m) + \lambda_1^- u(k) + \lambda_2^- u(m)\big) q_{k,m} &= \\ \lambda u(k) q_{k-1,m} + \mu_1 u(m) q_{k+1,m-1} + \mu_2(1-\theta) q_{k,m+1} &+ \\ \mu_2 \theta u(k) q_{k-1,m+1} + \lambda_1^- q_{k+1,m} + \lambda_2^- q_{k,m+1}&, \\ 0 \leq k+m \leq L-1&, \qquad (1)\end{aligned}$$

$$\begin{aligned}\big(\lambda + \mu_1 u(L-m) + \mu_2 u(m) + \lambda_1^- u(L-m) +& \\ \lambda_2^- u(m)\big)\, p_{0,m} &= \\ \lambda u(L-m) q_{L-m-1,m} + \mu_1 u(m) p_{0,m-1} &+ \\ \mu_2(1-\theta) u(L-m) p_{1,m+1} + \mu_2 \theta u(L-m) p_{0,m+1} &+ \\ \lambda_1^- u(L-m) p_{1,m} + \lambda_2^- u(L-m) p_{1,m+1},\ m = \overline{0,L},& \quad (2)\end{aligned}$$

$$\begin{aligned}\big(\lambda + \mu_1 u(L-m) + \mu_2 u(m) + \lambda_1^- u(L-m) +& \\ \lambda_2^- u(m)\big)\, p_{n,m} &= \\ \lambda p_{n-1,m} + \mu_1 u(m) p_{n,m-1} &+ \\ \mu_2(1-\theta) u(L-m) p_{n+1,m+1} &+ \\ \mu_2 \theta u(L-m) p_{n,m+1} + \lambda_1^- u(L-m) p_{n+1,m} &+ \\ \lambda_2^- u(L-m) p_{n+1,m+1}, \quad n > 0,\ m = \overline{0,L},& \quad (3)\end{aligned}$$

with the normalizing condition

$$\sum_{0 \leq k+m \leq L-1} q_{k,m} + \sum_{n=0}^{\infty} \sum_{m=0}^{L} p_{n,m} = 1. \qquad (4)$$

Defining the following generating functions

$$F_m(z) = \sum_{n=0}^{\infty} p_{n,m} z^n, \ |z| \leq 1, \ \ m = \overline{0,L},$$

the system $(1) - (3)$ can be written as

$$\begin{aligned}(a(z) - z\mu_2^*)\, F_0(z) - b(z) F_1(z) &= f_0(z), \\ -z\mu_1 F_{m-1}(z) + a(z) F_m(z) - b(z) F_{m+1} &= f_m(z), \\ -z\mu_1 F_{L-1}(z) + \big(a(z) - z\mu_1^* + \lambda_1^-\big) F_L(z) &= 0, \quad (5)\end{aligned}$$

where

$$\begin{aligned} a(z) &= z\,(\lambda(1-z) + \mu_1^* + \mu_2^*) - \lambda_1^-, \\ b(z) &= \mu_2^* - \mu_2\theta(1-z), \\ f_m(z) &= \lambda z q_{L-m-1,m} - (\mu_2^* - \mu_2\theta)\, p_{0,m+1} \\ &- \mu_2\theta p_{0,m+1} - \lambda_1^- p_{0,m}, \quad m = \overline{0, L-1}, \end{aligned}$$

and

$$\mu_1^* = \mu_1 + \lambda_1^-, \ \ \mu_2^* = \mu_2 + \lambda_2^-.$$

The solution of system (5) in the domains $|z| \leq 1$ is easily obtained by the Kramer's rule and takes the form

$$F_m(z) = \frac{S_m(z, \vec{q})}{D_L(z)}, \quad m = \overline{0,L}, \qquad (6)$$

where

$$\begin{aligned} S_m(z,\vec{q}) = U_{L-m-1}(z) \sum_{k=0}^{m-1} (z\mu_1)^{m-k} D_{k-1}(z) f_k(z) &+ \\ D_{m-1}(z) \sum_{k=m}^{L-1} (b(z))^{k-m} U_{L-k-1}(z) f_k(z).& \quad (7)\end{aligned}$$

The polynomials $U_k(z)$ and $D_k(z)$ are defined by the following recurrent relations:

$$\begin{aligned} U_{-1}(z) &= 1, \\ U_0(z) &= a(z) - z\mu_1^* + \lambda_1^-, \\ U_k(z) &= a(z)U_{k-1}(z) - z\mu_1 b(z)U_{k-2}(z), \end{aligned}$$
$k = \overline{1, L-1}$;

$$\begin{aligned} D_{-1}(z) &= 1, \\ D_0(z) &= a(z) - z\mu_2^*, \\ D_k(z) &= a(z)D_{K-1}(z) - z\mu_1 b(z)D_{k-2}(z), \end{aligned} \tag{8}$$
$k = \overline{1, L-1}$,

$$D_L(z) = \left(a(z) - z\mu_1^* + \lambda_1^-\right) D_{L-1}(z) - z\mu_1 b(z)D_{L-2}(z).$$

Notice that $S_m(z, \vec{q})$ in expression (7) depends on unknown probabilities $p_{0,m+1}$ and $q_{L-m-1,m}$, $m = \overline{0, L-1}$. These will be found using the fact that generating functions are analytical in the domain $|z| < 1$, continuous on $|z| = 1$ and using the properties of the roots of polynomials D_k, $k = \overline{-1, L}$. These properties are resumed in *Lemma* 1.

Lemma 1. *Let $z_{j,k}$ be the $j-th$ real root of the polynomial $D_k(z)$ in increasing order; $deg(D_k)$ – the degree of D_k and $sgn(D_k(z))$ – the sign of $D_k(z)$ in the point z. Then*

1. $deg(D_k) = 2k + 2, \quad k = \overline{-1, L}$.
2. $D_{2k}(0) < 0$, $D_{2k+1}(0) > 0, \quad k \geq 0$.
3. $sgn(D_{2k}(\infty)) = -1, \quad sgn(D_{2k+1}(\infty)) = 1$.
4. $D_k(1) = \mu_1^{k+1}, k = \overline{0, L-1}, \quad D_L(1) = 0, \quad D_L(0) = 0$.
5. *The roots of $D_k(z)$, $k < L$, are real, positive and simple.*
6. *$D_k(z)$ has $k+1$ roots over $(0, 1)$, $k < L$.*
7. *The roots of $D_j(z)$ and $D_{j-1}(z)$ alternate in the following manner:*
   $0 < z_{1,j} < z_{1,j-1} < z_{2,j} < z_{2,j-1} < \ldots < z_{k+1,j} < 1$,
   $1 < z_{k+2,j} < z_{k+1,j-1} < z_{k+3,j}. < \ldots < z_{2k+2,j} < \infty$.

Now, let us consider separately the polynomial $D_L(z)$. From assertion 4 of Lemma 1, because 1 is a root of $D_L(z)$, we can define

$$(1 - z)g_L(z) = D_L(z), |z| < 1, \quad L = 1, 2, \ldots \tag{9}$$

and using (8) we can prove the following assertion.

Lemma 2.

$$\begin{aligned} g_0(z) &= \lambda z, \\ g_1(z) &= a(z)g_0(z) + z\mu_1\mu_2\theta - z\mu_1^*\mu_2^*, \\ g_k(z) &= a(z)g_{k-1}(z) - z\mu_1 b(z)g_{k-2}(z), \end{aligned} \tag{10}$$
$k \geq 2$.

In order to complete the analysis of the principal properties of polynomials $D_L(z)$, $L \geq 1$, we are going to specify the value of polynomials $g_k(z)$ at the point $z = 1$. With this objective in mind, we define new generating function

$$H(z, w) = \sum_{k=0}^{\infty} g_k(z)w^k =$$
$$g_0(z) + g_1(z)w + \sum_{k=2}^{\infty} g_k(z)w^k, \quad |w| < 1. \tag{11}$$

Using this definition and recurrence (7) we obtain

$$H(z, w) = \frac{z\lambda(1 - a(z)w) + g_1(z)w}{1 - a(z)w + z\mu_1 b(z)w^2}. \tag{12}$$

Next, evaluating $H(z, w)$ in $z = 1$,

$$H(1, w) = \frac{\lambda\,(1 - a(1)w) + g_1(1)w}{1 - a(1)w - \mu_1 b(1)w^2}.$$

Using that

$$a(1) = \mu_1^* + \mu_2^* - \lambda_1^-,$$
$$b(1) = \left(\mu_1^* + \mu_2^* - \lambda_1^-\right)\lambda + \mu_1\mu_2\theta - \mu_1^*\mu_2^*,$$

we have

$$H(1, w) = \frac{\lambda + \left(\mu_1\mu_2\theta - \mu_1^*\mu_2^*\right) w}{\left(1 - \mu_1 w\right)\left(1 - \mu_2^* w\right)}.$$

Now, $H(1, w)$ can be expanded in power series to obtain

$$H(1, w) = \sum_{K=0}^{\infty} \left[\left(\lambda\mu_1 - \mu_1^*\mu_2^* + \mu_1\mu_2\theta\right)\mu_1^k + \left(\mu_1^*\mu_2^* - \mu_1\mu_2\theta - \lambda\mu_2^*\right)\mu_2^{*k}\right] / (\mu_1 - \mu_2^*) w^k, \tag{13}$$
$$\mu_1 \neq \mu_2^*.$$

From (13) we get for $\mu_1 \neq \mu_2^*$

$$g_k(1) = \frac{\left(\lambda\mu_1 - \mu_1^*\mu_2^* + \mu_1\mu_2\theta\right)\mu_1^k + \left(\mu_1^*\mu_2^* - \mu_1\mu_2\theta - \lambda\mu_2^*\right)\mu_2^{*k}}{\mu_1 - \mu_2^*}.$$

For the case, when $\mu_1 = \mu_2^* = \mu$ can be considered directly recurrence (10) from which we obtain

$$g_k(1) = \mu_1^k \left[(L + 1)\lambda - L(\mu_1^* - \mu_2\theta)\right].$$

Now, we shall study the properties of $D_L(z)$ evaluated in $z = 1$.

Lemma 3. *Let $z_{j,k}$ be the $j - th$ real root of the polynomial $D_k(z)$ in increasing order. Then the following holds:*

1. Over the intervals $(0, z_{L,L-1})$ and $(z_{L+1,L-1}, \infty)$ the roots of the polynomials $D_L(z)$ and

$D_{L-1}(z)$ *alternate similar to property* 7. *of Lemma 1.*

2. (*a*) $D_L^{'}(1) > 0, \Rightarrow z_{L+1,L} = 1,$

(*b*) $D_L^{'}(1) < 0, \Rightarrow z_{L+2,L} = 1,$

(*c*) $D_L^{'}(1) = 0, \Rightarrow z_{L+1,L} = z_{L+2,L} = 1.$

From here, we claim the following assertion.

Theorem. *The following statements are equivalent:*
1. There exits an equilibrium distribution of the considered system.
2. $D_L(z)$ *has exactly* $L+1$ *roots over the interval* $[0,1]$.
3. $D_L^{'}(1) > 0$.
4. $g_L < 0$.

Proof. It follows directly from Lemma 3 and (9)

Corollary 1. *For the ergodicity of the Markov process on consideration it is necessary and sufficient that the following conditions are completed:*

$$\lambda < \frac{L\,(\mu_1^* - \mu_2\theta)}{L+1}, \quad \text{for } \mu_1 = \mu_2^*, \qquad (14)$$

$$\lambda < \frac{(\mu_1^*\mu_2^* - \mu_1\mu_2\theta)\left(1-\gamma^k\right)}{\mu_1(1-\gamma^{k+1})}, \qquad (15)$$

for $\mu_1 \neq \mu_2^*$ *and* $\gamma = \mu_2^*/\mu_1$.

Let us pass now to the determination of the $L(L+3)/2$ unknowns $q_{k,m},\ \ 0 \le k+m \le L-1$, and $q_{L-m,m} \equiv p_{0,L-m,m},\ \ m = \overline{1,L}$.
It is easy to show that substitution of $z=0$ in (5) give us the connection equation

$$-(\lambda+\mu_2^*)q_{0,L} + \mu_1 q_{1,L-1} = 0 \qquad (16)$$

and that the root $z=1$ in $D_L(z)$ is a root of the function $S_m(z)$ too. This way, the remaining $L-1$ roots z, $|z|<1$ give us a system of $L-1$ equations for the determination of the unknowns $q_{k,m}$ of the form:

$$S_0(z_{L,j},\vec{q}) = 0, \quad j = \overline{1,L-1}. \qquad (17)$$

Let us see now the system of equations (1). To solve them we use the approach proposed in (Bocharov et al. 1975). With this purpose we introduce the $(L-m)\times(L-m+1)$ matrices A_m, $(L-m-1)\times(L-m+1)$ matrices B_m, $m=\overline{0,L-2}$, and $(L-m)\times(L-m)$ matrices C_m with the following form:

$$A_0 = \begin{pmatrix} -\lambda & \lambda_1^- & \cdots & 0 & 0 \\ \lambda & -(\lambda+\mu_1^*) & \cdots & 0 & 0 \\ 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -(\lambda+\mu_1^*) & \lambda_1^- \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -(\lambda+\mu_2^*) & \lambda_1^- & \cdots & 0 \\ \lambda & -(\lambda+\mu_1^*+\mu_2^*) & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_1^- \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0 & \mu_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mu_1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_1 & 0 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} -(\mu_2\bar{\theta}+\lambda_2^-) & \cdots & 0 \\ -\mu_2\theta & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -(\mu_2\bar{\theta}+\lambda_2^-) \end{pmatrix}.$$

Matrices A_m, $m=\overline{2,L-1}$, B_m, $m=\overline{1,L-2}$, and C_m, $m=\overline{1,L-1}$, are obtained from A_{m-1}, B_{m-1}, and C_{m-1}, respectively, omitting the last row and the last column. Additionally let us define

$$\vec{q}_0 = (q_{00}, q_{10}, ..., q_{L0})^T$$

and

$$\vec{q}_m = (q_{0,m}, q_{1,m}, ..., q_{L-m,m})^T.$$

Then (1) may be represented in the matrix form

$$C_m\vec{q}_{m+1} = A_m\vec{q}_m + u(m)B_{m-1}\vec{q}_{m-1},$$
$$m = \overline{0,L-1}. \qquad (18)$$

Now we introduce the sequence of W_m matrices, $m=\overline{0,L}$, defined by the following recurrence formulas:

$$\begin{array}{rcl} W_0 & = & I, \\ W_1 & = & C_0^{-1}A_0, \\ W_m & = & C_{m-1}^{-1}\left[A_{m-1}W_{m-1} + B_{m-2}W_{m-2}\right], m=\overline{2,L}. \end{array} \qquad (19)$$

From (18) and (19) it is evident that

$$\vec{q}_m = W_m\vec{q}_0 \quad m = \overline{1,L}. \qquad (20)$$

After this we make

$$B_{L-1} = (0, \mu_1)$$

and

$$W_{L+1} = -(\lambda+\mu_2^*)W_L + B_{L-1}W_{L-1}. \qquad (21)$$

Then from (20) and (21) we obtain that the connection equation (16) may be written as

$$W_{L+1}\vec{q}_0 = 0. \quad (22)$$

This way, the L homogeneous equations (17) and (22) along with the normalization equation permit us to determine uniquely the unknowns $q_{0,0}, q_{1,0}, ..., q_{L,0}$. Probabilities $q_{L-m-1,m}$ and $p_{0,m+1}$, $m = \overline{0, L-1}$, can be constructed using (20) and then, substituting in (6), we can construct the generating functions (7). These generating functions are rational functions that may be decomposed into partial fractions and expanded into series by the powers of z. Consequently the problem of finding the stationary distribution $p_x, x \in X$, is solved.

NUMERICAL RESULTS

We present the solutions obtained in this work for several values of L and λ and fixed

$$\mu_1 = 4/5, \quad \mu_2 = 2/3,$$

$$\lambda_1^- = 2/5, \quad \lambda_2^- = 2/5,$$

$$\theta = 1/6.$$

In table 1, for each case we present the blocking probability $P^B_{L,\lambda}$, as calculated from

$$P^B_{L,\lambda} = 1 - \sum_{0 \le k+m < L} q_{k,m}.$$

In table 2 we present the mean length of the external queue as a function of L and λ as obtained from

$$M_{L,\lambda} = \sum_{n=0}^{n'} n p_{n,\cdot}.$$

We selected n' in such a way that the sum of the probabilities were near unity.
In table 3 we present mean waiting time, as obtained from

$$W_{L,\lambda} = \frac{M_{L,\lambda}}{\lambda}.$$

For the considered values the λ_{max}, i.e. the limit intensity value for arriving positive customers supported for the network was near 1.1.
The algorithm was programmed in Maple.

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Table 1. Mean length of the external queue

$L \setminus \lambda$	.8	.9	1
25	$.1119e^{-2}$	.0298	.7127
30	$.2112e^{-3}$	.0101	.4107
35	$.3987e^{-4}$	$.3458e^{-2}$	.2366
40	$.7524e^{-5}$	$.1176e^{-2}$	.1363
45	$.1420e^{-5}$	$.3997e^{-3}$	.0785

Table 2. Blocking probability

$L \setminus \lambda$	.8	.9	1
25	$.3174e^{-3}$	$.5801e^{-2}$	.0745
30	$.5990e^{-4}$	$.1972e^{-2}$	.0429
35	$.1130e^{-4}$	$.6709e^{-3}$	.0247
40	$.2133e^{-5}$	$.2819e^{-3}$	.0192
45	$.1420e^{-6}$	$.7760e^{-4}$	.0082

Table 3. Mean waiting time

$L \setminus \lambda$	.8	.9	1
25	$.1399e^{-2}$	.0331	.7127
30	$.2640e^{-3}$	.0112	.4107
35	$.4983e^{-4}$	$.3842e^{-2}$	.2366
40	$.9405e^{-5}$	$.1306e^{-2}$	.1363
45	$.1775e^{-5}$	$.4444e^{-3}$	.0785

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TRANSIENT ANALYSIS OF A SINGLE SERVER SYSTEM IN A COMPACT STATE SPACE

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ABSTRACT

We consider applications of transient analysis to single server systems and demonstrate the flexibility of the method for discrete time GI/G/1/(*N*) servers. Evaluations of the probabilities of system states are based on transition equations starting from some predefined initial situation. When only arrival and departure instants are considered, the state space representing the system can be made compact and thus the computation becomes more efficient. Besides convergence to steady state distributions of the queue size, the length of busy and idle periods are analyzed for systems with finite and infinite waiting room.

I. INTRODUCTION

Transient analysis of stochastic systems is a basic method which is related to steady state analysis as well as simulation. Matrix analytic methods [4] and factorization approaches [7] provide efficient computation schemes for steady state solutions of systems with regular structures in the transition equations. In addition, transient analysis shows the evolution of the system from a predefined starting situation over a period of time or in long term development. When the system is ergodic, then convergence to a steady system state is observed over time. Steady state analysis is included in the usual transient behaviour, but the computational effort is often essentially higher than for direct steady state solutions depending on the convergence properties and on the relevant state space of the system. On the other hand, the transient analysis is more flexible in order to include exceptional cases and non-regular structures in the transition matrix.

In comparison to simulation, transient analysis provides complete distribution functions of system states at embedded points in time, whereas simulation follows randomly chosen paths of system development yielding results subject to statistical deviations within confidence levels. State space limitation is an important precondition to make transient analysis feasible. In a large state space, only those states e.g. with probability $>10^{-k}$ may be considered for an approximate transient analysis in a feasible state space. This generates a tendency to extract a set of states surrounding the maximum likelihood behaviour of the system. On the other hand, it restricts the ability to indicate regions of very small probability included in transient distributions of the complete state space. Equilibrium point analysis [5][6] is another method to estimate the relevant points of stable system behaviour, which may be complemented by transient analysis to get better insight into their neighbourhood.

In the next section we consider GI/G/1 systems in discrete time and shortly derive their basic transition equations in section III. The evaluation of the queue size over a sequence of arrival events is considered in the following sections. Sections VI and VII investigate the length of busy and idle periods as well as systems with finite waiting room.

II. COMPACT STATES SPACE FOR GI/G/1 SERVERS IN DISCRETE TIME

We consider the queue size of a classical discrete time GI/G/1 service system. Let A and S denote i.i.d. random variables for the interarrival and the service times with corresponding distributions $a(k) = \Pr(A = k)$ and $s(k) = \Pr(S = k)$. We assume distributions with finite support $1 \le A \le g$ and $1 \le S \le h$. The difference between a service and an interarrival time is denoted by $U = S - A$ where $u(k) = \Pr(U = S - A = k)$ is in a range $-g \le U \le h$. As a stability condition for a system with unlimited queue size, the mean service time is assumed to be smaller than the mean interarrival time

$$E(S) = \Sigma_k\, k \cdot s(k) < E(A) = \Sigma_k\, k \cdot a(k) \;\Leftrightarrow\; E(U) < 0.$$

The service discipline is non-preemptive and the order of service is independent of the service time, e.g. first come first served or random.

In order to comprehend the system state at an arbitrary point in time, we have to include three components

- the number N of customers in the system;
- the stage R_S of the service process represented by the residual service time, i.e. the time until the next departure. This component is only relevant for $N > 0$ and not when the system is empty;
- the stage of the arrival process R_A represented by the time until the next arrival instant. Note, that R_A could be equivalently defined to indicate the time span since the last arrival.

In a discrete time system with finite interarrival and service times, the latter two components are also finite. Let $N_{\max}$ denote the maximum number of customers considered in the transient analysis. This may be due to a systems with finite waiting room as considered in section VII. For unlimited waiting room, $N_{\max}$ may represent a truncation bound, such that the part of the state space beyond $N_{\max}$ is seldomly entered and can be neglected. Then the 3-component state space includes all the combinations

$$(1 \le N \le N_{\max}) \times (1 \le R_S \le h) \times (1 \le R_A \le g)$$
$$\cup \ (N = 0) \times (1 \le R_A \le g)$$

and has the size $g \cdot h \cdot N_{\max} + g$.

In the sequel, we observe the system only when arrival or service events are happening. Then a two-dimensional state space is sufficient. Besides a component for the number N of customers, the second component R comprises the stage of the arrival and service process. R is defined in the range $-g \le R \le h$ such that

- $R < 0$ if the current event is departure. Then $|R|$ is the time until the next arrival.
- $R > 0$ if the current event is an arrival. Then $|R|$ is the time until the next departure.
- $R = 0$ if an arrival and a departure coincide at the same point in time.

Starting from a system state Z_0 at time 0, the state Z_k is entered afterwards when the k-th instant of an event is encountered, where a coincidence of an arrival and departure at the same time is counted as a single event instant. The index of a system state is also used for its components

$$Z_k = (N_k, R_k) \in \aleph_0 \times \{-g, \ldots, h\} \quad \text{for } k \in \aleph_0,$$

where the current event(s) is (are) included in N_k, such that N_k represents the system state immediately after the considered event time.

The 2-dimensional state model at arrival and departure instants requires only $(g+h+1) \cdot N_{\max} + g$ states. When the number of steps g and h to represent the arrival and service time distribution are large, then a reduction from $g \cdot h$ to $g+h+1$ becomes essential. Comparable state models have been used by [2][4] for discrete time as well as phase type Ph/Ph/1 systems.

III. BASIC TRANSITION EQUATIONS

Based on the introduced state model, we obtain the transition equations associated with each event. Let

$$P_k(n, r) = \Pr\{N_k = n, R_k = r\} = \Pr\{Z_k = (n, r)\}$$

denote the state probabilities at the k-th event instant. When the probabilities $P_k(n, r)$ are known for all states, then the following equations are derived to determine the state probabilities $P_{k+1}(n, r)$ at the next event:

$$\begin{aligned} P_{k+1}(n,-r) = & \sum_{j=1}^{\min(h,g-r)} s(j) \cdot P_k(n+1,-r-j) + \\ & \sum_{j=1}^{\min(g-r,h)} a(r+j) \cdot P_k(n+1,j) + \\ & u(-r) \cdot P_k(n+1,0) \quad \text{for } n \ge 0; 1 \le r \le g; \end{aligned}$$

$$\begin{aligned} P_{k+1}(n,r) = & \sum_{j=1}^{\min(g,h-r)} a(j) \cdot P_k(n-1,r+j) + \\ & \sum_{j=1}^{\min(h-r,g)} s(r+j) \cdot P_k(n-1,-j) + \\ & u(r) \cdot P_k(n-1,0) \quad \text{for } n \ge 2; 1 \le r \le h; \end{aligned}$$

$$P_{k+1}(1,r) = P_{k+1}(0,r) = 0 \qquad \text{for } 1 \le r \le h;$$

$$\begin{aligned} P_{k+1}(n,0) = & \sum_{j=1}^{\min(g,h)} a(j) \cdot P_k(n,j) + \\ & \sum_{j=1}^{\min(h,g)} s(j) \cdot P_k(n,-j) + \\ & u(0) \cdot P_k(n,0) \quad \text{for } n \ge 2; \end{aligned}$$

$$\begin{aligned} P_{k+1}(1,0) = & \sum_{j=1}^{\min(h,g)} s(j) \cdot P_k(1,-j) + \\ & u(0) \cdot P_k(1,0) + \\ & \sum_{j=1}^{g} P_k(0,-j) \quad \text{and } P_{k+1}(0,0) = 0. \end{aligned}$$

Remarks:

A state $Z_{k+1} = (n, r)$ with negative second component $r < 0$ is entered from a previous state $Z_k = (n+1, \ldots)$, when the next event is a departure. The states $(0, r)$ with $r < 0$ determine the idle time distribution.

A state $Z_{k+1} = (n, r)$ with positive second component $r > 0$ is entered from a previous state $Z_k = (n-1, \ldots)$ when an arrival comes next. The states $(1, r)$ with $r > 0$ are never entered, since a transition from an empty system always starts a new service and interarrival time at state $(1, 0)$: $Z_k = (0, \ldots) \Rightarrow Z_{k+1} = (1, 0)$.

The transition equations are used for an iterative evaluation of the probability distribution $P_k(n, r)$ being observed k event instants after the beginning. As an initiation of the system, any arbitrary distribution $P_0(n, r)$ of the system states may be considered, which will develop in the course of events towards the stationary distribution of the system, if preconditions for convergence to a steady state are fulfilled. Usually we will start at state $Z_0 = (1, 0)$, which is entered after an idle period and corresponds to a deterministic distribution.

IV. TRANSITION EQUATIONS AT ARRIVAL INSTANTS

The transition probabilities at arrival instants can again be determined using the basic transition equations of section III by evaluating them in a different order. Let

$$P_k^{(A)}(n, r) = \Pr\left\{N_k^{(A)} = n, R_k^{(A)} = r\right\}$$

denote the state probabilities immediately before the k-th arrival instant for $r \leq 0$ where $|r|$ accounts for the time until the next departure. The corresponding probabilities $P_k^{(A)}(n, r)$ $(r > 0)$ for departures are still included as intermediate computation results, representing state probabilities for departures being encountered after the k-th and before the (k+1)-th arrival.

After the distribution $P_k^{(A)}(n, r)$ at the k-th arrival has been determined for all $r \leq 0$, we next compute all transition probabilities $P_k^{(A)}(n, r)$ with $r > 0$, which correspond to departures. Therefore we still make use of the first transition equation of section III without increment to k+1 on the left hand, since k now accounts for arrivals and remains unchanged for departures.

The queue size n reduces with each departure, thus allowing for an iterative computation of $P_k^{(A)}(n, r)$ in decreasing order of n, which even can capture a series of several departures occurring before the next arrival.

$$\begin{aligned} P_k^{(A)}(n,-r) = & \sum_{j=1}^{\min(h, g-r)} s(j) \cdot P_k^{(A)}(n+1,-r-j) + \\ & \sum_{j=1}^{\min(g-r, h)} a(r+j) \cdot P_k^{(A)}(n+1, j) + \\ & u(-r) \cdot P_k^{(A)}(n+1, 0) \quad \text{for} \quad n \geq 0; 1 \leq r \leq g; \end{aligned}$$

Thereafter, all remaining transition equations of section III are applied without any change to compute $P_{k+1}^{(A)}(n, r)$ for all $r \geq 0$. Thus we only have to add the superscript (A) in the notation of the probabilities in order to rewrite the transition equations for arrivals. With probability $P_{k+1}^{(A)}(n, 0)$ the (k+1)-th arrival coincides with a departure. In this state transition, the queue size n is unchanged corresponding to the situation immediately after both simultaneous events while k is incremented due to the arrival event.

In this way, the adapted transition equations enable to compute the transient state distributions at the k-th arrival instant for k = 1, 2, 3, … while preserving the normalization constraint

$$\forall k: \quad \sum_n \sum_{r=-h}^{0} P_k^{(A)}(n, r) = 1 \quad \Rightarrow \sum_n \sum_{r=-h}^{0} P_{k+1}^{(A)}(n, r) = 1.$$

Instead of focusing on arrivals, the distribution of system states can also be evaluated at the k-th departure. Therefore the basic transition equations can again be evaluated in a modified order, where k is incremented only at departure events instead of arrivals.

V. EXAMPLE OF TRANSIENT SYSTEM EVALUATION

In order to illustrate the transient behaviour, we consider a simple discrete time GI/G/1 example. The interarrival and service time distribution are given by:

$$a(1) = a(5) = 0.5 \text{ and } s(1) = s(4) = 0.5$$

with corresponding difference distribution

$$u(-4) = u(-1) = u(0) = u(3) = 0.25 \quad \text{and}$$

with mean $E(A) = 3$, $E(S) = 2.5$ and $E(U) = -0.5$. The utilization level of the system is $\rho \approx 83.3\%$. Figure 1 shows the results of the transient analysis, when the starting point is an empty system ($N_0 = 0$). For comparison, the a second graph shows results for initialization with $N_0 = 5$. For both cases, the queue sizes distribution is computed at points of an arrival. In particular, we consider the probabilities of a customer joining k other customers at the server when he arrives.

Both graphs of Figure 1 show probabilities for small queue sizes $\Pr(N_k^{(A)} = 0)$, …, $\Pr(N_k^{(A)} = 3)$ immediately before the k = 1., 2., …, 60. customer arrives. The distributions turn from different initial starting conditions to a steady state

$$p_k^{(A)} \stackrel{\text{def}}{=} \lim_{m \to \infty} \Pr(N_m^{(A)} = k)$$

when preconditions for a stationary system are valid. The steady state distribution can be analysed by polynomial factorization [2][7]. In the example we obtain

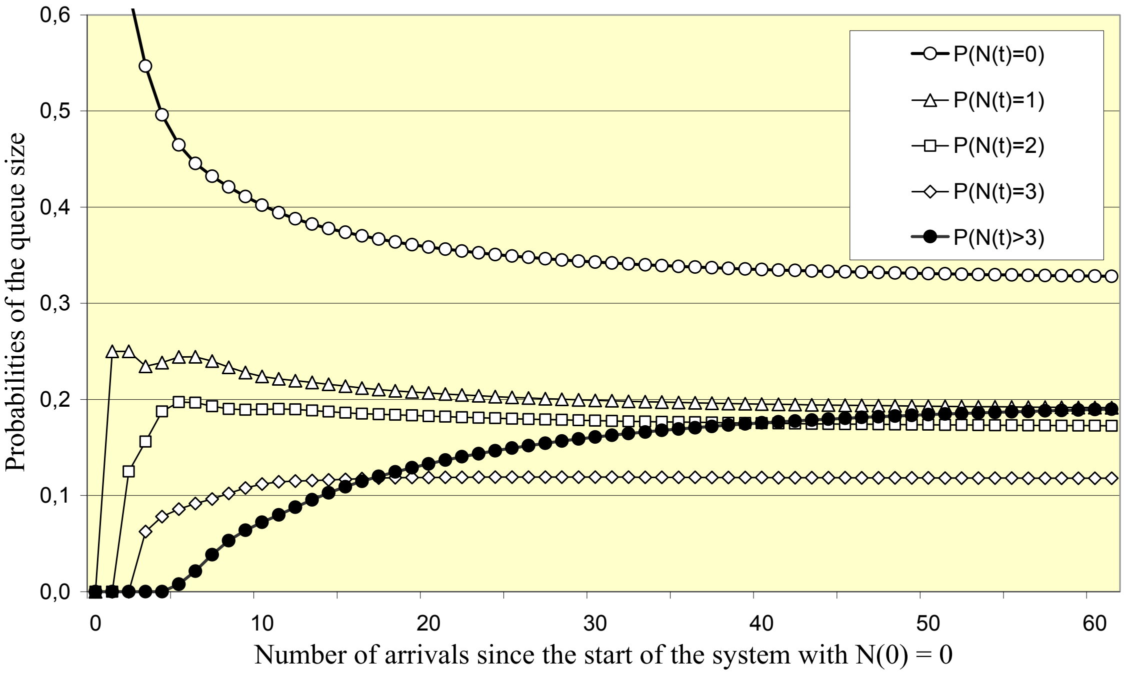
$p_0^{(A)} \approx 0.323$; $p_1^{(A)} \approx 0.189$; $p_2^{(A)} \approx 0.171$ and $p_3^{(A)} \approx 0.117$. The probabilities $\Pr(N_{60}^{(A)} = k)$ after 60 arrivals still deviate from the stationary distribution for $k = 0, …, 3$ by about 2% and $\Pr(N_{60}^{(A)} > 3)$ deviates by 5% from $p_k^{(A)}$, see last column of both graphs of Figure 1.

VI. ANALYSIS OF BUSY PERIODS AND DEPARTURES

The previous analysis of the transition equations at arrival instants can be modified to obtain the distribution of the number of arrivals per busy period. Therefore we start in the state $(N_0^{(A)}, R_0^{(A)}) = (1, 0)$ at the begin of a busy period. The computation of state distributions at arrivals is again carried out as derived in section IV, with the exception of transitions from states (0, r). When the system becomes empty, a new busy period starts. We simply omit transitions from empty system states, which all would lead to (1, 0):

$$\widetilde{P}_{k+1}^{(A)}(1,0) = \sum_{j=1}^{\min(h, g)} s(j) \cdot \widetilde{P}_k^{(A)}(1, j) + u(0) \cdot \widetilde{P}_k^{(A)}(1, 0).$$

Then the resulting distributions $\widetilde{P}_k^{(A)}(n, r)$ for $r \leq 0$ only account for arrivals being observed in the same busy period.



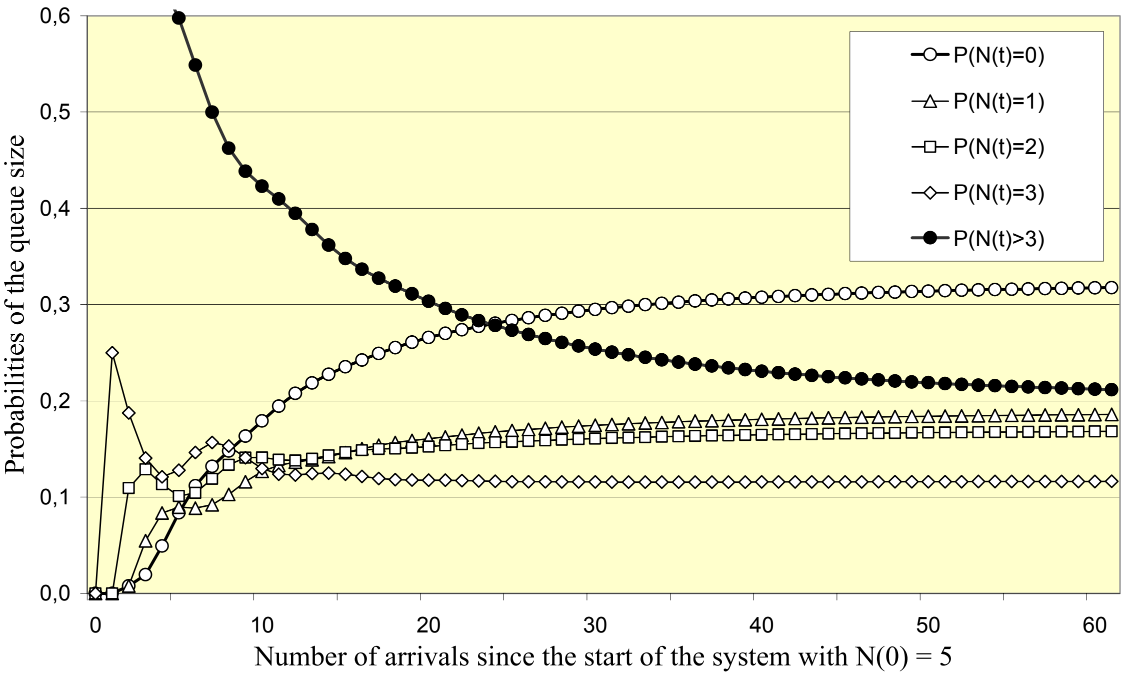


Figure 1: Transient behaviour of a GI/G/1 system with two different initial states

Thus we obtain defective distributions $\widetilde{P}_k^{(A)}(n, r)$, where the states $(N_k^{(A)}, R_k^{(A)}) = (n,r)$ are reached if the busy period includes at least k arrivals.

We can determine the probabilities $b_k^{(A)}$ that exactly k customers arrive and are served in a busy period:

$$\sum_n \sum_{r=-h}^{0} \widetilde{P}_k^{(A)}(n, r) = \sum_{m>k} b_m^{(A)} \Rightarrow$$

$$b_{k+1}^{(A)} = \sum_n \sum_{r=-h}^{0} \left(\widetilde{P}_k^{(A)}(n, r) - \widetilde{P}_{k+1}^{(A)}(n, r) \right).$$

Figure 2 shows an evaluation of the distribution of the number of customers $b_k^{(A)}$ per busy period for the same GI/G/1 example as considered in section 5. In addition, we can determine the distribution of the number of customers in the system at the arrival of the k-th customer in a busy period by normalizing the defective distribution $\widetilde{P}_k^{(A)}(n,\ r)$ and we can determine the distribution of length I_k of an idle period, which follows a busy period with k arrivals, by normalization of the zero state probabilities:

$$\Pr(I_k = m) = \widetilde{P}_k^{(A)}(0,\ m) / \sum_r \widetilde{P}_k^{(A)}(0,\ r) \cdot$$

The results are a basis for output process modeling, together with the moments of the busy period [3].

VII. GI/G/1/N SERVERS WITH FINITE WAITING ROOM

The previous analysis for GI/G/1 systems with infinite waiting room can be restricted to GI/G/1 systems with unlimited waiting room $n \leq N$. We assume new arrivals to be rejected and lost, when N customers are in the system, while the arrival process continues without interruption. Then all transitions from a state (N, r) with N customers to a state $(N+1, s)$ are redirected to (N, s). This is taken into account by the following modification in the basic transition equations of section III:

$$\begin{aligned} P_{k+1}(N,r) = & \sum_{j=1}^{\min(g,h-r)} a(j)\cdot\left(P_k(N-1,r+j)+P_k(N,r+j)\right)+ \\ & \sum_{j=1}^{\min(h-r,g)} s(r+j)\cdot\left(P_k(N-1,-j)+P_k(N,-j)\right)+ \\ & u(r)\cdot\left(P_k(N-1,0)+P_k(N,0)\right) \quad \text{for } 1\leq r\leq h. \end{aligned}$$

Finally we have $\forall\ n > N:\ P_{k+1}(n,s) = 0$. Since a state $(N, -r)$ would be reachable only from previous states $(N+1, s)$, we obtain $P_{k+1}(N,-r) = 0$ for $1 \leq r \leq g$. The computation of $P_{k+1}(N,r)$ for $1 \leq r \leq h$ can be carried out iteratively for decreasing r, i.e. starting from $P_{k+1}(N,h),\cdots,P_{k+1}(N,1)$.

Finally, the modifications of section IV in order to analyse the state probabilities $P_k^{(A)}(n,\ r)$ only at arrival (or departure) epochs are directly applicable to systems with finite waiting room. The busy period analysis of section V is also directly transferable.

Our implementation of the transient solution for infinite systems in fact also makes use of a finite system analysis, where the number N is chosen to be sufficiently large to make the probabilities of the omitted states and their influence on the relevant system states negligible. The alternative by truncating states with more than N customers and ignoring transitions over the truncation bound may also lead to negligible deviation, but in contrast to the finite system analysis this does not exactly preserve the normalization constraint.

Figure 3 shows evaluations of a system with limitation to $N = 10$ customers. The service time distribution is given by $s(1) = s(10) = 0.5$. Three cases are considered for the interarrival time distribution, with $a(1)$ varying from $a(1) = 0.6$, 0.5 to 0.4 and $a(10) = 1 - a(1)$.

We analyse the state probabilities $P_k^{(A)}(n,\ r)$ immediately before arrivals as in section IV. The system is initialized to be empty ($n = 0$) or full ($n = 10$). In both cases, the mean number of customers is traced over 80 arrival instants while converging to steady state.

VIII. CONCLUSION

Transient analysis can be used to obtain manifold properties of server systems including

- the development of the system over time from arbitrary starting conditions e.g. a system state or an initial distribution, converging to steady state behaviour, if conditions for a stable or stationary system are valid,
- view on system at arrival and/or departure epochs,
- characterization of the departure process by analysing the distribution of the length of a busy period and the following idle period,
- first passage times, i.e. the time until a predefined set of states is reached for the first time etc.

The underlying transition equations of the system can be flexibly adapted to modified boundary conditions for systems with finite waiting room and many other special cases. Challenges of the method can be seen in the computational complexity and numerical accuracy. A compact representation of the state space can keep the analysis tractable for a wide range of server systems including the renewal case GI/G/1/(N) studied in this paper and extensions to semi-Markov servers [7] or even small queueing networks. Investigations by [7] also show, how the accuracy level can be evaluated by verified computation techniques based on interval arithmetic enclosing the exact result or by sensitivity analysis with regard to input parameters.

If the state space becomes too large, transitions may be followed only for a limited set of transient states, e.g. those with probabilities above a minimum threshold or some other criterion for their relevance. This allows an approximate analysis of a wider spectrum of systems.

The transition equations in this paper collect all transitions to a state at the next considered epoch. When only a subset of states of the previous epoch is selected to be followed in the further development of the system, the transition equations should be rewritten to collect transitions from a state of the previous epoch. This view makes the implementation more efficient, since a marked set of relevant states at the previous epoch can be passed through and all their transitions at the next epoch are taken into account as contributions to the state probabilities at the next embedded point in time. The different views on transition equations from the previous or from the next epoch provides another source of flexibility for the transient system analysis.

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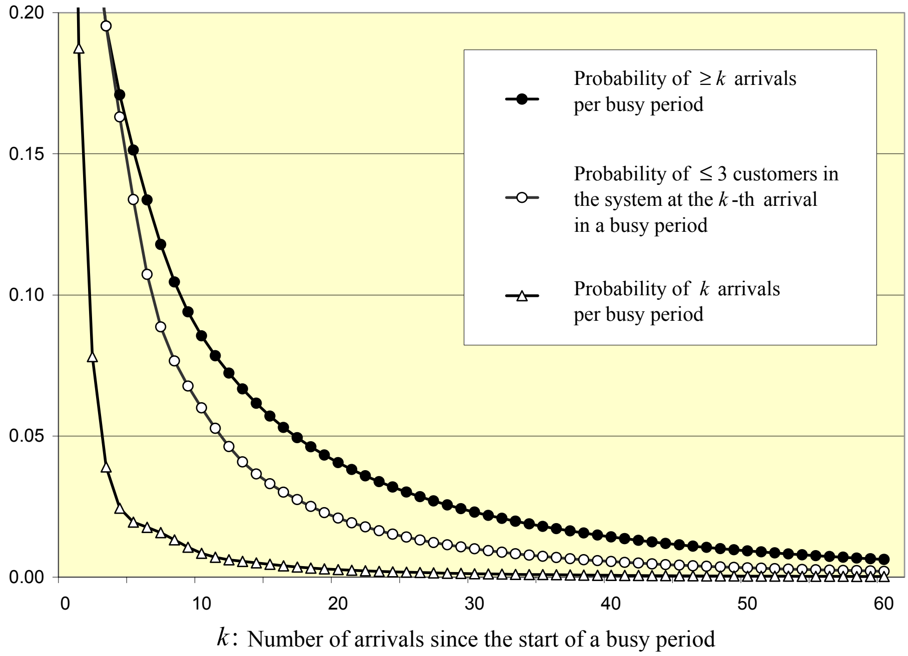

Figure 2: Transient analysis of characteristic distributions of the busy period

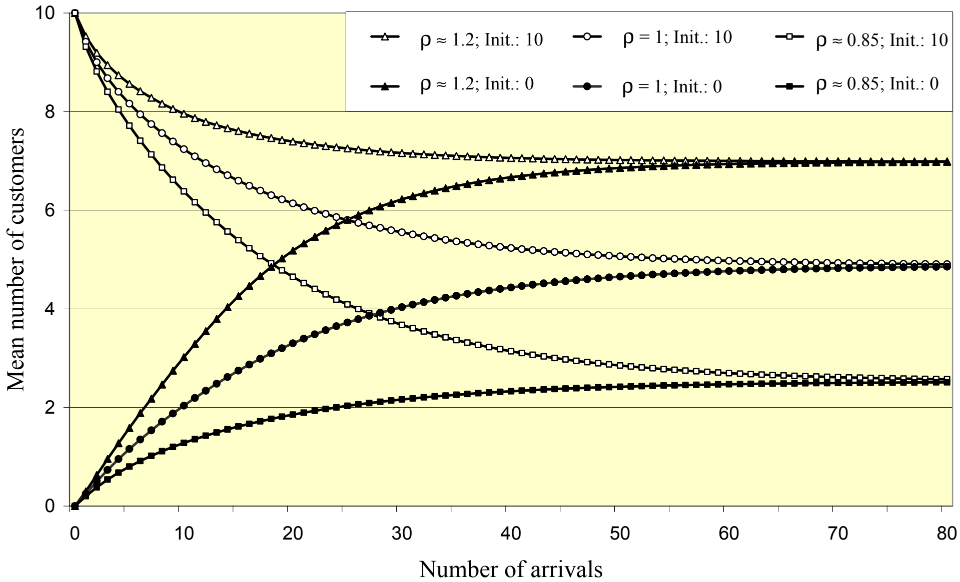

Figure 3: Transient analysis of a system with finite capacity

OPTIMIZATION OF BUFFERS CAPACITY IN TANDEM QUEUEING SYSTEMS WITH BATCH MARKOVIAN ARRIVALS

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KEYWORDS

Stochastic Models, Queueing Networks, Queueing Systems, Markov Models, Optimization.

ABSTRACT

Tandem queueing systems well suits for modeling many telecommunication systems. Recently, very general $BMAP|G|1|N \rightarrow \cdot|PH|1|M-1$ type tandem queues were constructively studied. In this paper we illustrate application of the obtained results for optimization of a buffer pool design.

INTRODUCTION

Open queueing networks and tandem queues as their important special case are widely used in capacity planning and performance evaluation of computer and communication systems, service centers, manufacturing systems, etc. Theory of tandem queues is developed rather well, for references see, e.g. (Gnedenko and Koenig 1983).

Apart from approximate analysis of the tandems used, e.g. (Heindl 2001; Heindl 2003; Ferng and Chang 2001a; Ferng and Chang 2001b), there are papers presented, that use analytical methods for such investigation (Breuer et al. 2004; Gomez-Corral 2002a; Gomez-Corral 2002b; Klimenok et al. 2005).

This paper exploits results on $BMAP|G|1|N \rightarrow \cdot|PH|1|M-1$ type tandems with losses and blocking at the second phase, see (Breuer et al. 2004; Klimenok et al. 2005), to calculate performance characteristics of the tandem queues in case of finite buffer at the first phase (i.e. $N < +\infty$). The analytical results were implemented in the software package "Sirius-C". This allows authors to perform optimization of some real-life networks or their fragments.

In the following sections we outline major analytical results necessary to calculate system characteristics used in optimization criterion as well as numerical examples of optimization in these tandem systems.

MATHEMATICAL MODEL

We consider tandem queue consisting of two queues (phases). The first queue is of the $BMAP|G|1|N$ type, i.e., it has a single server, finite buffer of capacity N, general service time distribution function $B(t)$ having the finite initial moment $b_1 = \int_0^\infty t dB(t)$. The input flow is described by the *BMAP(Batch Markovian Arrival Process)*. The $BMAP$ was introduced by D. Lucantoni (Lucantoni 1991) as extension of the models of versatile flows by M. Neuts (Neuts 1989) and N flows by (Ramaswami 1980). The class of the $BMAP$ includes many considered previously input flows such as, e.g., the stationary Poisson (M), Erlangian (E_k), Hyper-Markovian (HM), Phase-Type (PH), Markov Modulated Poisson Process ($MMPP$), etc. In opposite to recurrent (GI) flows and PH flow in particular, the $BMAP$ flow is correlated one. It makes it extremely useful for modeling the real flows in the modern telecommunication networks.

The $BMAP$ is defined by means of the directing (underlying) process $\tilde{\nu}_t, t \geq 0$, which is a continuous time Markov chain with the state space $\{0, ..., W\}$. Arrival of customers occurs in batches at the epochs when the process $\tilde{\nu}_t, t \geq 0$, has jumps. The intensities of jumps from one state into another one, which are accompanied by arrival of a batch consisting of k customers, are combined into the matrices $D_k, k \geq 0$, of size $(W+1) \times (W+1)$. The matrix generating function of these matrices is $D(z) = \sum_{k=0}^{\infty} D_k z^k, |z| \leq 1$. The matrix $D(1)$ is an infinitesimal generator of the process $\tilde{\nu}_t, t \geq 0$. The vector $\vec{\theta}$ of this process stationary distribution satisfies equations $\vec{\theta} D(1) = \vec{0}, \vec{\theta}\mathbf{e} = 1$. Here and in the sequel $\vec{0}$ is zero row vector and $\mathbf{e}$ is column vector of appropriate size consisting of units. In case the dimensionality of the vector is not clear from context, it is indicated as a subscript, e.g. $e_{\overline{W}}$ denotes the unit column vector of dimensionality $\overline{W} = W + 1$.
The average intensity λ (fundamental rate) of the $BMAP$ is defined as

$$\lambda = \vec{\theta} D'(z)|_{z=1}\mathbf{e}.$$

The intensity λ_g of groups arrival is defined as

$$\lambda_g = \vec{\theta}(-D_0)\mathbf{e},$$

variance v of intervals between the groups arrival is calculated as:

$$v = 2{\lambda_g}^{-1}\vec{\theta}(-D_0)^{-1}\mathbf{e} - {\lambda_g}^{-2},$$

the correlation coefficient c_{cor} of intervals between the successive groups arrival is calculated as

$$c_{cor} = ({\lambda_g}^{-1}\vec{\theta}(-D_0)^{-1}(D(1) - D_0)(-D_0)^{-1}\mathbf{e} - {\lambda_g}^{-2})/v.$$

For more information about the $BMAP$ and related research see (Lucantoni 1991) and the overviewing paper by S. Chakravarthy (Chakravarthy 2001).

The second queue has a finite buffer of capacity $M - 1, M \geq 1$. The single server is characterized by the PH- type service time distribution having an irreducible representation (β, S). Here β is the stochastic row-vector of dimension K and S is $K \times K$ matrix having the negative diagonal and non-negative non-diagonal entries, such as the column vector $S_0 = -S\mathbf{e}$ is non-negative and has at least one positive entry. The average service time is defined as $\beta(-S)^{-1}\mathbf{e}$. For more information about PH see (Neuts 1981).

In case the entering batch of customers finds insufficient number of places in a buffer (or the buffer is already full at all), the suitable number of customers from the batch joins a queue while the rest (or the whole group) leaves the system forever (is lost at the first phase). In case the customer completes the service at first phase and meets the buffer before the second phase be busy, we consider two cases:

- this customer leaves the system forever and is considered to be lost at the second phase.
- this customer waits for a buffer to have empty cell and prevents during waiting the other customers from being served at the first phase (i.e. service is blocked).

For optimization tasks we calculate the steady state distribution of the number of customers and loss probability in the tandem system.

Embedded Markov Chain

Following (Klimenok et al. 2005) for tandem with losses at the second phase, we consider the process $\varsigma_t^{(1)} = \{\tilde{i}_t, \tilde{j}_t, \tilde{\nu}_t, \tilde{\eta}_t\}, t \geq 0, 0 \leq \tilde{i}_t \leq N + 1, 0 \leq \tilde{j}_t \leq M$, $0 \leq \tilde{\nu}_t \leq W$, $1 \leq \tilde{\eta}_t \leq K$, where $\tilde{i}_t$ is the number of customers at the first phase, $\tilde{j}_t$ is the number of customers at the second phase, $\tilde{\nu}_t$ is the state of the $BMAP$ directing process, $\tilde{\eta}_t$ is the state of the directing process of the PH service at epoch $t, t \geq 0$.

Following (Breuer et al. 2004) for tandem with blocking at the second phase we consider the process $\varsigma_t^{(2)} = \{\tilde{i}_t, \tilde{j}_t, \tilde{\nu}_t, \tilde{\eta}_t, \tilde{\chi}_t\}, t \geq 0, 0 \leq \tilde{i}_t \leq N + 1, 0 \leq \tilde{j}_t \leq M$, $0 \leq \tilde{\nu}_t \leq W$, $1 \leq \tilde{\eta}_t \leq K$, $\tilde{\chi}_t = 0, 1$, where $\tilde{i}_t$, $\tilde{j}_t$, $\tilde{\nu}_t$, $\tilde{\eta}_t$ have the same meanings as in the tandem with losses, and $\tilde{\chi}_t = 0$ if the first server is working or waiting for a customer at moment t or $\tilde{\chi}_t = 1$ if it is blocked.

The processes $\varsigma_t^{(1)}, t \geq 0$, and $\varsigma_t^{(2)}, t \geq 0$, are non-Markovian. Thus, to investigate these processes, first we consider the embedded (at the service completion epochs $t_n, n \geq 1$, at the first phase) Markov chains. Due to fact that, in tandem with blocking a customer, which causes a blocking, is not counted neither at the second nor at the first phases, the multi-dimensional embedded Markov chains will have the same components

$$\xi_n^{(1)}, \xi_n^{(2)} = \{i_n, j_n, \nu_n, \eta_n\}, n \geq 1,$$

where

$$i_n = \tilde{i}_{t_n+0}, 0 \leq i_n \leq N, j_n = \tilde{j}_{t_n-0}, 0 \leq j_n \leq M,$$

$$\nu_n = \tilde{\nu}_{t_n}, 0 \leq \nu_n \leq W, \eta_n = \tilde{\eta}_{t_n-0}, 1 \leq \eta_n \leq K.$$

Note that if $j_n = 0$, then value of the component η_n is not defined.

Introduce the stationary state probabilities of these Markov chains as:

$$\pi(i, 0, \nu) = \lim_{n \to \infty} P\{i_n = i, j_n = 0, \nu_n = \nu\},$$

$$\pi(i, j, \nu, \eta) = \lim_{n \to \infty} P\{i_n = i, j_n = j, \nu_n = \nu, \eta_n = \eta\},$$

$$0 \leq i \leq N, 0 \leq j \leq M, 0 \leq \nu \leq W, 0 \leq \eta \leq K.$$

Vectors of stationary state probabilities are defined as follows:

$$\pi(i,0) = \{\pi(i,0,0), \pi(i,0,1), ..., \pi(i,0,W)\}, i = \overline{0,N},$$

$$\pi(i,j) = \{\pi(i,j,0,1), \dots, \pi(i,j,0,K), \pi(i,j,1,1), ..., \pi(i,j,1,K), \dots, \pi(i,j,W,1), \dots, \pi(i,j,W,K)\}, \quad i \geq 0, j > 0$$

$$\pi_i = \{\pi(i,0), \pi(i,1), \dots, \pi(i,M)\},$$

Conditions for this stationary distribution existence as well as definition of transition probabilities matrices can be found in (Breuer et al. 2004; Klimenok et al. 2005).

Steady State Distribution at Arbitrary Epochs

Having calculated vectors π_i, $i = \overline{0,N}$, by means of algorithms suggested in (Breuer et al. 2004; Klimenok et al. 2005), we can calculate steady-state distributions at arbitrary epochs.

For the tandem with losses at the second phase, these stationary probabilities and their vectors we define as follows:

$$p^{(1)}(i,0,\nu) = \lim_{t\to\infty} P\{\tilde{i}_t = i, \tilde{j}_t = 0, \tilde{\nu}_t = \nu\},$$

$$p^{(1)}(i,j,\nu,\eta) = \lim_{t\to\infty} P\{\tilde{i}_t = i, \tilde{j}_t = j, \tilde{\nu}_t = \nu, \tilde{\eta}_t = \eta\},$$

$$0 < j \leq M, 0 \leq i \leq N+1, 0 \leq \nu \leq W, 1 \leq \eta \leq K.$$

Enumerate the states of process $\varsigma_t^{(1)}, t \geq 0$, in the lexicographic order and form the probability vectors $p^{(1)}(i,j)$, $i = \overline{0,N+1}$, $j = \overline{1,M}$, of corresponding probabilities and $\mathbf{p}_i^{(1)} = \{p^{(1)}(i,0), \dots, p^{(1)}(i,M)\}$, $i = \overline{0,N+1}$.

For the tandem with blocking at the second phase these stationary probabilities and their vectors we define as follows:

$$p^{(2)}(i,0,\nu) = \lim_{t\to\infty} P\{\tilde{i}_t = i, \tilde{j}_t = 0, \tilde{\nu}_t = \nu\},$$

$$p^{(2)}(i,j,\nu,\eta) = \lim_{t\to\infty} P\{\tilde{i}_t = i, \tilde{j}_t = j, \tilde{\nu}_t = \nu, \tilde{\eta}_t = \eta\},$$

$$p^{(2)}(i,M,\nu,\eta,\chi) = \lim_{t\to\infty} P\{\tilde{i}_t = i, \tilde{j}_t = j, \tilde{\nu}_t = \nu, \tilde{\eta}_t = \eta, \tilde{\chi}_t = \chi\},$$

$0 \leq i \leq N+1$, $0 < j < M$, $0 \leq \nu \leq W$, $1 \leq \eta \leq K$, $\chi = \overline{0,1}$.

Enumerate the states of process $\varsigma_t^{(2)}, t \geq 0$, in the lexicographic order and form the probability vectors $p^{(2)}(i,j)$, $i = \overline{0,N+1}$, $j = \overline{1,M-1}$, and $p^{(2)}(i,M,0)$, $p^{(2)}(i,M,1)$ of corresponding probabilities and denote $\mathbf{p}_i^{(2)} = \{p^{(2)}(i,0), \dots, p^{(2)}(i,M-1), p^{(2)}(i,M,0) + p^{(2)}(i,M,1)\}$, $i = \overline{0,N+1}$.

We refer to the papers (Breuer et al. 2004; Klimenok et al. 2005), where the analytical formulas for calculation of steady-state probability vectors $\mathbf{p}_i^{(r)}$, $i = \overline{0,N+1}$, $r = 1,2$, are presented.

Tandem Performance Characteristics

The most important characteristics for investigated systems are:

- the loss probability at the first phase $P_{loss1}^{(1)}$, $P_{loss1}^{(2)}$ for tandems with losses and blocking respectively;
- the loss probability at the second phase $P_{loss2}^{(1)}$ for the tandem with losses only.

These loss probabilities can be calculated as follows:

$$P_{loss1}^{(1)} = 1 - \lambda^{-1} \sum_{i=0}^{N} \mathbf{p}_i^{(1)} \sum_{k=0}^{N+1-i} (k+i-N-1)\tilde{D}_k \mathbf{e},$$

$$P_{loss1}^{(2)} = 1 - \lambda^{-1} \sum_{i=0}^{N} \mathbf{p}_i^{(2)} \sum_{k=0}^{N+1-i} (k+i-N-1)\tilde{D}_k \mathbf{e},$$

$$P_{loss2}^{(1)} = \sum_{i=0}^{\infty} \pi(i,M)\mathbf{e},$$

where

$$\tilde{D}_k = \begin{bmatrix} D_k & 0 \\ 0 & I_M \otimes D_k \otimes I_K \end{bmatrix}, k \geq 0,$$

$\otimes$ - denotes Kroneker product of matrices, I_* denotes identity matrix of appropriate dimension.

For the system with blocking it is not possible to lose the customer at the second phase, thus we consider $P_{loss2}^{(2)} = 0$ in numerical examples for optimization criterion without loss of generality.

OPTIMIZATION CRITERION AND NUMERICAL EXAMPLE

We consider two strategies of buffers capacity planning:

- buffer capacities at the first and at the second phases are independent;
- memory is shared between first and second phases, thus total buffers capacity at both phases is constant.

To perform optimization task, we use the following cost criterion

$$C = C_1 N + C_2(M-1) + C_{loss1}\lambda P_{loss1} +$$
$$+C_{loss2}\lambda \left(1-P_{loss1}\right) P_{loss2},$$

where C_{loss1} is penalty for a customer loss at the first phase, C_{loss2} is penalty for a customer loss at the second phase, C_1 is cost of the cell at the first buffer; C_2 is cost of the cell at the second buffer.

For the case of shared buffer capacities we used the following criterion

$$C = C_{loss1}\lambda P_{loss1} + C_{loss2}\lambda \left(1-P_{loss1}\right) P_{loss2}.$$

Parameters of Tandems for Numerical Examples

We use the following parameters for numerical examples. Input flow is a $BMAP$-flow of intensity $\lambda = 10$, intensity of groups $\lambda_g = 5$, with correlation $c_{cor} = 0.199999$, and variation $c_{var} = 12.2732$. Matrices defining the BMAP are as follows

$$D_0 = \begin{bmatrix} -6.74538 & 5.45412 \times 10^{-6} \\ 5.45412 \times 10^{-6} & -0.219455 \end{bmatrix},$$

$$D_1 = D_3 = \begin{bmatrix} 2.01021 & 0.0134084 \\ 0.036728 & 0.0291068 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 2.68027 & 0.0178778 \\ 0.0489707 & 0.038809 \end{bmatrix},$$

and $D_k = 0$, $k \geq 4$.

Service time distribution at the first phase is degenerate with mean $T = 0.07$.

Phase-type service time distribution at the second phase has the following parameters

$$S = \begin{bmatrix} -10 & 0 \\ 0 & -40 \end{bmatrix},$$

$$\vec{\beta} = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix}.$$

These parameters make mean service time at the second phase equal to 0.0775.

First Numerical Example

In this example we consider independent buffer capacities at the first and at the second phases. Cost parameters in the criterion are specified as follows. Cost C_1 of a buffer cell at the first phase is 1, cost C_2 of a buffer cell at the second phase is 2, penalty C_{loss1} for customer loss at the first phase is 9, penalty C_{loss2} for the customer loss at the second phase is 15.

Figures 1 and 2 present values of the optimization criterion for various values of buffer size at the first and the second phases.

It can be seen that, for the fixed above values of the cost parameters, the value of the criterion is

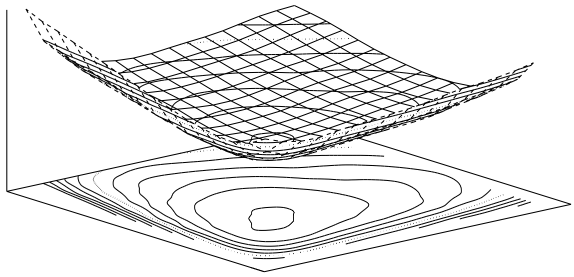

Figure 1: Dependency of Criterion Value on the Buffers Capacity for Tandem with Losses

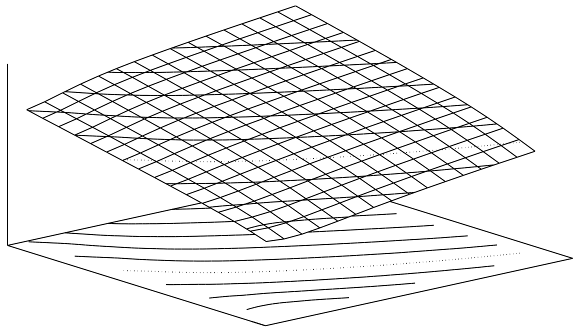

Figure 2: Dependency of Criterion Value on the Buffers Capacity for Tandem with Blocking

smaller for the model with customers loss at the second phase.

Optimal value of the criterion is equal to 39.334 at point $N = 5$, $M = 7$.

Second Numerical Example

In this numerical example we consider the same costs as in the previous example and shared buffer capacities, i.e. it is assumed, that the total capacity $M + N$ of two buffers is 16 and we share it between two phases.

The result is presented on figure 3.

Optimal value of the criterion is equal to 17.90 and M and N are equal to 7 and 9 respectively.

CONCLUSION

This paper gives examples of optimization of the tandem queueing models, that can be used to support operational maintenance of the telecommunication networks or their fragments. Also it can be used to verify approximating results during investigation of the real-life network objects.

Definite advantage of this paper is the possibility to analyze networks with correlated traffic, which is the case in modern telecommunication systems.

Figure 3: Dependency of Criterion Value on the first buffer capacity for $N + M = 16$

Given examples, show that application of such models can save up to 200% of operational costs.

This work addresses practical applicability of the analytical formulas and is aimed to help to further application of the queueing networks to investigation of modern telecommunication systems like cellular phones networks, Internet service providers' networks and many others under considerations of the complex heterogeneous nature of the traffic.

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SESSION 5

METHODOLOGY AND SOLUTIONS

TRAINING HIDDEN NON-MARKOV MODELS

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KEYWORDS

Hidden Markov Models, Discrete Phase-type Distributions, Stochastic Petri Nets .

ABSTRACT

Hidden Markov models (HMM) are well known in speech recognition, where they are trained to recognize spoken words and even whole sentences. They are used to find the parameters of a so-called hidden model (usually a DTMC) by training it with observed output sequences. This paper introduces an approach to train stochastic Petri nets with the methods of HMM. As opposed to a DTMC, a stochastic Petri net can model time-continuous stochastic processes with generally distributed state transitions. The training algorithm finds the parameters of the hidden models distribution functions only by training the model with the observed output. By using a more general modelling paradigm, more realistic models can be analysed using the methods of HMM. An experiment verifies the functioning of the method for an example model.

INTRODUCTION

Motivation

Finding the parameters of discrete stochastic models is often a tedious process: Collecting data, matching distributions, implementing approximations. Sometimes one cannot observe the actual process that one wants to model, but rather the results or the output of that process, since the actual model states are hidden. This observable output could be for example temperature records from the 18th century when the actual state of the system is the weather condition, which was not recorded. Observable output could also be the visible symptoms of a patient, where the hidden state is the actual disease or physical state. Diagnosing a disease or estimating the effects of medication based on the symptoms of the patient is very interesting.

Hidden Markov models (HMM) possess the ability to model hidden processes with observable outputs. One can find out the probability or the generating path of a specific output sequence (trace) and also train a hidden model to produce a specific output. The latter is equivalent to the parameterization of the model. The hidden model of HMM is usually a discrete-time Markov chain (DTMC). This restricts the modeling capabilities of the paradigm for example to geometric state duration distributions. We propose a method to train stochastic Petri nets (SPN) with the methods of HMM. SPNs are dynamic models that contain generally distributed state transitions and are used to model continuous processes with a time-dependent stochastic behavior.

Using SPNs, one can write down the structure of the hidden model with specific outputs of the model states and then find the transitions distribution parameters by training them with observed output sequences. With this new approach dynamic and more realistic hidden models can be parameterized with the methods of HMM, which expands the range of possible application areas.

Previous Work

Hidden Markov Models and their application in speech recognition were first published around 1970, one of the first papers is (Baum et al. 1970). A comprehensive summary of the theory including algorithms and application examples can be found in (Rabiner 1989). In order to make the models more flexible, some research was done on explicit state duration densities (hidden semi-Markov models), which however complicated the solution algorithms (Russel and Moore 1985). Expanding all HMM states to a sub-HMM was also used to realize more general state duration distributions (expanded state HMM) (Russel and Cook 1987). This increased the number of free parameters, the sub-HMM topologies were not very flexible, and the performance tuned to speech-recognition systems. In (Wickborn et al. 2006) a method is described to find the output sequence probability and the corresponding internal state sequence for continuous stochastic models with generally distributed transitions, but the method is not applicable for the training of models. An idea that was presented there, how general models could be trained, is further investigated in this paper.

BACKGROUND

Hidden Markov Models

Hidden Markov Models (HMM) are sometimes also called signal models, and are basically discrete-time Markov chains (DTMC) that emit signals in every step.

They are widely used in speech recognition systems and sometimes in pattern recognition.

A Hidden Markov model can be described by a 5-Tupel (S,V,A,B,π). S is the set of states of the DTMC. V is the set of output symbols. A is the transition probability Matrix of the DTMC. B is the output probability Matrix, with b_{ij} being the probability to output symbol v_j in state s_i. π is the initial probability vector of the DTMC. A sequence of states of the DTMC is denoted as $Q=\{ q_1, q_2, q_3,\ldots, q_T\}$ and an observed output sequence as $O=\{ o_1, o_2, o_3,\ldots, o_T\}$ with T being the maximum number of steps of the DTMC. A parameterization of a specific HMM is often denoted by $\lambda=(A,B,\pi)$ which fully defines the model.

There are three basic questions that can be answered for a HMM and a given output sequence (trace).

1. What is the probability of producing the given output sequence O with the given model λ?

2. What is the most probable state sequence Q of the model λ that produced the observed output sequence O?

3. Maximize the probability with which the model λ produces the output sequence O by training the parameters of the model.

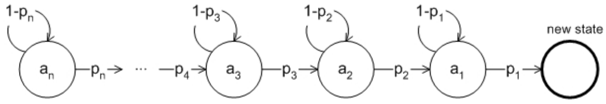
The first question can be efficiently answered by the Forward Algorithm (Rabiner 1989). The most likely state sequence can be determined by the Viterbi Algorithm. These two tasks will not be of interest in this paper and are only mentioned for completeness.

The third task of training a HMM model is solved by a kind of EM (Expectation Maximization) algorithm, the so-called Baum-Welch Algorithm (Baum et al. 1971). This algorithm takes an initial model parameterization $\lambda=(A,B,\pi)$ and iteratively improves it. The probability to produce the given output sequence O is increased in every step, but the algorithm does not necessarily find a globally optimal solution. Therefore the initial parameterization is of great importance.

Discrete Phases

Discrete Phase-type distributions are a method to describe a non-Markovian probability distribution function through a discrete-time Markov chain segment. They were first thoroughly described and formalized in (Neuts 1981). It is also possible to approximate general distribution functions through such discrete phase-type distributions. This makes it possible to turn a discrete stochastic model (for example a Petri net) containing general transitions into a Markov chain. An approximation algorithms has been proposed in (Isensee and Horton 2005).

The structure of a discrete phase-type approximation as used in this paper is shown in Figures 1. The phases $1\ldots n$ represent the initial state with their initial probabilities a_i. The p_i represent the state transition probabilities of the DTMC segment. The time to absorption in the *new state* mimics the non-Markovian distributions function.



Figures 1 : Structure of Discrete Phase-Type Approximation

TRAINING NON-MARKOVIAN MODELS

General Idea

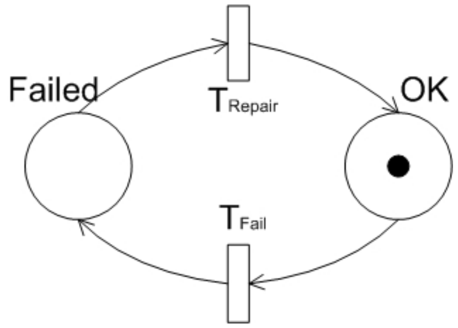
The general idea for the training of non-Markovian models is a stepwise approach. First the model to be trained is to be specified in its structure with transitions and output probabilities for the symbols and the states. The better this initial guess for the transitions distributions is, the better is the chance to find a good fit later. Then the model is turned into a DTMC by replacing the non-Markovian distributions with DPH of a chosen order.

The now Hidden Markov Model can be trained using the well known Baum-Welch Algorithm and a given output sequence or set of output sequences. The algorithm can be modified to not change the output probabilities and to leave the system structure intact by preserving zero entries in the transition probability matrix. Whether some state transitions might be deleted due to the unsupervised training process or whether states might change their roles, has to be investigated.

When the model has been successfully trained and the model structure preserved, one should be able to extract the also trained phase type distribution, which corresponds to a particular general transition in the model. One can now investigate the time to absorption in the DPH and even try and match a known general distribution function to it, which can then be used as the parameterization for the original model.

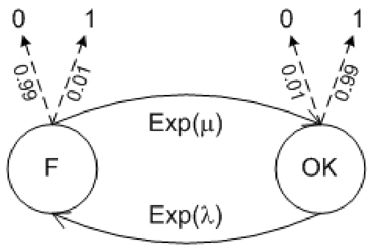
Model Initialization

The presented approach is mainly a possibility to train parameters of real life models with a defined structure. Therefore the general structure of the model with its states and transitions needs to be known to obtain useful results. The system structure can be modeled for example using stochastic Petri nets with generally distributed firing times as described in (Bobbio et al. 1998). One restriction is that the model needs to have a finite state space, since it will later be turned into a Markov chain. An example Petri net of a machine model with the states *OK* and *Failed* corresponding to the places of the net is shown in Figures 2. The transitions are T_{Fail} and T_{Repair}.



Figures 2 : Example Petri Net of a Machine Model

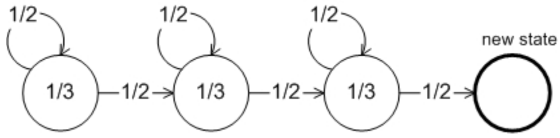
If additional information about the transitions is available, these should be initialized as good as possible, since the training algorithm used later is dependent on the initial model conditions. Furthermore, output symbols with probabilities need to be specified on the basis of the states of the system. This can also happen on the reachability graph of the Petri net, where the discrete states of the system are explicitly visible. The reachability graph with specified output probabilities for the output symbols *0* and *1* and two exponential distributions as initial parameterization is shown in Figures 3.



Figures 3 : Initialized Reachability Graph of Example Petri Net

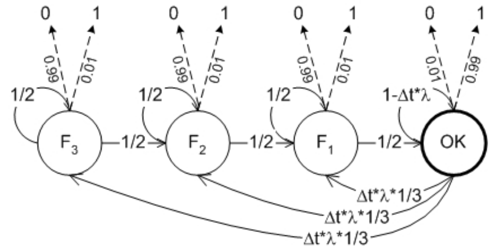
Replacing General Distributions

When all timed transitions of the SPN have an exponential distributions, as in GSPNs (Bobbio et al. 1998), then the reachability graph of the Petri net is equivalent to a Markov chain. In order to map the non-Markovian transitions to a Markov chain, one can replace them by so called discrete phase-type distributions (DPH). We use one minimal structure, since any acyclic DPH of the same order (number of phases) can be transformed into one of this specific structure with the minimum necessary number of parameters (Bobbio et al. 2002). Figures 4 shows a DPH of order three with uniformly chosen initial parameters. Theoretically the condition holds, that the larger the order of the DPH is, the better is the fit of the general distribution. Nevertheless does the number of phases probably have an influence on the time needed for the fit, or the amount of data needed, since by adding one phase, the number of independent variables to be trained is increased by two. Furthermore do some types of distributions require fewer phases to be fit accurately than others (Isensee and Horton 2005).



Figures 4 : Example DPH with Arbitrary Parameters

Some discrete states of the model will be replaced by several states of the DTMC. Since the probabilities of the output symbols are specified on the discrete state level of the SPN, these same values can just be copied to each corresponding state in the DTMC. The resulting DTMC for the model (Figures 3) and the DPH (Figures 4) is shown in Figures 5, assuming that the transition T_{Fail} is exponential and T_{Repair} is non-Markovian. The rates of the exponential distributions were transformed into transition probabilities by discretizing them with a chosen time step $\Delta t=0.1$.

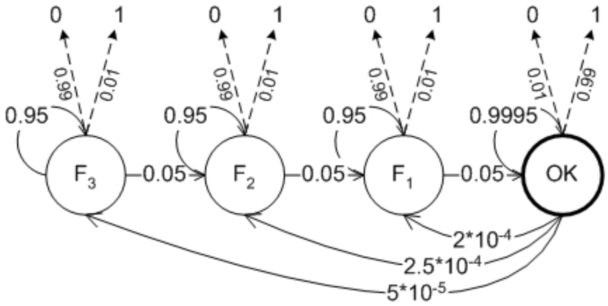


Figures 5 : Resulting HMM of Example Petri Net

Baum-Welch

The resulting DTMC with output probabilities for certain symbols can be interpreted as a hidden Markov model. One can now use the existing Baum-Welch Algorithm to train the HMM to produce the given output sequence or set of output sequences with a maximum probability. As opposed to the applications of HMM so far, the roles of the states in the DTMC need to stay fixed during the fitting process. Only if this condition is met, one can find and extract the DPH parameters from the trained model. Since the Baum-Welch Algorithm is an unsupervised training algorithm, it can also change the roles of DTMC states, which is a known property of the algorithm.

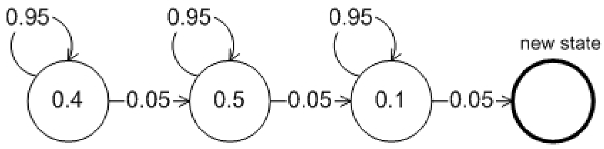
An existing implementation of the Baum-Welch Algorithm was modified to preserve the output probabilities. This was found necessary in some initial experiment to fix the roles of the states. The system structure is also left intact by not changing any zero entries in the transition probability matrix of the DTMC. This means, that no new state transitions are introduced into the model. Whether existing state transitions are preserved or might be destroyed in the trained model, needs to be investigated further. A deleting of transitions would significantly change the structure of the model and would complicate interpretation if not make it impossible. The result of the training process could be a HMM as shown in Figures 6.



Figures 6 : Trained Example HMM

Extraction

The extraction of the DPH from the trained HMM is basically the reverse step of the replacement of general transitions. If the model structure and state roles have been preserved during the previous training process, the location of the DPH in the DTMC is known, and its parameters can be extracted by normalizing the incoming transition probabilities. The extracted DPH of the trained HMM in Figures 6 is shown in Figures 7.

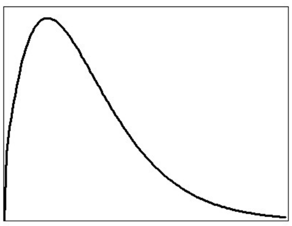


Figures 7 : Extracted DPH from Example HMM

The reachability graph of the original model can contain several state transitions that refer to the same transition in the Petri net. If these are replaced by DPH, the resulting trained DPH need to be combined in order to find a representation for the Petri net transition. This could for example be done by weighting the different fits parameters.

Backfitting

The last step in determining the parameterization of the general transitions in the Petri net is to investigate the time to absorption in the extracted DPH. The shape of the output of the example DPH in Figures 7 is shown in Figures 8. The shape clearly resembles a PDF (probability density function) of a Weibull distribution.



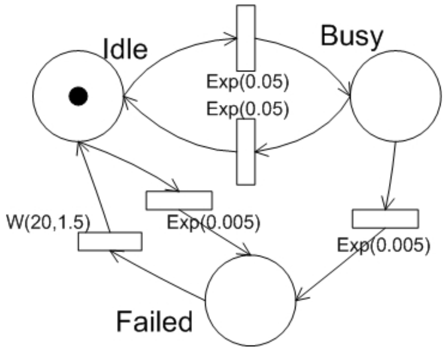
Figures 8 : Shape of Time to Absorption in Example DPH

The more interesting analysis is to try and match the phase-type distribution to a known general distribution function with specific parameters. Assuming a discretization time step of $\Delta t=0.1$, which was also used for the discretization, the closest match is a Weibull distribution with a scale parameter of approximately $\alpha=5$ and a shape parameter of about $\beta=1.5$. This finding of distribution type and parameters can be done by using common input modeling methods.

Here the optimization algorithm for the fitting of the DPH parameters, described in (Isensee and Horton 2005), was adapted and used to find the parameters of a general distribution function for a specific DPH output. The algorithm can also perform the fit for several known distribution types, compare the resulting error values and return the most likely distribution type and parameters. The distribution can now be used for the corresponding transition in the initial Petri net.

EXPERIMENTS

The experiments were performed with the model shown in Figures 9. The Petri net shows a web server that can be either working in the states *Idle* or *Busy* and that can be in state *Failed*, and under repair. The output that the system produces is actually the response to a ping request sent once every time unit. To the outside user in the web the actual state of the web server is not visible, he can only observe the answer to the ping request. By analyzing the observed traces he tries to determine the actual probability or distribution characteristics of the server being offline.

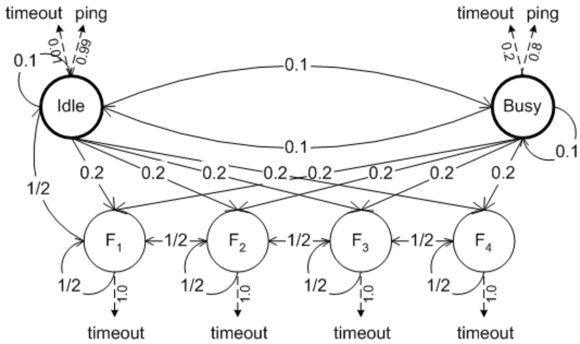


Figures 9 : Petri Net of Web Server Model

It is known, that in state *Idle* the web server responds with a probability of *0.99* and in state *Busy* only with a probability of 0.8. When it is *Failed*, the web server does not answer at all, so the probability for a timeout is 1.0. Only the repair time is assumed to have a non-Markovian distribution.

The output sequences for the training of the model were created by running a discrete event simulation of model using the parameters shown in Figures 9. According to the state of the model the simulation produces an output symbol every $\Delta t=1$ time units, and records it in an output file. Five stochastically independent runs were performed to produce five independent output sequences. They have a maximum possible length of $T=1,410,100$ symbols.

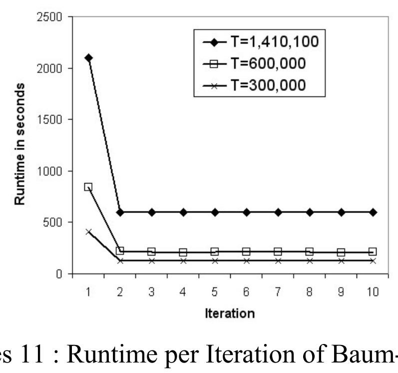
The reachability graph of the Petri net was turned into a Markov chain by replacing the general transition with a phase-type distribution of order four. The initial parameterization of the HMM for the model training process was chosen as shown in Figures 10. The initial state probabilities were set uniformly to *1/6.*



Figures 10 : Initial Parameters of Example HMM

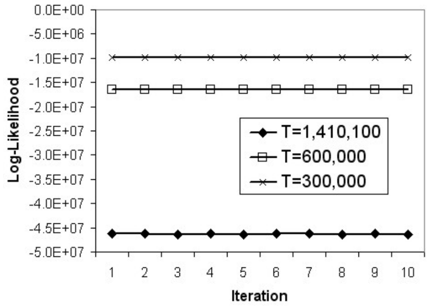
This initial model was trained iteratively using the Baum-Welch Algorithm with fixed output probabilities. Even though the algorithm itself is an iterative algorithm (iterations are called epochs), one training run did not result in a good fit, but several successive calls of the algorithm improved the model fitness. The chosen implementation terminates when the relative difference between the current likelihood and the mean likelihood of the previous fits falls below a threshold of *1e-6.* It was not tested, whether a stricter stop criterion would eliminate the need of several successive runs.

The experiments were performed using the five stochastically independent output sequences of different length. *Exp1* used the maximum possible length of *T=1,410,100*, *Exp2* used traces of length *T=600,000* and *Exp3* length *T=300,000.* The results for the processing time of the first 10 iterations are shown in Figures 11. The runtime decreases significantly with the length of the symbol sequences, as expected. The average runtime for *Exp1* is *600s*, for *Exp2 200s* and for *Exp3* about *120s*. The first iteration is more expensive in all cases, since this trains an arbitrarily initialized model, which requires more so-called epochs.



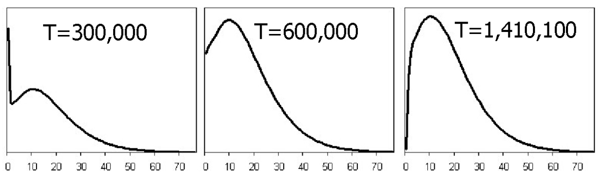
Figures 11 : Runtime per Iteration of Baum-Welch Algorithm Using Different Trace Length

The development of the log-likelihood through the first 10 iterations is shown in Figures 12. A larger absolute value represents a lower log-likelihood. The first call of Baum-Welch leads to an initially steep improvement in the initial log-likelihood of *4e-1*, which is not shown in the graph, since this contains the resulting trained models log-likelihood values. The log-likelihood for the longer traces is much lower than that of the shorter ones, since it is computed from a product of more probability values. The further iterations do not seem to improve the log-likelihood of the models, but the resulting fits for the non-Markovian distributions do. The cycling through the traces decreases the danger of getting stuck in a local minimum for all of the three experiments.



Figures 12 : Log-Likelihood for Successive Baum-Welch Iterations for Different Trace Length

The shape of the non-Markovian distribution for the repair time after 50 iterations can be seen in Figures 13. The fit that resulted from the training with the longest traces is the best one. It already resembles the Weibull distribution, which was used for creating the output symbol sequences. Further experiments showed that about 100 iterations were needed to find a good approximation for the W(20,1.5) distribution for the longest traces in *Exp1*. For the medium length traces *Exp2* about 300 iterations were needed to find a good fit. The shortest traces in *Exp3* did not yield comparable fits to the other experiments after 300 iterations, and did not improve further.



Figures 13 : Fits for Repair Time after 50 Baum-Welch Iterations with Different Sequence Length

This behavior of the algorithm is due to the following fact. The longer the traces, the more information is contained in them. The shortest traces obviously do not carry enough to train the tested model. The training using medium length traces needed more iterations, but about as much time in total as the training with the long traces.

The experiments show, that the approach above described does work. It is possible to train non-Markovian models using the methods of HMM. The use of several output traces instead of just one increases the probability to find a globally good solution.

CONCLUSION

Summary

This paper introduced an approach to find the parameters of non-Markovian stochastic Petri nets using methods of hidden Markov models, specifically the Baum-Welch algorithm. By replacing the state transition by a DPH and training its parameters, a known mathematical distribution function can be found for the transition in the Petri net. This makes it possible to parameterize continuous stochastic processes by using some observable output of their hidden states, where before the hidden models were usually restricted to DTMCs. This expands the range of possible applications to more realistic models beyond speech recognition. The experiments showed that the approach works, and that only output sequences, which contain a sufficient amount of data, work for training the models.

Outlook

The approach presented here is still new and opens up exciting new prospects and questions. Possible application areas of a working training method for non-Markovian models are any kind of hidden stochastic systems that can only by observed via their output: Disease recognition in a patient on the basis of the visible symptoms, consumer behavior estimation or failure protocols of running systems. Once trained these models could help in diagnosing, predicting behaviors and finding errors. Less spectacular, but also useful are applications, where noisy data is used for finding model parameters. The data does not need to be filtered, it can be used as is.

There are also several open issues regarding the training algorithm. The implementation of the training algorithm itself needs to be modified to be able to train with more than one trace, which was done artificially in this paper. More experiments are needed that test the feasibility on larger and more complex models. The replacing of several non-Markovian transitions will be needed for real life models. The merging of several trained DPH when a transition had to be replaced at several positions in a Markov chain is also interesting problem. The influence of the length of the DPH used for replacing on the training process and results is also interesting. Another open problem is finding the right amount of data needed for the training, so that it is sufficient and does not slow the training process too much. It needs to be tested whether it is possible to also train the output probabilities of the model. Under which conditions the Baum-Welch algorithm preserves the model structure also needs to be investigated.

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RELIABLE COMPUTATION OF WORKLOAD DISTRIBUTIONS USING SEMI-MARKOV PROCESSES

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ABSTRACT

In stochastic traffic modeling, the computation of workload distributions plays a prominent role since they influence the quality of service parameters. InterVerdiKom is a workload analysis tool that uses two different techniques: the polynomial factorization approach and the Wiener-Hopf factorization to determine the workload distributions of GI/GI/1 and SMP/GI/1 service systems accurately.

KEYWORDS

Reliable algorithms, stochastic traffic modeling

INTRODUCTION

The computation of workload distributions of service systems in telecommunication networks is essential for determining the quality of service parameters for various types of data transfer traffic. Depending on the model, there are different ways to determine workload distributions. We investigate polynomial and Wiener-Hopf factorization as efficient approaches for classical general independent GI/GI/1 and discrete time semi-Markovian server systems. We show that subproblems can be non-trivially solved by a computer program. Both methods are integrated into a recent tool InterVerdiKom (Interval-based numerical software with result verification of the workload distribution for Internet traffic). Our approach is first to obtain a numerical solution of the steady state workload distribution using an extension of the algorithm of Grassmann and Jain (1989). After a verification step, the guaranteed workload distribution can be computed. Both steps are carried out by using InterVerdiKom.

VERIFIED COMPUTATION OF THE WORKLOAD DISTRIBUTION

In (Fausten et al., 2004a) we have presented a short introduction to traffic modeling in telecommunication networks. The classical approach in queueing and service systems is to consider random variables for the inter-arrival times of events corresponding to the arrivals of packets, flows, connections or other units relevant for network elements. Two basic characteristics of the stochastic behavior of traffic are the distribution function of considered random variables and the autocorrelation of the process. To model the distribution function of arriving data as well as the autocorrelation function, semi-Markov processes can be used. These processes extend the well-known Markov models with finite state space. Furthermore, they provide memoryless states with exponential or geometrical distribution, such that each state is associated with an arbitrary state specific distribution. For a process $SMP(m)$ with m states in the underlying chain, the autocorrelation has the form of a superposition of $m-1$ geometrical terms including complex coefficients. $SMP(m)$ models have been successfully used within a fitting procedure for a given video trace to handle video traffic multiplexing as described in (Traczinski et al., 2005). Since the results have influence on service level agreements, the analysis of such models and the modeling itself should be done as exactly as possible. Consequently, interval arithmetic is applied to guarantee the results of the analysis and to validate the method. Hence, we are able to obtain reliable information about data delay and loss probabilities.

Polynomial factorization approach

In the following, we consider a GI/GI/1 service system in the Kendall notation with general independent (GI) arrival and service processes and only one service station. A_n, S_n denote the interarrival and service times of the n-th arrival,

$$0 \leq A_n \leq g, 0 \leq S_n \leq h,$$

$$U_n = S_n - A_n, W_n = \max(W_{n-1} + U_{n-1}, 0)$$

W_n the waiting time of the n-th request. Then the fact that for $i > 0$

$$Pr(W_n = i) = \sum_k Pr(W_{n-1} = k) Pr(U_{n-1} = i - k)$$

leads to

$$w(i) := \lim_{n \to \infty} Pr(W_n = i)$$

the stationary distribution. The workload of a GI/GI/1 server denotes the time required to process all requests present.

For the stationary distribution w we obtain the characteristic system equation (Traczinski et al., 2005)

$$w(k) = \sum_{-h \le i \le g} w(k+i)u(-i) \text{ for } k \ge h$$

$$w(k) = \sum_{-k \le i \le g} w(k+i)u(-i) \text{ for } h-1 \ge k \ge 1$$

$$w(0) = \sum_{0 \le i \le g} w(i) \sum_{i \le j \le g} u(-j)$$

Using the representation $w(k) = \alpha\beta^k$ we can derive the characteristic function as

$$p(z) = S(z^{-1})A(z) - 1 = U(z^{-1}) - 1$$

with the generating functions $A(z)$ and $S(z)$ of the inter-arrival and service time

$$U(\beta^{-1}) = \sum_{-g \le i \le h} u(i)\beta^{-i} = 1$$

$$E(U) = \sum_{-g \le i \le h} iu(i) = -\sum_{-h \le i \le g} iu(-i)$$

with the stability condition $E(U) < 0$ and

$$w(k) = \sum_{1 \le j \le h} \alpha_j \beta_j^k, k > -h$$

for simple zeros $\beta_1, \ldots, \beta_h$ of $z^h p(z)$ inside the unit circle.

Since the degree of the characteristic system equation is $g+h$, we need the roots exclusively inside the unit circle. Now we use the following algorithm:

- Input: characteristic polynomial
- Determine the zeros $\beta_j, j = 1, \ldots, h$, inside the unit circle. All roots except zero have to be simple
- Solve the Vandermonde system

$$\sum_{j=1}^{h} \alpha_j \beta_j^k = 0 \text{ for } k = -h+1, \ldots, -1$$

$$\sum_{j=1}^{h} \alpha_j = w(0)$$

for α_j using the β_j

- Compute enclosures for $w(0)$ and $w(k), k > 0$
- Output: verified workload w

This algorithm has the following problems: For huge values g it is difficult to localize h zeros in the unit circle and to find tight enclosures for the zeros of the large system. Only tight enclosures lead to a suitable solution of the Vandermonde system for the coefficients α_j and to tight enclosures for the workload $w(k)$. So the major goal of this paper is to derive theoretical results that guarantee the existence of h zeros inside the unit circle and to compute initial values that are close to these zeros before starting a numerical solver.

In queueing theory, Rouché's theorem is very important to prove ergodicity conditions or the existence of solutions for polynomial equations building the generating functions. Generally, the theorem is used to show the existence of a certain number of zeros in the analyticity domain of a complex function. However, in certain cases it is very difficult to verify the assumptions of the theorem. Thus, it is of general interest to provide an alternative approach to obtaining the zero distributions of the system by using numerical methods or properties of the underlying stochastic process (i.e., the Markov chain).

In the GI/GI/1 case, it can be shown with the extended Rouché theorem (Klimenok, 2001) that exactly h roots of $z^h p(z)$ are inside the unit disc.

Klimenok's theorem. Let the functions $f(z)$ and $\varphi(z)$ be analytic in the open disk $|z| < 1$ and continuous on the boundary $|z| = 1$, and the following relations hold:

$$|f(z)|_{|z|=1, z\neq 1} > |\varphi(z)|_{|z|=1, z\neq 1},$$
$$f(1) = -\varphi(1) \neq 0.$$

Also, let the functions $f(z)$ and $\varphi(z)$ have derivatives at the point $z = 1$ and assume that the following inequality holds:

$$(f'(1) + \varphi'(1))/f(1) < 0.$$

Then the numbers $N_{f+\varphi}$ and N_f of zeros of the functions $f(z) + \varphi(z)$ and $f(z)$ in the domain $|z| < 1$ are related as follows:

$$N_{f+\varphi} = N_f.$$

In the present case we set

$$\varphi(z) := z^h \sum_{-h \le i \le g} u(-i) z^i, f(z) = -z^h$$

and find

$$(f'(1) + \varphi'(1))/f(1) = (h - \sum_{-h \le i \le g} u(-i)(h+i))$$
$$= -\sum_{-h \le i \le g} u(-i)i < 0.$$

If we assume a uniformly distributed U, it is possible to derive the asymptotic behavior of the zeros $\beta, |\beta| < 1, g \to \infty$. Assume

$$u(-i) = 1/c, c := g + h + 1.$$

Then

$$cU(\beta^{-1}) = \beta^g + \cdots + \beta^{-h} = c \Rightarrow \beta^{g+1} - \beta^{-h} = c(\beta - 1)$$
$$\Rightarrow \beta^{-h} \approx c(1-\beta), \beta = r\exp(i\theta).$$

Then it follows that

$$cr^h = 1/(1 + r^2 - 2r\cos\theta)^{1/2}, r = c^{-1/h}(1 + O(1/h))$$
$$\arg\beta = 2\pi k/h + 1/h \arctan(r\sin\theta/(1 - r\cos\theta)).$$

Furthermore,

$$|(\beta^{h+g+1} - 1)/(\beta - 1)| \ge (r^{g+h+1} - 1)/(r+1) := cr^h \Leftrightarrow$$
$$1 = r^h(c + cr + r^{g+1}) \Leftrightarrow$$
$$r = (c + cr + r^{g+1})^{-1/h} \Rightarrow$$
$$r > (c + c^{1-1/h} + c^{-(g+1)/h})^{-1/h},$$

since $r < c^{-1/h}$, which provides a lower bound for r. The asymptotic behavior can be used to generate suitable initial values automatically as an input for standard root finder tools.

Theorem. Assume that a closed expression for

$$z^h S(z^{-1})A(z)$$

can be obtained such that the asymptotic relation

$$z^h S(z^{-1})A(z) \approx T(z,h)$$

for $g \to \infty$ and $|z| < 1$ holds true. Then the roots w inside the unit circle fulfill $w^h/T(w,h) \approx 1$.

Example 1. In the equidistributed case we have

$$z^h S(z^{-1})A(z) = \frac{(1-z^{h+1})(1-z^{g+1})}{(g+1)(h+1)(1-z)^2},$$

$$T(z,h) := \frac{(1-z^{h+1})}{(g+1)(h+1)(1-z)^2}.$$

Note that the degree of the numerator is substantially reduced.

Next, we want to discuss the more general SMP/GI/1 systems and a Markov chain with m states. We use

$$p_m(z) = z^{mh}\det(U^T(z^{-1}) - I)$$

to denote the characteristic polynomial of the SMP/GI/1 system and consider the following determinant function:

$$p_m(z) = \det\begin{pmatrix} s_{11} - z^h & s_{12} & \dots & s_{1m} \\ s_{21} & s_{22} - z^h & \dots & s_{2m} \\ \dots & \dots & \dots & \dots \\ s_{m1} & s_{m2} & \dots & s_{mm} - z^h \end{pmatrix}$$

with

$$s_{ab} = \sum_{j=-h}^{g} u_{ab}(-j)z^{j+h},$$

$$u_{ij}(k) = Pr(U_{n+1} = k, \sigma_{n+1} = j|\sigma_n = i)$$

and

$$\sum_{-g\le k\le h}\ \sum_{1\le j\le m} u_{ij}(k) = 1.$$

As in the previous case, we apply Klimenok's theorem. This can be done successfully for $m = 2$ and 3, but for greater m it is only possible when further restrictive conditions are applied to the distribution $\{u_{ij}(k)\}$. To derive the asymptotic behavior of the zeros, we write

$$p_m(z) = \det\begin{pmatrix} a_{11}(z) - z^h & \dots & a_{1m}(z) \\ a_{21}(z) & \dots & a_{2m}(z) \\ \dots & \dots & \dots \\ a_{m1}(z) & \dots & a_{mm}(z) - z^h \end{pmatrix}.$$

For $m = 2$, we infer

$$|p_2(z)| \ge |1 - a_{11}||1 - a_{22}| - a_{12}a_{21} = 0$$

with $a_{ii} = \max_{|z|=1}|a_{ii}(z)| = a_{ii}(1)$, $i = 1, 2$, and $a_{12} = 1 - a_{11}, a_{21} = 1 - a_{22}$. Therefore, the assumptions in Klimenok's theorem are fulfilled since the first $\ge$-sign can be replaced by the $>$- sign if $z \ne 1$.

We put

$$f(z) = (a_{11}(z) - z^h)(a_{22}(z) - z^h),$$

$$\varphi(z) = -a_{12}(z)a_{21}(z)$$

and find

$$\begin{aligned} f(1) &= (1-a_{11})(1-a_{22}) > 0 \\ f'(1) &= (\sum_{j=-h}^{g}(j+h)u_{11}(-j) - h)(a_{22}-1) \\ &\quad + (\sum_{j=-h}^{g}(j+h)u_{22}(-j) - h)(a_{11}-1), \\ \varphi'(1) &= -(\sum_{j=-h}^{g}(j+h)u_{12}(-j))a_{21} \\ &\quad - (\sum_{j=-h}^{g}(j+h)u_{21}(-j))a_{12}, \\ f'(1) + \varphi'(1) &= \\ &\quad a_{21}(h - \sum_{j=-h}^{g}(j+h)(u_{11}(-j) + u_{12}(-j))) \\ &\quad + a_{12}(h - \sum_{j=-h}^{g}(j+h)(u_{22}(-j) + u_{21}(-j))) \\ &\quad = -a_{21}\sum_{j=-h}^{g} j(u_{11}(-j) + u_{12}(-j)) \\ &\quad - a_{12}\sum_{j=-h}^{g} j(u_{22}(-j) + u_{21}(-j)) < 0. \end{aligned}$$

The stochastic matrix $P = (a_{ij}(1))_{m\times m}$ has m eigenvalues $z_1, \dots, z_m$. The greatest equals one, and all the others are located in the unit circle.

We define $c\gamma(z) := (z^c - 1)/(z - 1)$ and evaluate the determinant in the case of equidistribution to

$$p_m(z) = c^{-m}(1-z)^{-m}\prod_{i=1}^{m}(z_i - c(1-z)z^h),$$

$$c = g + h + 1, g \to \infty, |z| < 1.$$

Thus, the m relevant asymptotic relations as g tends to infinity are given by

$$c(1-z)z^h = z_i, i = 1, \dots, m,$$

z_i denoting the eigenvalues of the stochastic matrix. This fact can be used to generate the initial suggestion. After that, the roots of polynomials with complex interval coefficients are enclosed by an interval iteration. A verification step concludes the process. Figure 1 (taken from Traczinski et al. (2005)) shows the zero distribution for an $SMP(5)$ adaptation in the case $m = 5$, $h = 23$, $g = 64$. We clearly discern five circles with a deformation near the point 1, just as to be expected from our considerations above.

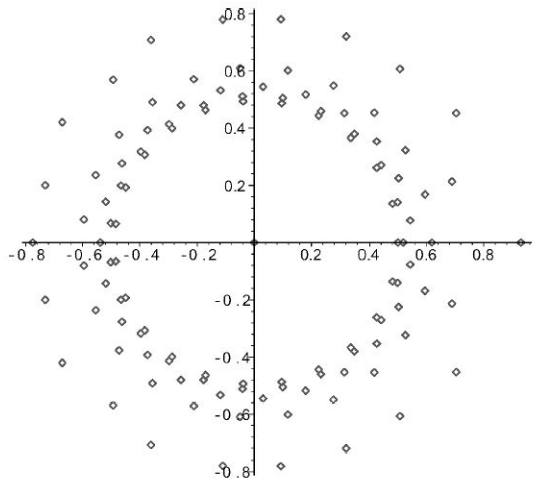

Fig. 1. The 115 zeros of the characteristic polynomial

Example 2. A SMP/GI/1 system modeled as a SSMP(3) (Traczinski, 2005) is given by :

$$\begin{array}{ll} P(A_1=0)=0.01, & P(A_1=50)=0.01, \\ P(A_1=k)=0.02, & k=1,...,49, \\ P(A_2=0)=0.02, & P(A_2=25)=0.02, \\ P(A_2=k)=0.04, & k=1,...,24, \\ P(A_2=k)=0, & k=26,...,50, \\ P(A_3=0)=0.02, & P(A_3=50)=0.02, \\ P(A_3=k)=0, & k=1,...,25, \\ P(A_3=k)=0.04, & k=26,...,49, \end{array}$$

and the transition probabilities

$$P=(p_{ij}):=\begin{pmatrix} 0.2 & 0.6 & 0.2 \\ 0.6 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}.$$

The generating function of the service time distribution is

$$S(z)=0.05z+0.3z^2+0.6z^3+0.05z^8.$$

Thus

$$z^8S(z^{-1})=T(z)=0.05+0.6z^5+0.3z^6+0.05z^7.$$

We put $g=50, h=8$ and the arrival process is modeled as a three step SMP with

$$\begin{aligned} A_1(z)&=0.02\frac{z^{51}-1}{z-1}-0.01\left(1+z^{50}\right) \\ (1-z)A_1&=0.01\left(1+z-z^{50}-z^{51}\right)\approx \\ &\quad 0.01(1+z)=:c_1(z), |z|<1, \\ A_2(z)&=0.04\frac{z^{26}-1}{z-1}-0.02\left(1+z^{25}\right) \\ (1-z)A_2&=0.02(1+z-z^{25}-z^{26})=:c_2(z) \\ A_3(z)&=0.02\left(1+z^{50}\right)+0.04z^{26}\frac{z^{24}-1}{z-1} \\ (1-z)A_3&=0.02\left(1-z+2z^{26}-z^{50}-z^{51}\right)\approx \\ &\quad 0.02\left(1-z+2z^{26}\right)=:c_3(z), |z|<1. \end{aligned}$$

The asymptotic characteristic polynomial $p_\infty(z)$ has a triple zero at $z=1$ and is given by

$$p_\infty(z)=\det\left(p_{ij}(z)\right)$$

with

$$p_{ij}(z)=p_{ij}c_i(z)T(z)-\delta_{ij}z^8(1-z)$$

and

$$\delta_{ij}=\begin{cases}1, & i=j\\ 0, & \text{otherwise}\end{cases}$$

It can be truncated at degree 27. If we drop the three zeros near 1, we get a good suggestion for the 24 zeros inside the unit circle. Even the linear approximations to the polynomials $A_i(z)$ provide initial values of the same quality. Define

$$q(z)=\det\left(q_{ij}\right)$$

with

$$q_{ij}=p_{ij}d_iT(z)-\delta_{ij}z^8(1-z)$$

and

$$d_1=0.01(1+z), d_2=0.02(1+z), d_3=0.02(1-z).$$

Then

$$q(z)=z^{27}-2.9999z^{26}+3.0010998z^{25}+\dots$$

The results from our tool InterVerdiKom (IVK) show that the derived zeros are correct up to 12 decimal places, as you can see in the following example:
$-0.2401372483116056+0.2306542641305137i$ (IVK),
$-0.2401372483115994+0.2306542641304819i$ (q(z))

The Grassmann-Jain algorithm

Alternatively, we can apply the Grassmann-Jain algorithm to solve the SMP/GI/1 matrix case. The settings for the transition probabilities of the Markov process
- $u_{ij}(k)=Pr(U_{n+1}=k,\sigma_{n+1}=j|\sigma_n=i)$; $i,j\in\{1,\dots,m\}; \mathbf{u}(k)=(u_{ij}(k)), -g\le k\le h$
- $U_n=S_n-A_n$ service and interarrival times of the n-th arrival
- $\sigma\in\{1,\dots,m\}$ states of the underlying Markov chain
- $l_{ij}(k)=Pr(I=k,\sigma_E=j|\sigma_A=i)$ the probability for an idle period I of duration k
- σ_A and σ_E initial and final state of a given busy period imply the following relationships among the probabilities $v_{ij}(k)$ and $l_{ij}(k)$

$$\mathbf{v}(k)=\mathbf{u}(k)+\sum_{m=1}^{\min(h-k,g)}\mathbf{v}(k+m)\mathbf{l}(m) \qquad (1)$$
$$k=0,\dots,h.$$

$$\mathbf{l}(k)=\mathbf{u}(-k)+\sum_{m=0}^{\min(g-k,h)}\mathbf{v}(m)\mathbf{l}(m+k) \qquad (2)$$
$$k=1,\dots,g.$$

Here $v_{ij}(k)$ denotes the probability that a phase with level k and initial state j is observed within a busy period having initial state $\sigma_A = i$.
Relation 1 defines a recursion for the probability interval matrices $\mathbf{v}(k)$ with known $\mathbf{u}$ and $\mathbf{l}$:

$$\mathbf{v}(h) = \mathbf{u}(h), \mathbf{v}(h-1) = \mathbf{u}(h-1) + \mathbf{v}(h)\mathbf{l}(1), \ldots,$$

the relation 2 represents an equation system for the probability interval matrices $\mathbf{l}(k), k = 1, \ldots, g$.
In (Fausten et al., 2004a) the software environment is described that computes the Wiener-Hopf factorization numerically using the method developed originally for GI/GI/1 systems by Grassmann and Jain and now adapted to the SMP/GI/1 case. In order to solve equations 1 and 2, the iteration process introduced by Grassmann and Jain takes initial approximations for $\mathbf{l}(k)$. Thus, a sequence of interval matrix distributions $\mathbf{l}^{(n)}(k)$, $\mathbf{v}^{(n)}(k)$ for $n = 0, 1, \ldots$ is obtained. After this, a verification step is performed. If the condition $[\mathbf{v}^{(n+1)}(k)] \subset [\mathbf{v}^{(n)}(k)]$ holds for all $k = 0, \ldots, h$, Brouwer's fixed point theorem guarantees that the correct solution $\mathbf{v}(k)$ is contained in $[\mathbf{v}^{(n+1)}(k)]$ for all $k = 0, \ldots, h$. With this verified result for $\mathbf{v}(k)$ we compute a verified enclosure of $\mathbf{l}(k)$.

THE INTERVERDIKOM-TOOL

The newly developed 'Interval-based numerical software with result verification of the workload distribution for Internet traffic' (InterVerdiKom) provides a unified approach via a graphic user interface to the verified computation of the workload distribution using both presented methods, the polynomial factorization and the Wiener-Hopf method for GI/GI/1 and SMP/GI/1 service systems (Traczinski, 2005).
The user can choose a method to determine initial values for the equation solver (Muller, Bauhuber or external file with results from asymptotic methods) and - in a forthcoming version - directly influence the Grassmann-Jain iteration process. The results of the numerical analysis are visualized and also stored in a text file (See figure 2 for polynomial factorization results and figure 4 for a screenshot of the tool displaying results of the Wiener-Hopf factorization).
Thus the user has easy access to the distribution of the zeros of the characteristic polynomial as well as to the workload distribution, quantiles and mean values. The main advantage of this approach is the guaranteed accuracy of the results even when rigorous proofs are inaccessible, which is important for quality of service evaluation.

Further examples

The developed tool is used to calculate a broad range of cases. Chaudhry (Chaudhry, 1993) gives some examples to illustrate the lack of numerical stability of classical approaches to root determination. For instance, consider example no. 4 (Chaudhry, 1993, page 1042). It is

$$P(A = 15) = 2/5, P(A = 30) = 3/5;$$
$$P(S = 10) = 4/5, P(S = 50) = 1/5.$$

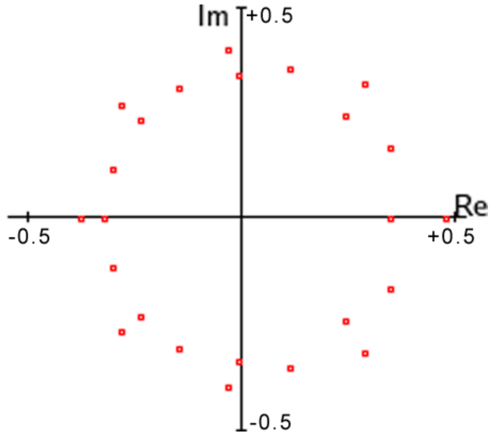

Fig. 2. Zeros by polynomial factorization, SMP/GI/1 example

with $g = 30$ and $h = 50$, all other probabilities are set to zero. As the multiplicity of the root zero of the characteristic polynomial is 15, 35 roots different from zero inside the unit circle are needed.

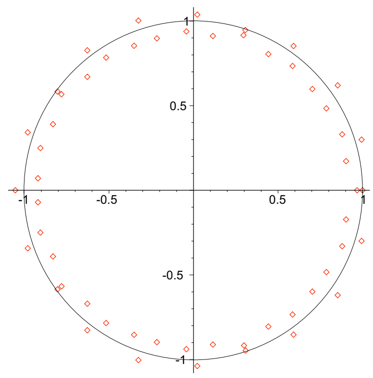

Fig. 3. Determined roots by the Bauhuber algorithm, Chaudhry example

The Bauhuber method implemented in (Engeln-Müllges and Uhlig, 1996) yields 39 roots with an absolute value less than one shown in figure 3. Without verification, it is not possible to find out the correct ones. Using a verifying root algorithm, 35 intervals within the unit circle can be determined. Even in this non equidistributed case the method described in our theorem gives initial values with two or three correct decimal places.
Some results of the polynomial and the Wiener-Hopf factorization are shown in table 1. The original values by Chaudhry are thus corrected in some minor rounding errors. The probabilities $w(k)$, k not divisible by 5, are of special interest, as these must have value zero. Using the Wiener-Hopf method, the point interval $[0, 0]$ is returned. The polynomial factorization yields intervals of diameters less than 10^{-13}, each containing zero.
The calculated results for the previously mentioned example 2 using Wiener-Hopf factorization are shown in table 2, a screenshot of the InterVerdiKom-Tool displaying the workload distribution is shown in figure 4. Further examples with large state spaces (parameters g and h up

w	Polynomial	Wiener-Hopf	Chaudhry
$w(0)$	$4.94385103765_{4937}^{5348} \cdot 10^{-1}$	$4.943851037655_{087}^{260} \cdot 10^{-1}$	$4.944 \cdot 10^{-1}$
$w(5)$	$2.23565255710_{1442}^{9583} \cdot 10^{-2}$	$2.235652557105_{334}^{589} \cdot 10^{-2}$	$2.235 \cdot 10^{-2}$
$w(10)$	$3.49771797201_{3460}^{9679} \cdot 10^{-2}$	$3.497717972016_{389}^{686} \cdot 10^{-2}$	$3.497 \cdot 10^{-2}$
$w(15)$	$5.58455796074_{1512}^{6142} \cdot 10^{-2}$	$5.58455796074_{3643}^{4008} \cdot 10^{-2}$	$5.584 \cdot 10^{-2}$
$w(20)$	$7.79101579562_{5164}^{8584} \cdot 10^{-2}$	$7.79101579562_{6674}^{7110} \cdot 10^{-2}$	$7.791 \cdot 10^{-2}$
$w(90)$	$6.45962880511_{4227}^{6276} \cdot 10^{-3}$	$6.45962880511_{4758}^{5573} \cdot 10^{-3}$	-

Table 1: Verified results, Chaudhry example

w	Polynomial	Wiener-Hopf
$w(0)$	$9.3420008_{04974016}^{10422118} \cdot 10^{-1}$	$9.34200080769_{7686}^{8442} \cdot 10^{-1}$
$w(1)$	$2.4496759_{20020237}^{46977175} \cdot 10^{-2}$	$2.449675933498_{618}^{791} \cdot 10^{-2}$
$w(3)$	$1.16685832_{2022162}^{6455210} \cdot 10^{-2}$	$1.166858324238_{638}^{737} \cdot 10^{-2}$
$w(10)$	$7.8832706_{66859667}^{76088431} \cdot 10^{-5}$	$7.88327067147_{3747}^{4379} \cdot 10^{-5}$
$w(16)$	$1.4058038_{29968625}^{30572595} \cdot 10^{-6}$	$1.405803830270_{552}^{673} \cdot 10^{-6}$
$w(26)$	$6.10657080_{5768996}^{7908923} \cdot 10^{-10}$	$6.10657080683_{8710}^{9245} \cdot 10^{-10}$
$w(50)$	$1.14681943_{6928069}^{7228533} \cdot 10^{-17}$	$1.146819437078_{250}^{364} \cdot 10^{-17}$

Table 2: Verified results, SMP/GI/1 example

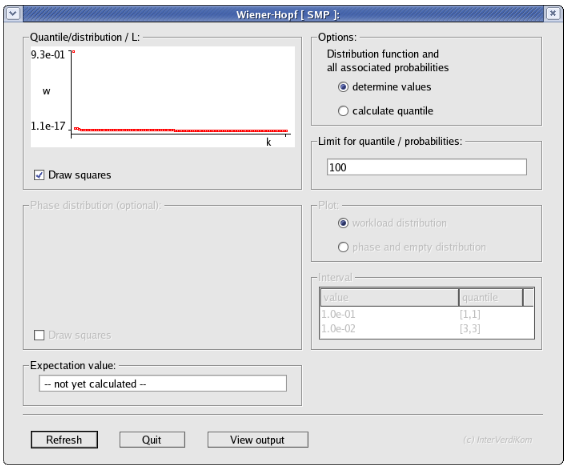

Fig. 4. Screenshot of Wiener-Hopf factorization, SMP/GI/1 example

to 4000) calculated with InterVerdiKom can be found in (Traczinski et al., 2005; Fausten et al., 2004b; Traczinski et al., 2004; Fausten et al., 2004a; Haßlinger, 2003; Haßlinger and Fausten, 2002).

FURTHER WORK

Currently, we work on several enhancements to the InterVerdiKom-Tool:

- Calculation of larger SMPs with more states
- Adaptation of the algorithm to transient distributions
- Computation of the time until the system reaches the equilibrium
- Analysis of arrival distributions (Convolution, superposition, aggregation of SMPs)
- Allowance of forwarding strategies (Reservation, prioritization, fair bandwidth sharing, packet drop policy)

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MULTIPLICATIVE SOLUTION FOR EXPONENTIAL G-NETWORKS WITH DEPENDENT SERVICE AND PREEMPTIVE RESUME OF SERVICE OF KILLED CUSTOMERS

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KEYWORDS
Exponential networks, negative customers, dependent service, product form solution.

ABSTRACT

G-networks with Poisson flow of positive customers, multi-server exponential nodes, and dependent service at the different nodes are studied. Every customer arriving at the network is defined by a set of random parameters: customer route, the length of customer route, customer volume and his service time at each route stage as well. A killed positive customer is removed at the last place in the queue and quits the network just after his remaining service time will be elaborated. Product form solution for multidimensional stationary distribution of the network state is derived.

INTRODUCTION

A central place in new advances in the theory of multiplicative networks belongs to G-networks introduced by Gelenbe (Gelenbe 1989a; Gelenbe 1990a; Gelenbe 1989b; Gelenbe 1990b). G-networks are open queueing networks for which, along with usual (positive) customers, additional flows of negative customers are considered (Gelenbe 1991; Gelenbe et al. 1991). A negative customer, unlike positive customers, upon arrival at network node, without receiving any service, kill one positive customer if there is any at this node (thereby the number of positive customers at the node is reducing by one) and then quits the network.

G-networks with negative customers, trigers, and signals are studied in (Gelenbe and Pujolle 1998). Reviews on G-networks are also given in (Artalejo 2000) and (Bocharov and Vishnevskii 2003).

A non-traditional open queueing networks with negative customers and dependent service have been studied relatively recently in (Bocharov et al. 2004a; Bocharov et al. 2004b; Bocharov et al. 2004c; Bocharov et al. 2004). In such a network every positive customer arriving at the network is defined by a set of random parameters: customer route, route's length, and customer volumes and service times at every stage of the route as well. For G-network with dependent service and with nodes of BCMP-types a product form for the stationary state probabilities of underlying Markov process was obtained in (Bocharov et al. 2004a; Bocharov et al. 2004c). Analogous result was obtained in (Bocharov et al. 2004b) for exponential G-network with dependent service in which customers can change their type. G-network with dependent service and with nodes of BCMP-types, except exponential nodes, in which a customer killed at a node quits the network only after full completion of his service at this node was considered in (Bocharov et al. 2004), where product solution for stationary state distribution was also derived.

In this paper, we extend the results in (Bocharov et al. 2004) for G-networks with dependent service and, unlike (Bocharov et al. 2004), with exponential multi-server nodes in which a killed positive customer is removed at the last place in the queue and quits the network just after his remaining service time will be elaborated. For such G-networks, the multidimensional stationary distribution of the network state probabilities is represented in product form.

NETWORK DESCRIPTION

We consider an open queueing network with M multi-server exponential nodes with common infinite buffers.

We denote by c_s the number of servers in sth node, $s=\overline{1,M}$, and by $\mathcal{M}$ the set of network nodes.

A Poisson flow of (usual, positive) customers of intensity λ enters the network. Each customer arriving at the network is characterized by a set of random variables $(L,\vec{R},\vec{Y},\vec{X})$, which depend neither on analogous random variables for other customers nor on network pre-history, where:

— L is a customer route random length, i.e. the number of stages (nodes) at which he will be served;

— $\vec{R}=(R_1,\ldots,R_L)$ is a random route comprising an assembly of node numbers nodes (the same nodes at different stages are allowed) the customer subsequently passes at all L stages;

— $\vec{Y}=(Y_1,\ldots,Y_L)$ are customer random volumes at route stages the customer subsequently passes (the case when these volumes are different at different stages are considered too);

— $\vec{X}=(X_1,\ldots,X_L)$ are customer random service times at the route stages the customer subsequently passes.

It is only natural that under this network description the volume Y_n and the service time X_n define the service of a customer at node R_n. Let us recall that the routes $\vec{R}$ where the numbers can be repeated are allowed, i.e. a customer can be served at the same node s several times (but probably with different customer lengths and volumes).

Service time X_n at node $R_n=s$ depends on neither service processes of other customers, nor the route $\vec{R}$, nor any of the volumes Y_k (including the volume Y_n), nor any of the times X_k of customer service at other route stages and has exponential probability distribution function (PDF) with parameter μ_s.

Stochastic characteristics of a random variable $(L,\vec{R},\vec{Y})$ are given by the joint PDF

$$G(l,\vec{r},\vec{y})=\mathbf{P}\{L=l,\,R_n=r_n,\,Y_n\le y_n,\quad n=\overline{1,l}\}.$$

We shall make an additional technical assumption on PDF $G(l,\vec{r},\vec{y})$. Namely, we suppose that the PDF $G(l,\vec{r},\vec{y})$ are absolutely continuous, and denote by $g(l,\vec{r},\vec{y})$ its density, i.e.

$$g(l,\vec{r},\vec{y})=\frac{\partial^l}{\partial y_1\cdots\partial y_l}G(l,\vec{r},\vec{y}).$$

This assumption could be easily neglected if we interpret derivatives as generalized ones.

Besides of the flow of positive customers described above, flows of negative customers arrive at the network. These flows are defined in the following way.

A 2.1. The flows arriving at different nodes are independent.

A 2.2. A customer flow arriving at node s is Poisson one of intensity $c_s\gamma_s$.

A 2.3. A negative customer arriving at node s with the same probability $1/c_s$) chooses one of the servers. If at the chosen server a not "killed" customer is being served, the negative customer immediately "kills" him and quits the network. But the killed (here and in what follows we shall omit the quotation marks) positive customer passes at the end of the queue (or if there are no customers in the queue he continues his service) and quits the network just only after his remaining service time will be elaborated. If at the chosen server an already killed positive customer is being served he simply passes at the end of the queue (or continues his service if there is no queue) and, again, quits the network just after his remaining service time will be elaborated. Finally, if at the moment of a negative customer's arrival into some node there is no positive customers there, then the negative customer quits the network without inducing any action.

AUXILIARY FUNCTIONS

Let us set

$$\omega_n(l,\vec{r},\vec{y})=\frac{\mu_{r_n}}{\mu_{r_n}+\gamma_{r_n}}\,,\quad n=\overline{1,l},\tag{1}$$

$$\omega_n^*(l,\vec{r},\vec{y})=\prod_{i=1}^{n-1}\omega_i(l,\vec{r},\vec{y}),\quad n=\overline{1,l+1},\tag{2}$$

$$g_n^*(l,\vec{r},\vec{y})=\omega_n^*(l,\vec{r},\vec{y})\,g(l,\vec{r},\vec{y}),\quad n=\overline{1,l},\tag{3}$$

These functions have a very transparent physical meaning:

— $\omega_n(l,\vec{r},\vec{y})$ is the probability that a positive customer with parameters $(l,\vec{r},\vec{y})$ not killed until the nth stage will not be killed at this stage (at node r_n);

— $\omega_n^*(l,\vec{r},\vec{y})$ is the probability that a positive customer with parameters $(l,\vec{r},\vec{y})$ will not be killed until the nth stage.

Let us set for the sth node

$$\rho_s=\frac{\lambda}{\mu_s}\sum_{l=1}^{\infty}\sum_{1\le r_1,\ldots,r_l\le M}\int\limits_{R^l}\sum_{n=1}^{l}g_n^*(l,\vec{r},\vec{y})\,\delta_{s-r_n}\,\vec{dy},\tag{4}$$

$$\rho_s^+=\frac{\lambda}{\mu_s+\gamma_s}\sum_{l=1}^{\infty}\sum_{1\le r_1,\ldots,r_l\le M}\int\limits_{R^l}\sum_{n=1}^{l}g_n^*(l,\vec{r},\vec{y})\,\delta_{s-r_n}\,\vec{dy},\tag{5}$$

$$\rho_s^-=\frac{\lambda\gamma_s}{\mu_s(\mu_s+\gamma_s)}\lambda\sum_{l=1}^{\infty}\sum_{1\le r_1,\ldots,r_l\le M}\times\int\limits_{R^l}\sum_{n=1}^{l}g_n^*(l,\vec{r},\vec{y})\,\delta_{s-r_n}\,\vec{dy}=\rho_s^+-\rho_s^-,\tag{6}$$

where δ_j is the Kronecker symbol. For the sake of brevity we shall use the notations

$$\int\limits_{R^l}\ldots\,\vec{dy}=\int\limits_{R^l}\!\!\ldots\!\!\int\ldots\,dy_1\cdots dy_l.$$

MARKOV PROCESS

Let us define the Markov process that describes the operation of our queueing network.

We shall denote a network state by an assembly $\vec{z} = (\vec{z}_1, \ldots, \vec{z}_M)$, where the assembly $\vec{z}_s = (k_s, \vec{z}_{s1}, \ldots, \vec{z}_{sk_s})$, $s = \overline{1, M}$, in turn, describes the state of the sth node in the following way: k_s is the number of customers at the sth node and the assembly $\vec{z}_{si}$, $s = \overline{1, M}$, $i = \overline{1, k_s}$, with components $\vec{z}_{si} = (l_{si}, \vec{r}_{si}, \vec{y}_{si}, n_{si}, w_{si})$, stores the information $(l_{si}, \vec{r}_{si}, \vec{y}_{si})$ on the ith customer at the sth node, and his position (n_{si}, x_{si}, w_{si}) in the network:

— l_{si} is the route length;

— $\vec{r}_{si} = (r_{si1}, \ldots, r_{sil_{si}})$ is the route;

— $\vec{y}_{si} = (y_{si1}, \ldots, y_{sil_{si}})$ are customer volumes at his route stages;

— n_{si} is the number of the route's stage which the customer exists (while being served or waiting for service); clearly, $n_{si} \le l_{si}$;

— w_{si} is the function which shows customer state; we set $w_{si} = 0$ if the customer is not killed, and $w_{si} = 1$ if the customer is killed (but is being served).

Evidently that due to the notations introduced above, we have $r_{sin_{si}} = s$. It is also clear that the vector $\vec{z}_s = 0$ if $k_s = 0$, i.e. when there are no customers at the sth node, and the vector $\vec{z} = \mathbf{0} = (0, \ldots, 0)$ in the case, when there are no customers in the network at all.

In that follows, customers at nodes are numbered in order to their arrival at the node. Thereby the last number is assigned to the positive customer (even if there are no customers in the queue and this customer is still being served).

The set of states of the network is denoted by $\mathcal{Z} = \{\vec{z}\}$.

To describe the operation of our queueing network, let us consider the process

$\vec{Z}(t) = \vec{z}$, if the network exists in state $\vec{z}$ at instant t.

It is obviously a Markov process.

In the sequel, we also consider the (non-Markovian) processes

$$\vec{Z}_s(t) = \vec{z}_s, \quad \text{if the network exists in state } \vec{z} \text{ at instant } t, \quad s = \overline{1, M},$$

describing the operation of individual network nodes.

STATIONARY DISTRIBUTION OF THE MARKOV PROCESS: PRODUCT FORM

The stationary density of the state probability distribution of the process $\vec{Z}(t)$ is denoted by $p(\vec{z})$.

By direct construction, we prove the existence of this density under an obvious constraint on the network load.

Theorem. *If for all nodes the condition $\rho_s < c_s$ is fulfilled, then there exists a limiting (stationary) distribution of probabilities of states of the process $\vec{Z}(t)$ with probability distribution density*

$$p(\vec{z}) = \prod_{s=1}^{M} p_s(\vec{z}_s), \tag{7}$$

thereby

$$p_s(\vec{z}_s) = p_s(0)\, d_s(k_s) \prod_{i=1}^{k_s} \Big(\frac{\lambda(\delta_{w_{si}} \mu_s + \delta_{1-w_{si}} \gamma_s)}{\mu_s(\mu_s + \gamma_s)} \Big) \times g^*_{n_{si}}(l_{si}, \vec{r}_{si}, \vec{y}_{si}), \tag{8}$$

where

$$p_s(0) = \Big(\sum_{i=0}^{c_s} \frac{\rho_s^i}{i!} + \frac{\rho_s^{c_s+1}}{c_s!\,(c_s - \rho_s)} \Big)^{-1}, \tag{9}$$

$$d_s(k_s) = \begin{cases} 1/k_s! & \text{for } k_s \le c_s, \\ 1/(c_s!\, c_s^{k_s - c_s}) & \text{for } k_s > c_s. \end{cases} \tag{10}$$

Proof. It is easily seen that the Markov process $\vec{Z}(t)$ is regular and positive Harris recurrent, because the state $\mathbf{0}$ is a positive recurrent atom for it. Therefore, to prove the theorem it is sufficient to show that the function $p(\vec{z})$, defined by theorem assentations satisfies the system of equations for the stationary probability density of states of the process $\vec{Z}(t)$. In analogy with the discrete case, we refer to this system of equations as the system of equilibrium equations.

Let us associate every state $\vec{z} \in \mathcal{Z}$ with a state set $\widetilde{\mathcal{Z}}(\vec{z})$. The states belonging to the state $\widetilde{\mathcal{Z}}(\vec{z})$ are called the predecessors of the state $\vec{z}$. Predecessor states are introduced for the reason that a direct transition to the state $\vec{z}$ is only possible from these states.

In turn, let us subdivide all predecessors of a state $\vec{z}$ into 4 disjoint classes. To every class, there corresponds a definite type of transition to the state $\vec{z}$.

The first class $\mathcal{Z}^+(\vec{z}) = \mathcal{Z}_1^+(\vec{z}) \cup \mathcal{Z}_2^+(\vec{z})$ consists of two (intersecting) subclasses $\mathcal{Z}_1^+(\vec{z})$ and $\mathcal{Z}_2^+(\vec{z})$.

The first subclass $\mathcal{Z}_1^+(\vec{z}) = \{\vec{z}^+(\vec{z}, i, l, \vec{r}, \vec{y})\}$, is a set of network states from which a transition to the state $\vec{z}$ is possible owing to the exit of a customer from the network upon completion of his service (at the last node in the route) and contains only the states $\vec{z}^+(\vec{z}, i, l, \vec{r}, \vec{y})$ of the form

$$\vec{z}^+(\vec{z}, i, l, \vec{r}, \vec{y}) = (\vec{z}_1, \ldots, \vec{z}_{s-1}, \vec{z}^*_s, \vec{z}_{s+1}, \ldots, \vec{z}_M),$$

where

$$s = r_l, \quad \vec{z}^*_s = (k_s + 1, \vec{z}^*_{s1}, \ldots, \vec{z}^*_{s,k_s+1}),$$

$$i = \overline{1, \min\{k_s + 1, c_s\}}$$

and

$$\vec{z}^*_{sj} = \begin{cases} \vec{z}_{sj}, & j < i, \\ (l, \vec{r}, \vec{y}, l, 0), & j = i, \\ \vec{z}_{s,j-1}, & j > i, \end{cases}$$

and $(l, \vec{r}, \vec{y})$ are the parameters of the customer that leaves the ith place at the sth node upon full completion of his service. These parameters may take any (possible) value.

The second subclass $\mathcal{Z}_2^+(\vec{z}) = \{\vec{z}^+(\vec{z}, i, l, \vec{r}, \vec{y}, n)\}$ is a set of network states from which a transition to the state $\vec{z}$ is possible owing to the service completion and the exit of a killed customer from the network from ith place at the sth node at the nth route's stage and contains only the states $\vec{z}^+(\vec{z}, i, l, \vec{r}, \vec{y}, n)$ of the form

$$\vec{z}^+(\vec{z}, i, l, \vec{r}, \vec{y}, n) = (\vec{z}_1, \ldots, \vec{z}_{s-1}, \vec{z}^*_s, \vec{z}_{s+1}, \ldots, \vec{z}_M),$$

where

$$n = \overline{1, l}, \quad s = r_n, \quad \vec{z}^*_s = (k_s + 1, \vec{z}^*_{s1}, \ldots, \vec{z}^*_{s,k_s+1}),$$

$$i = \overline{1, \min\{k_s + 1, c_s\}}$$

and

$$\vec{z}^*_{sj} = \begin{cases} \vec{z}_{sj}, & j < i, \\ (l, \vec{r}, \vec{y}, n, 1), & j = i, \\ \vec{z}_{s,j-1}, & j > i. \end{cases}$$

The parameters $(l, \vec{r}, \vec{y})$ may also take any (possible) value.

Now let us consider for each $s \in \mathcal{M}$ for which $k_s > 0$ the coordinate $\vec{z}_s$ of the vector $\vec{z}$.

Let $n_{sk_s} = 1$ and $w_{sk_s} = 0$. Then we say that the node s belongs to the node set $\mathcal{S}^-(\vec{z})$. Let $\vec{z}^-(\vec{z}, s)$ denote the state

$$\vec{z}^-(\vec{z}, s) = (\vec{z}_1, \ldots, \vec{z}_{s-1}, \vec{z}^*_s, \vec{z}_{s+1}, \ldots, \vec{z}_M),$$

where

$$\vec{z}^*_s = (k_s - 1, \vec{z}_{s1}, \ldots, \vec{z}_{s,k_s-1}).$$

The states $\vec{z}^-(\vec{z}, s)$ are called predecessor states of the class $\mathcal{Z}^-(\vec{z})$. Obviously, the class $\mathcal{Z}^-(\vec{z})$ consists of states from which a transition to the state $\vec{z}$ is possible due to the arrival of a new customer, and the set $\mathcal{S}^-(\vec{z})$ consists of nodes in which the not killed customer that had arrived last at the network exists at the last place under the state $\vec{z}$.

If $n_{sk_s} > 1$ and $w_{sk_s} = 0$, then we say that the node s belongs to state set $\mathcal{S}_1(\vec{z})$. The class $\mathcal{Z}_1(\vec{z})$ defines the predecessor states from which a transition to the state $\vec{z}$ takes place when a (not killed) customer upon completion of service in a node jockeys to another node (at the last place). This class consists of disjoint subclasses $\mathcal{Z}_1(\vec{z}, s)$, which are defined as follows. Let $\mathcal{Z}_1(\vec{z}, s) = \{\vec{z}(\vec{z}, s, j)\}$, and every state $\vec{z}(\vec{z}, s, j)$ is of the form

$$\vec{z}(\vec{z}, s, j) = (\vec{z}_1, \ldots, \vec{z}_{s'-1}, \vec{z}^*_{s'}, \vec{z}_{s'+1}, \ldots, \vec{z}_{s-1}, \vec{z}^*_s, \vec{z}_{s+1}, \ldots, \vec{z}_M)$$

(obviously, s can precede s' and even coincide with s', if a customer once again arrives at the sth node from the sth node), where

$$s' = r_{s,k_s,n_{sk_s}-1}, \quad \vec{z}^*_{s'} = (k_{s'} + 1, \vec{z}^*_{s'1}, \ldots, \vec{z}^*_{s',k_{s'}+1}),$$

$$j = \overline{1, \min\{k_{s'} + 1, c_{s'}\}},$$

$$\vec{z}^*_{s'i} = \begin{cases} \vec{z}_{s'i}, & i < j, \\ (l_{sk_s}, \vec{r}_{sk_s}, \vec{y}_{sk_s}, n_{sk_s} - 1, 0), & i = j, \\ \vec{z}_{s',i-1}, & i > j, \end{cases}$$

$$\vec{z}^*_s = (k_s - 1, \vec{z}_{s1}, \ldots, \vec{z}_{s,k_s-1}).$$

Finally, if $w_{sk_s} = 1$ then the node s belongs to state set $\mathcal{S}_2(\vec{z})$. The class $\mathcal{Z}_2(\vec{z})$ consists of disjoint subclasses $\mathcal{Z}_2(\vec{z}, s)$, which are defined as follows. Let $\mathcal{Z}_2(\vec{z}, s) = \{\vec{z}(\vec{z}, s, j, w)\}$, and every state $\vec{z}(\vec{z}, s, j, w)$ is of the form

$$\vec{z}(\vec{z}, s, j, w) = (\vec{z}_1, \ldots, \vec{z}_{s-1}, \vec{z}^*_s, \vec{z}_{s+1}, \ldots, \vec{z}_M),$$

where

$$\vec{z}^*_s = (k_s, \vec{z}^*_{s1}, \ldots, \vec{z}^*_{s,k_s}), \;\; j = \overline{1, \min\{k_s, c_s\}}, \;\; w = 0, 1,$$

$$\vec{z}^*_{si} = \begin{cases} \vec{z}_{si}, & i < j, \\ (l_{sk_s}, \vec{r}_{sk_s}, \vec{y}_{sk_s}, n_{sk_s}, w), & i = j, \\ \vec{z}_{s,i-1}, & i > j. \end{cases}$$

The state $\vec{z}(\vec{z}, s, j)$ is a predecessor state from which a transition to the state $\vec{z}$ is possible due to the arrival of a new negative customer at the jth server of the node s and the removal of a customer existing there at last place in the node (and his killing if he had not been killing yet).

Let us once again note that each of the classes $\mathcal{Z}^-(\vec{z})$, $\mathcal{Z}_1(\vec{z})$ and $\mathcal{Z}_2(\vec{z})$ can be empty. In particular, all classes $\mathcal{Z}^-(\vec{z})$ $\mathcal{Z}_1(\vec{z})$ and $\mathcal{Z}_2(\vec{z})$ are empty for the state $\mathbf{0} = (0, \ldots, 0)$.

Let us write the global balance equations for a state $\vec{z}$. For this, let us compute the probability flows arriving at and departing from the state $\vec{z}$.

The departing flow from the state $\vec{z}$ consists of two parts.

First, departure from the state $\vec{z}$ takes place upon arrival of a new customer at the network (with intensity λ). The probability flow departing from the state $\vec{z}$ due to the arrival of a new customer is $\lambda p(\vec{z})$.

Second, departure from the state $\vec{z}$ also takes place upon completion of service of a customer at the ith server at the sth node (with intensity μ_s which not depends on this customer is killed or not) or arrival of a negative customer at this server at this node and removal of the customer existing there at the last place at the node, and his killing if he had not been killed yet (with intensity γ_s). The total probability flow departing from the state $\vec{z}$ due to completion of service of customers or arrivals of negative customers is

$$\sum_{s=1}^{M}\sum_{i=1}^{\min\{k_s,c_s\}}(\mu_s+\gamma_s)\,p(\vec{z})$$

(let us note that in the sum over s nodes for which $k_s=0$ are not included).

Arrival at a state $\vec{z}$ may take place from the state $\vec{z}^{+}(\vec{z},i,l,\vec{r},\vec{y})\in\mathcal{Z}_1^{+}(\vec{z})$ upon completion of service of the not killed customer at the ith server at the r_lth node (at the last route's node). Since the intensity of completion service at the r_lth node is μ_{r_l}, the total probability flow of arrivals at the state $\vec{z}$ from the states of the subclass $\mathcal{Z}_1^{+}(\vec{z})$ is

$$\sum_{s=1}^{M}\sum_{i=1}^{\min\{k_s+1,c_s\}}\sum_{l,\vec{r}}\int\limits_{R^l}\mu_s\delta_{s-r_l}p(\vec{z}^{+}(\vec{z},i,l,\vec{r},\vec{y}))\,\vec{dy}.$$

Moreover, the state $\vec{z}$ can be reached from the state $\vec{z}^{+}(\vec{z},i,l,\vec{r},\vec{y},n)\in\mathcal{Z}_2^{+}(\vec{z})$ upon completion of service of the customer killed at the ith server of the r_nth node at the nth route's stage (with the intensity of completion of service in the r_nth node equal to μ_{r_n}). Thus, the total probability flow of arrivals to the state $\vec{z}$ from the states of the subclass $\mathcal{Z}_2^{+}(\vec{z})$ is

$$\sum_{s=1}^{M}\sum_{i=1}^{\min\{k_s+1,c_s\}}\sum_{l,\vec{r},n}\int\limits_{R^l}\mu_s\delta_{s-r_n}p(\vec{z}^{+}(\vec{z},i,l,\vec{r},\vec{y},n))\,\vec{dy}.$$

Further, the state $\vec{z}$ can be reached from the state $\vec{z}^{-}(\vec{z},s)\in\mathcal{Z}^{-}(\vec{z})$ if a new customer arrives at the network at the last place of the node s (with intensity $\lambda g(l_{sk_s},\vec{r}_{sk_s},\vec{y}_{sk_s})$). The total probability flow of arrivals to the state $\vec{z}$ due to such transitions is

$$\sum_{s\in\mathcal{S}^{-}(\vec{z})}\lambda g(l_{sk_s},\vec{r}_{sk_s},\vec{y}_{sk_s})p(\vec{z}(\vec{z},s)).$$

The state $\vec{z}$ can be also reached from $\vec{z}(\vec{z},s,i)\in\mathcal{Z}_1(\vec{z})$ upon completion of service of the not killed customer at ith server of the node $s'=r_{s,k_s,n_{sk_s}-1}$ (with intensity $\mu_{s'}$) and his transition at the sth node at last place. The total probability flow of arrivals to the state $\vec{z}$ from the states of the subclass $\mathcal{Z}_1(\vec{z})$ is

$$\sum_{s\in\mathcal{S}_1(\vec{z})}\sum_{i=1}^{\min\{k_{s'}+1,c_{s'}\}}\mu_{s'}p(\vec{z}(\vec{z},s,i)).$$

Finally, the state $\vec{z}$ can be also reached from the state $\vec{z}(\vec{z},s,i,w)\in\mathcal{Z}_2(\vec{z})$ upon arrival of a negative customer and transition (killed or not killed yet) of the customer at ith server of the node s at the last place (with intensity γ_s). Then the total probability flow of arrivals to the state $\vec{z}$ from the states of the subclass $\mathcal{Z}_2(\vec{z})$ is

$$\sum_{s\in\mathcal{S}_2(\vec{z})}\sum_{i=1}^{\min\{k_s,c_s\}}\sum_{w=0}^{1}\gamma_s p(\vec{z}(\vec{z},s,i,w)).$$

Now we can write the system of equilibrium equations for the state $\vec{z}$.

$$\lambda p(\vec{z})+\sum_{s=1}^{M}\sum_{i=1}^{\min\{k_s,c_s\}}(\mu_s+\gamma_s)\,p(\vec{z})$$
$$=\sum_{s=1}^{M}\sum_{i=1}^{\min\{k_s+1,c_s\}}\sum_{l,\vec{r}}\int\limits_{R^l}\mu_s\delta_{s-r_l}p(\vec{z}^{+}(\vec{z},i,l,\vec{r},\vec{y}))\,\vec{dy}$$
$$+\sum_{s=1}^{M}\sum_{i=1}^{\min\{k_s+1,c_s\}}\sum_{l,\vec{r},n}\int\limits_{R^l}\mu_s\delta_{s-r_n}p(\vec{z}^{+}(\vec{z},i,l,\vec{r},\vec{y},n))\,\vec{dy}$$
$$+\sum_{s\in\mathcal{S}^{-}(\vec{z})}\lambda g(l_{sk_s},\vec{r}_{sk_s},\vec{y}_{sk_s})p(\vec{z}(\vec{z},s))$$
$$+\sum_{s\in\mathcal{S}_1(\vec{z})}\sum_{i=1}^{\min\{k_{s'}+1,c_{s'}\}}\mu_{s'}p(\vec{z}(\vec{z},s,i))$$
$$+\sum_{s\in\mathcal{S}_2(\vec{z})}\sum_{i=1}^{\min\{k_s,c_s\}}\sum_{w=0}^{1}\gamma_s p(\vec{z}(\vec{z},s,i,w)). \qquad (11)$$

Let us verify whether function $p(\vec{z})$ defined by the formulas (7)–(10) satisfies the equations (11). For this purpose, we rewrite the equations (11) as

$$\Bigg[\lambda p(\vec{z})$$
$$-\sum_{s=1}^{M}\sum_{i=1}^{\min\{k_s+1,c_s\}}\sum_{l,\vec{r}}\int\limits_{R^l}\mu_s\delta_{s-r_l}p(\vec{z}^{+}(\vec{z},i,l,\vec{r},\vec{y}))\,\vec{dy}$$
$$-\sum_{s=1}^{M}\sum_{i=1}^{\min\{k_s+1,c_s\}}\sum_{l,\vec{r},n}\int\limits_{R^l}\mu_s\delta_{s-r_n}p(\vec{z}^{+}(\vec{z},i,l,\vec{r},\vec{y},n))\,\vec{dy}\Bigg]$$
$$+\sum_{s\in\mathcal{S}^{-}(\vec{z})}\Bigg[\sum_{i=1}^{\min\{k_s,c_s\}}(\mu_s+\gamma_s)\,p(\vec{z})$$
$$-\lambda g(l_{sk_s},\vec{r}_{sk_s},\vec{y}_{sk_s})p(\vec{z}(\vec{z},s))\Bigg]$$
$$+\sum_{s\in\mathcal{S}_1(\vec{z})}\Bigg[\sum_{i=1}^{\min\{k_s,c_s\}}(\mu_s+\gamma_s)\,p(\vec{z})$$
$$-\sum_{i=1}^{\min\{k_{s'}+1,c_{s'}\}}\mu_{s'}p(\vec{z}(\vec{z},s,i))\Bigg]$$
$$+\sum_{s\in\mathcal{S}_2(\vec{z})}\Bigg[\sum_{i=1}^{\min\{k_s,c_s\}}(\mu_s+\gamma_s)\,p(\vec{z})$$
$$-\sum_{i=1}^{\min\{k_s,c_s\}}\sum_{w=0}^{1}\gamma_s p(\vec{z}(\vec{z},s,i,w))\Bigg]=0. \qquad (12)$$

Lemma formulated below defines relations for the function $p(\vec{z})$ (7)–(10) and shows that each of the expressions in the formula (12) within square brackets is zero. Thus, the function $p(\vec{z})$ really satisfies equilibrium equations (11).

Lemma. *Function $p(\vec{z})$ defined by the formulas (7)–(10) satisfies for all states $\vec{z}$ the relations*
(1) the following identity holds:

$$\lambda p(\vec{z})$$

$$=\sum_{s=1}^{M}\sum_{i=1}^{\min\{k_s+1,c_s\}}\sum_{l,\vec{r}}\int_{R^l}\mu_s\delta_{s-r_l}p(\vec{z}^{+}(\vec{z},i,l,\vec{r},\vec{y}))\,d\vec{y}$$

$$+\sum_{s=1}^{M}\sum_{i=1}^{\min\{k_s+1,c_s\}}\sum_{l,\vec{r},n}\int_{R^l}\mu_s\delta_{s-r_n}p(\vec{z}^{+}(\vec{z},i,l,\vec{r},\vec{y},n))\,d\vec{y}; \tag{13}$$

(2) for any node $s\in\mathcal{S}^{-}(\vec{z})$

$$\sum_{i=1}^{\min\{k_s,c_s\}}(\mu_s+\gamma_s)\,p(\vec{z})=\lambda g(l_{sk_s},\vec{r}_{sk_s},\vec{y}_{sk_s})p(\vec{z}(\vec{z},s)); \tag{14}$$

(3) for any node $s\in\mathcal{S}_1(\vec{z})$

$$\sum_{i=1}^{\min\{k_s,c_s\}}(\mu_s+\gamma_s)\,p(\vec{z})=\sum_{i=1}^{\min\{k_{s'}+1,c_{s'}\}}\mu_{s'}p(\vec{z}(\vec{z},s,i)); \tag{15}$$

(4) for any node $s\in\mathcal{S}_2(\vec{z})$

$$\sum_{i=1}^{\min\{k_s,c_s\}}(\mu_s+\gamma_s)\,p(\vec{z})=\sum_{i=1}^{\min\{k_s,c_s\}}\sum_{w=0}^{1}\gamma_s p(\vec{z}(\vec{z},s,i,w)). \tag{16}$$

Proof of Lemma. It can be shown that a direct substitution of formulas (7)–(10) into the relations (13)–(16) leads to identities.

What remains to conclude the proof of the theorem is to verify that the probability distribution density $p(\vec{z})$ satisfies the normalizing condition. It is a simple matter to verify this condition. This completes the proof of the theorem.

Some important corollaries follow from this theorem. The proofs of these corollaries are omitted as they are easy to prove.

Corollary 1. *The marginal stationary distribution density of the process $\vec{Z}_s(t)$ is defined by formulas (8)–(10). The stationary state probability distribution of the process $\vec{Z}(t)$ has a product form, i.e. is representable as the product of stationary probability distributions of the process $\vec{Z}_s(t)$.*

Corollary 2. *The stationary distribution of the number of customers (without regard for their parameters) at a node s is of the form*

$$p_s(k)=p_s(0)\,d_s(k)\rho_s^k,$$

where ρ_s, $p_s(0)$, and $d_s(k)$ are given by the formulas (4), (9), and (10).

Corollary 3. *The stationary probabilities that a customer with parameters $(l,\vec{r},\vec{y})$ is not killed up to the nth stage and killed at the nth stage are $\omega_n^*(l,\vec{r},\vec{y})$ and $\omega_n(l,\vec{r},\vec{y})$, respectively, where $\omega_n^*(l,\vec{r},\vec{y})$ and $\omega_n(l,\vec{r},\vec{y})$ are given by formulas (2) and (1).*

Corollary 4. *The stationary intensity λ_s of the input flow at node s is defined by the following formula:*

$$\lambda_s=\lambda\sum_{l=1}^{\infty}\sum_{1\le r_1,\dots,r_l\le M}\int_{R^l}\sum_{n=1}^{l}g_n^*(l,\vec{r},\vec{y})\,\delta_{s-r_n}d\vec{y}.$$

From the theorem and corollaries given above follows that ρ_s is the traffic intensity at the sth node, ρ_s^+ is the traffic intensity of not killed customers at the sth node, ρ_s^- is the traffic intensity of killed but still being served customers at the sth node.

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EFFICIENT IMPLEMENTATION OF ALGORITHM FOR CALCULATING THE STATIONARY DISTRIBUTION FOR $M|PH|c$ SYSTEM WITH ADDRESSED STRATEGY OF RETRIALS

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ABSTRACT

We consider a multiserver retrial model in which arrivals occur according to a Poisson arrival process. The service time has phase-type distribution. The intensity of retrials linearly depends on the number of the customers in the orbit. The continuous-time multi-dimensional Markov chain describing the behavior of the system belongs to the class of multi-dimensional asymptotically quasi-Toeplitz Markov chains. Sufficient condition for stationary state distribution is proven.

1 Introduction

This paper presents a multiserver retrial queueing system with servers kept apart, thereby rendering it impossible for one to know the status (idle/busy) of the others. Customers proceeding to one channel will have to go to orbit if the server is busy and retry after some time to some channel, not necessarily the one already tried. Each orbital customer, independently of others, chooses the server randomly according to some specified probability distribution. Further this distribution is identical for all customers. We assume that the same 'orbit' is used by all retrial customers, between repeated attempts, to access the servers.

The simplest Markovian queueing model with such a strategy of access was investigated in recent work (Mushko et al., 2006). In (Mushko, 2005) this results were extended to the model with more general PH service process. In latter work, the behavior of the $M|PH|c$ system is described by the continuous-time multi-dimensional Markov chain $\zeta_t = \{i_t, m_t^{(1)}, \dots, m_t^{(c)}\}$, $t \geq 0$, where

- i_t, $i_t \geq 0$, be the number of customers presenting in the orbit at the epoch t;
- $m_t^{(r)}$ be the state of r-th server at the epoch t, $r = \overline{1, c}$:

$$m_t^{(r)} = \begin{cases} 0, & \text{if the } r\text{-th server is idle,} \\ m, & \text{if the } r\text{-th server operate the} \\ & \text{customer on phase } m,\ m = \overline{1, M}. \end{cases}$$

The alternative is to utilize an approach proposed in (Ramaswami and Lucantoni, 1985) and (Ramaswami, 1985). According to this approach, the numbers $h_t^{(m)}$ of servers being in phase m, $m = \overline{1, M}$, at the epoch t instead of the state of each server are considered.

In this paper we use the Ramaswami's approach to construct the continuous-time multi-dimensional Markov chain defining the behavior of the $M|PH|c$ system with addressed strategy of retrials.

2 Mathematical model

We consider a multiserver model having c identical servers. Service times distribution is of PH type. It means the following. The service process is directed by the continuous time Markov process m_t, $t \geq 0$. The state of this process at the service beginning epoch is defined according to the probabilistic row-vector $\boldsymbol{\beta} = \begin{pmatrix} \beta_1 & \cdots & \beta_M \end{pmatrix}$. Further, transitions of the process m_t, $t \geq 0$, are defined by the matrix S of dimension $M \times M$. The diagonal entries of the matrix are negative and $-S_{m,m}$ defines the parameter of the exponentially distributed sojourn time of process in the state m, $|S_{m,m}| < \infty$, $m = \overline{1, M}$.

The non-diagonal entries of the matrix S define the intensities of transitions of the process $m_t, t \geq 0$, in the state space $\{1, \ldots, M\}$. The value $-\sum\limits_{m'=1}^{M} S_{m,m'}$ defines the intensity of the transition of the process m_t, $t \geq 0$, from the state m into the absorbing state. The epoch of the transition of the process $m_t, t \geq 0$, into the absorbing state defines the service completion epoch. Denote $\mathbf{S}_0 = -S\mathbf{e}$. Here $\mathbf{e}$ is a column-vector of appropriate size consisting of units. It is assumed that all the entries of the column-vector $\mathbf{S}_0$ are non-negative and at least one of them is positive. The notion of the PH is given in (Neuts, 1981).

Arrival process to the system is a Poisson process with intensity λ, $\lambda > 0$.

At the epoch of arrival, the customer selects the r-th server for the service with probability $\frac{1}{c}, r = \overline{1, c}$. If the selected server is idle at the arrival epoch, the customer occupies the server and after the service it leaves the system forever. If the selected server is busy at the given arrival epoch, the customer moves to some virtual pool called the orbit and tries to get the service later on. Each customer staying in the orbit makes the repeated attempts in random intervals having length exponentially distributed with parameter α, $\alpha > 0$, independently of the other customers. At a given epoch of retrial, the customer selects the r-th server with probability $\frac{1}{c}, r = \overline{1, c}$. If that server is idle at the retrial epoch, the customer occupies the server and leaves the system after the service. If the server is busy at the retrial epoch, the customer returns to the orbit, even if some other servers are idle at that epoch. Every customer tries to get the service until it succeeds in occupying the server selected just before the current attempt is made.

3 Markovian process of the system states

Let

- i_t, $i_t \geq 0$, be the number of customers presenting in the orbit at the epoch t;
- n_t, $n_t = \overline{0, c}$, be the number of busy servers at epoch t;
- $h_t^{(m)}$, $h_t^{(m)} = \overline{0, c}$, $m = \overline{1, M}$, $\sum\limits_{m=1}^{M} h_t^{(m)} = n_t$, be the number of servers in phase m at the epoch t.

It is clear that the process $\zeta_t = \{i_t, n_t, h_t^{(1)}, \ldots, h_t^{(M)}\}$, $t \geq 0$, is the Markov chain.

Denote the stationary state probabilities of this Markov chain by:

$$p(i, n, h_1, \ldots, h_M) =$$

$$\lim_{t\to\infty} P\{i_t = i, n_t = n, h_t^{(1)} = h_1, \ldots, h_t^{(M)} = h_M\},$$

$$i \geq 0,\ n = \overline{0, c},\ h_m = \overline{0, c},\ m = \overline{1, M},\ \sum_{m=1}^{M} h_m = n.$$

Condition of these limits existence will be given below.

Let

$$N = \sum_{q=0}^{c} H(q),\ H(q) = \begin{pmatrix} q + M - 1 \\ M - 1 \end{pmatrix},\ q = \overline{0, c},$$

$$\mathbf{S}_0 = \begin{pmatrix} S_1^0 \\ S_2^0 \\ \vdots \\ S_M^0 \end{pmatrix},\ \mathbf{S}_d = \begin{pmatrix} S_{1,1} \\ S_{2,2} \\ \vdots \\ S_{M,M} \end{pmatrix}.$$

Enumerate the states of the process in direct lexicographic order of components i, n and then in reverse lexicographic order of components $h_1, \ldots, h_M$. Then, for instance

$$(f, g, 0, 0, \ldots, 0, 1) \prec \cdots \prec (f, g, 0, 0, \ldots, 0, s) \prec$$

$$(f, g{-}1, 1, 0, \ldots, 0, 1) \prec \cdots \prec (f, g{-}1, 1, 0, \ldots, 0, s) \prec$$

$$(f, g - 1, 0, 1, \ldots, 0, 1) \prec \cdots \prec$$

$$(f, g - 1, 0, 1, \ldots, 0, s) \prec \cdots \prec$$

$$(f, g{-}1, 0, 0, \ldots, 1, 1) \prec \cdots \prec (f, g{-}1, 0, 0, \ldots, 1, s) \prec$$

$$(f, 0, 0, 0, \ldots, g, 1) \prec \cdots \prec (f, 0, 0, 0, \ldots, g, s),$$

where $'\prec'$ is the precedence symbol according to the introduced order.

Form row vectors $\mathbf{p}_i$ of probabilities corresponding to the state i, $i \geq 0$, of the number of customers in the orbit:

$$\mathbf{p}_i = (p(i, 0, \mathbf{h}_1), p(i, 1, \mathbf{h}_2), p(i, 1, \mathbf{h}_3), \ldots,$$

$$p(i, 1, \mathbf{h}_{1+H(1)}), p(i, 2, \mathbf{h}_{2+H(1)}), p(i, 2, \mathbf{h}_{3+H(1)}), \ldots,$$

$$p(i, 2, \mathbf{h}_{1+H(1)+H(2)}), \ldots, p(i, c, \mathbf{h}_N)),$$

where

$$\mathbf{h}_v = \begin{pmatrix} h_1 & h_2 & \cdots & h_M \end{pmatrix},\ v = \overline{1, N},$$

$$h_m = \overline{0, c},\ m = \overline{1, M},\ \sum_{m=1}^{M} h_m = n.$$

Also form macro-vector

$$\mathbf{p} = \begin{pmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \cdots & \mathbf{p}_i & \cdots \end{pmatrix}.$$

Denote by Q, the infinitesimal generator of the Markov chain ζ_t, $t \geq 0$. Vector $\mathbf{p}$ satisfies the equilibrium equations:

$$\mathbf{p}Q = \mathbf{0}, \mathbf{pe} = 1, \qquad (1)$$

where $\mathbf{0}$ is a row vector consisting of zeroes. **Lemma.** The generator Q of the Markov chain ζ_t, $t \geq 0$, has the following form:

$$Q = \begin{pmatrix} Q_{0,0} & Q_{0,1} & O & O & \dots \\ Q_{1,0} & Q_{1,1} & Q_{1,2} & O & \dots \\ O & Q_{2,1} & Q_{2,2} & Q_{2,3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad (2)$$

where the blocks $Q_{i,j}$ are calculated as follows:

$$Q_{i,i} = A - i\alpha B,\ i \geq 0, Q_{i,i-1} = i\alpha L,\ i \geq 1,$$

$$Q_{i,i+1} = \lambda S,\ i \geq 0,$$

where

- $B = diag\{I,\ \frac{c-1}{c}I,\ \frac{c-2}{c}I,\ \dots,\ \frac{1}{c}I,\ O\}$,
- L is the block matrix having zero blocks except the first diagonal over the main diagonal. Nonzero blocks are the equal to

$$P_0,\ \frac{c-1}{c}P_1^{(M-2)},\ \frac{c-2}{c}P_2^{(M-2)},\ \dots,\ \frac{1}{c}P_{c-1}^{(M-2)},$$

- $S = diag\{O,\ \frac{1}{c}I,\ \frac{2}{c}I,\ \dots,\ \frac{c-1}{c}I,\ I\}$,
- A is the block matrix having zero blocks except the first diagonal over the main diagonal, the first diagonal under main diagonal and the main diagonal which consist of the following blocks correspondingly:

$$P_0,\ \frac{c-1}{c}P_1^{(M-2)},\ \frac{c-2}{c}P_2^{(M-2)},\ \dots,\ \frac{1}{c}P_{c-1}^{(M-2)};$$

$$A_0,\ A_1^{(M-2)},\ A_2^{(M-2)},\ \dots,\ A_{c-1}^{(M-2)};$$

$$O,\ R_0,\ R_1,\ \dots,\ R_{c-1},$$

where

- $diag\{a_1, \dots, a_c\}$ is the diagonal matrix with the diagonal entries $a_1, \dots, a_c$;
- the diagonal entries $a_{v,v}$, $v = \overline{1,N}$, of the matrix A have the form: $a_{v,v} = -\lambda - \mathbf{h}_v \mathbf{S}_d$, $v = \overline{1,N}$;
- $P_0 = \boldsymbol{\beta}$, $A_0 = \mathbf{S}_0$;
- for $j = \overline{1, c-1}$ matrices $P_j^{(0)}$ of dimension $(j+1) \times (j+2)$ have the form:

$$P_j^{(0)} = \begin{pmatrix} \beta_{M-1} & \beta_M & \cdots & 0 & 0 \\ 0 & \beta_{M-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_{M-1} & \beta_M \end{pmatrix};$$

- for $j = \overline{1, c-1}$ matrices $A_j^{(0)}$ of dimension $(j+2) \times (j+1)$ have the form:

$$A_j^{(0)} = \begin{pmatrix} (j+1)S_{M-1}^0 & 0 & \cdots & 0 \\ S_M^0 & jS_{M-1}^0 & \cdots & 0 \\ 0 & 2S_M^0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{M-1}^0 \\ 0 & 0 & \cdots & (j+1)S_M^0 \end{pmatrix};$$

- for $i = \overline{1, M-2}$, $j = \overline{1, c-1}$ matrices $G_i^{(j)}(0)$ of dimension $(j+1) \times (j+2)$ have the form:

$$G_i^{(j)}(0) = \begin{pmatrix} S_{i,M-1} & S_{i,M} & \cdots & 0 \\ 0 & S_{i,M-1} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & S_{i,M} \end{pmatrix};$$

- for $i = \overline{1, M-2}$, $j = \overline{1, c-1}$ matrices $T_i^{(j)}(0)$ of dimension $(j+2) \times (j+1)$ have the form:

$$T_i^{(j)}(0) = \begin{pmatrix} (j+1)S_{M-1,i} & 0 & \cdots & 0 \\ S_{M,i} & jS_{M-1,i} & \cdots & 0 \\ 0 & 2S_{M,i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{M-1,i} \\ 0 & 0 & \cdots & (j+1)S_{M,i} \end{pmatrix};$$

- for $j = \overline{1, c-1}$ matrices $E^{(j)}$ of dimension $(j+2) \times (j+2)$ have the form:

$$E^{(j)} = \begin{pmatrix} 0 & (j+1)S_{M-1,M} & \cdots & 0 \\ S_{M,M-1} & 0 & \cdots & 0 \\ 0 & 2S_{M,M-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{M-1,M} \\ 0 & 0 & \cdots & 0 \end{pmatrix};$$

- for $k = \overline{1, M-2}$, $j = \overline{1, c-1}$ and for $i = \overline{1, M-3}$, $k = \overline{1, M-i-2}$, $j = \overline{1, c-1}$ matrices $P_j^{(k)}$, $A_j^{(k)}$ and $G_i^{(j)}(k)$, $T_i^{(j)}(k)$ are calculated by the following recursive relations:

$$P_j^{(k)} = \begin{pmatrix} \beta_{M-k-1} & \mathbf{b} & O & O & \cdots & O \\ O & \beta_{M-k-1}I & P_1^{(k-1)} & O & \cdots & O \\ O & O & \beta_{M-k-1}I & P_2^{(k-1)} & \cdots & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & O & \cdots & P_j^{(k-1)} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} \beta_{M-k} & \ldots & \beta_M \end{pmatrix},$$

$$A_j^{(k)} = \begin{pmatrix} (j+1)S^0_{M-k-1} & O & O & \cdots & O \\ \mathbf{a} & jS^0_{M-k-1}I & O & \cdots & O \\ O & A_1^{(k-1)} & (j-1)S^0_{M-k-1}I & \cdots & O \\ O & O & A_2^{(k-1)} & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & S^0_{M-k-1}I \\ O & O & O & \cdots & A_j^{(k-1)} \end{pmatrix}, \mathbf{a} = \begin{pmatrix} S^0_{M-k} \\ \vdots \\ S^0_M \end{pmatrix},$$

$$G_i^{(j)}(k) = \begin{pmatrix} S_{i,M-k-1} & \mathbf{d} & O & \cdots & O \\ O & S_{i,M-k-1}I & G_i^{(1)}(k-1) & \cdots & O \\ O & O & S_{i,M-k-1}I & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & O \\ O & O & O & \cdots & G_i^{(j)}(k-1) \end{pmatrix}, \mathbf{d} = \begin{pmatrix} S_{i,M-k} & \ldots & S_{i,M} \end{pmatrix},$$

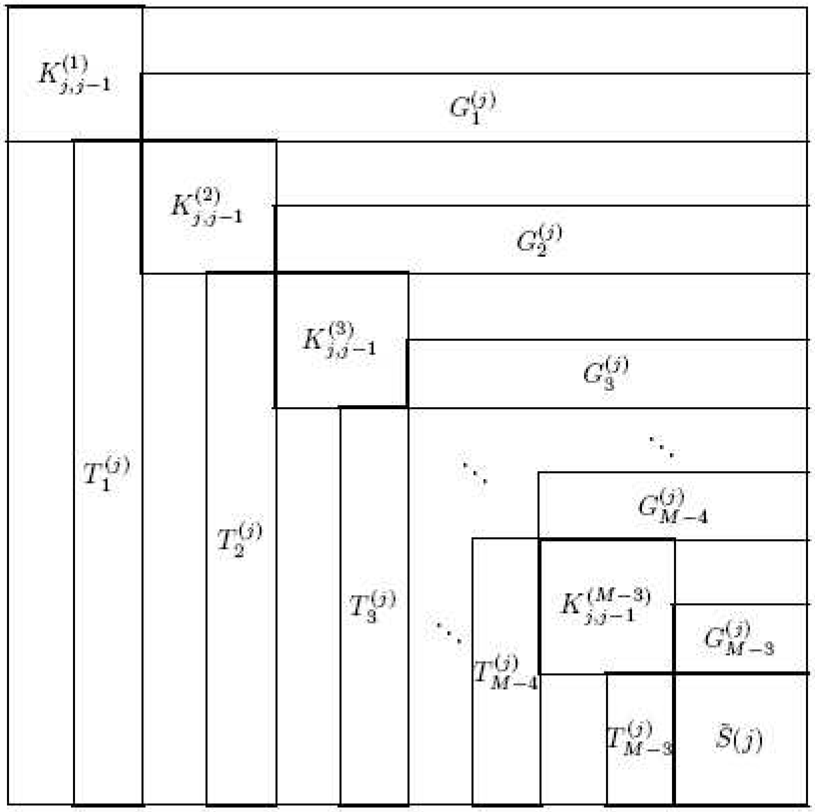
$$T_i^{(j)}(k) = \begin{pmatrix} (j+1)\cdot S_{M-k-1,i} & O & O & \cdots & O \\ \mathbf{c} & j\cdot S_{M-k-1,i}I & O & \cdots & O \\ O & T_i^{(1)}(k-1) & (j-1)\cdot S_{M-k-1,i}I & \cdots & O \\ O & O & T_i^{(2)}(k-1) & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & S_{M-k-1,i}I \\ O & O & O & \cdots & T_i^{(j)}(k-1) \end{pmatrix}, \mathbf{c} = \begin{pmatrix} S_{M-k,i} \\ \vdots \\ S_{M,i} \end{pmatrix};$$

- for $j = \overline{1, c-1}$ matrices $\tilde{S}(j)$ have the form:

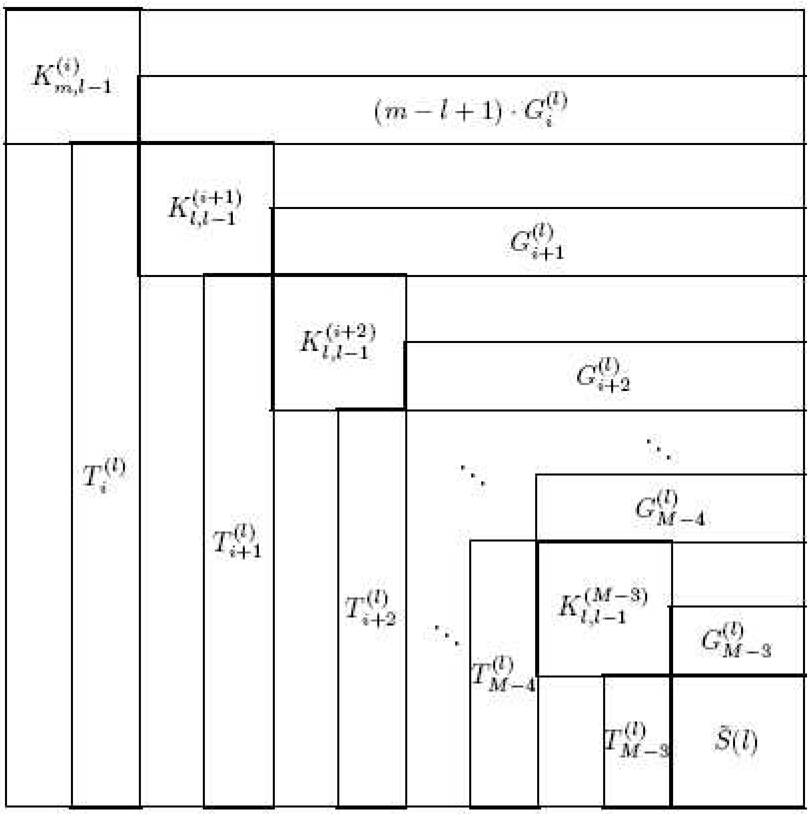
$$\tilde{S}(j) = \begin{pmatrix} \tilde{S}_j^{(M-2)} & jG^{(1)}_{M-2} & \cdots & O \\ T^{(1)}_{M-2} & E(1) & \cdots & O \\ O & T^{(2)}_{M-2} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & G^{(j)}_{M-2} \\ O & O & \cdots & E(j) \end{pmatrix};$$

- $T_i^{(j)} = T_i^{(j)}(M-i-2), i = \overline{1, M-2}, j = \overline{1, c-1}$;
- $G_i^{(j)} = G_i^{(j)}(M-i-2), i = \overline{1, M-2}, j = \overline{1, c-1}$;
- $R_0 = \tilde{S}_0^{(1)}$, $K^{(i)}_{m,0} = \tilde{S}^{(i)}_m$;
- for $j = \overline{1, c-1}$ matrices R_j and for $l = \overline{1, j-1}$, $i = \overline{1, M-3}$, $m = \overline{1, j}$ matrices $K^{(i)}_{m,l}$ are calculated by the following recursive relations:

$R_j =$


$K^{(1)}_{j,j-1}$
$G^{(j)}_1$
$K^{(2)}_{j,j-1}$
$G^{(j)}_2$
$K^{(3)}_{j,j-1}$
$G^{(j)}_3$
$T^{(j)}_1$
$T^{(j)}_2$
$T^{(j)}_3$
$G^{(j)}_{M-4}$
$K^{(M-3)}_{j,j-1}$
$G^{(j)}_{M-3}$
$T^{(j)}_{M-4}$
$T^{(j)}_{M-3}$
$\tilde{S}(j)$

$K^{(i)}_{m,l} =$


$K^{(i)}_{m,l-1}$
$(m-l+1) \cdot G^{(l)}_i$
$K^{(i+1)}_{l,l-1}$
$G^{(l)}_{i+1}$
$K^{(i+2)}_{l,l-1}$
$G^{(l)}_{i+2}$
$T^{(l)}_i$
$T^{(l)}_{i+1}$
$T^{(l)}_{i+2}$
$G^{(l)}_{M-4}$
$K^{(M-3)}_{l,l-1}$
$G^{(l)}_{M-3}$
$T^{(l)}_{M-4}$
$T^{(l)}_{M-3}$
$\tilde{S}(l)$

- for $i = 1$, $j = \overline{0, c-1}$ and $i = \overline{2, M-2}$, $j = \overline{1, c-1}$ matrices $\tilde{S}_j^{(i)}$ have the form:

$$\tilde{S}_j^{(i)} =$$

$$\begin{pmatrix} 0 & (j+1)S_{i,i+1} & \cdots & (j+1)S_{i,M} \\ S_{i+1,i} & 0 & \cdots & S_{i+1,M} \\ S_{i+2,i} & S_{i+2,i+1} & \cdots & S_{i+2,M} \\ \vdots & \vdots & \ddots & \vdots \\ S_{M,i} & S_{M,i+1} & \cdots & 0 \end{pmatrix};$$

- O is an zero matrix of appropriate dimension;
- I is an identity matrix of appropriate dimension.

Lemma is proved by means of analysis of transitions of the Markov chain $\zeta_t, t \geq 0$, during the infinitesimal time interval.

4 Stationary distribution of the Markov chain. Stability condition

The continuous-time multi-dimensional Markov chain $\zeta_t, t \geq 0$, describing the behavior of the system belongs to the class of multi-dimensional asymptotically quasi-Toeplitz Markov chains. So, as follows from (Klimenok and Dudin, 2005), stability condition for the Markov chain $\zeta_t, t \geq 0$, is expressed in terms of the matrix generating function $\tilde{Y}(z)$ of one-step transition probability matrices for the jump chain of the chain $\zeta_t, t \geq 0$, when the intensity of retrial is infinite one. It can be verified that this matrix generating function has the following form:

$$\tilde{Y}(z) = L + (I - B + \bar{I}F^{-1}A)z + \lambda\bar{I}F^{-1}Sz^2, |z| \leq 1.$$

Theorem. Stationary distribution of the Markov chain $\zeta_t, t \geq 0$, exists if the following inequality holds good:

$$\mathbf{X}[I - B + \bar{I}F^{-1}(A + 2\lambda S)]\mathbf{e} < 1, \quad (3)$$

where the vector $\mathbf{X}$ satisfies equations

$$\mathbf{X}[L - B + \bar{I}F^{-1}(A + \lambda S)] = \mathbf{0},\ \mathbf{X}\mathbf{e} = 1, \quad (4)$$

where $\bar{I} = I - \hat{I}$; F is diagonal matrix having the values $a_{v,v}$ as its diagonal entries, $a_{v,v}$, $v = \overline{1, N}$, is diagonal entry of the matrix A; the matrix $\hat{I}$ is obtained from the matrix I by replacing the diagonal entries corresponding to zero diagonal entries of the matrix B with 0.

Remark 1. It is easy to show that (3), (4) coincide with the condition for the existence of the stationary distribution of the Markov chain describing the behavior of the system from (Mushko, 2005).

5 Stationary distribution of the Markov chain. Algorithm for calculation

It was mentioned above that the continuous-time multi-dimensional Markov chain describing the behavior of the system belongs to the class of multi-dimensional asymptotically quasi-Toeplitz Markov chains. So, the effective and stable algorithm for solving system (1), (2), which is presented in (Klimenok and Dudin, 2005), can be applied for solving this system.

Remark 2. The positive feature of the approach proposed in (Mushko, 2005) is the simplicity of the analytical results at the expense of having to consider a large state space. The size N of the matrix involved in computations is equal to $(M+1)^c$. For $c = 8$, $M = 3$, $N = 65536$. Calculation with such a big matrices is practically impossible on a personal computer. Whereas the results of our section 3 allows to apply the Ramaswami's approach. It reduces significantly dimension of the macrostates: $N = 165$.

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SESSION 6

STATISTICS AND APPLICATIONS

REGRESSION MODEL OF SALES VOLUME

FROM WHOLESALE WAREHOUSE

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Sales forecast, stock inventory, regression model

ABSTRACT

The multiple regression model is offered for short-term forecasting of change in sales volumes from a wholesale warehouse. Testing of the model correctness was performed by means of Statistica 6.0 on the base of the company UNIFEX Ltd real sales data for two ranges of goods.
The estimation of change in sales volumes obtained by means of the offered model may be used by logistic and distribution specialists for further calculations, for example, for determination of volumes and frequency of orders for delivery of separate goods or a group of goods.

INTRODUCTION

As it is known, the logistic chain is rather complicated – starting with the research of demand pattern and ending by supply of the stock to the consumer with minimum total expenses (Kogan and Spiegel 2006, Mackay and Probert 2002).
The problem that very often comes up at logistic enterprises is precise determination of optimal volume of the ordered goods in one or some positions as well as orders frequency. Let us pick out the following task – optimal batch size determination for replenishment of the current stock so that the stock runs out or reaches some minimum (critical) level at the given moment of time. To solve this problem one needs to know the sales demand in the future, thus, the problem of demand forecast for a period of time arises. As is known, there is a lot of ways of forecasting of sales volumes and/or demand (Bayhan G. and Bayhan M 1998, Luxhoj, Riis and Stensballe 1996, Rao 1985, Profillidis 2000). There is one way of demand forecasting being considered in the given article – short-term forecasting using multiple factor regression model.
Let us consider a typical wholesale enterprise – UNIFEX Ltd that deals with purchasing of big batches of food- and non-provision goods from enterprises - producers and delivery of these goods to the shops (enterprises - customers). At UNIFEX Ltd sales volumes of each goods are recorded every week during the whole year. Actually it is the quantity of the goods delivered to all shops weekly.
For forecasting with the help of the regression model one must determine the set of independent quantitative factors (variables) supposedly effecting on sales quantity. It should be noted that a producer of a brand as usual deals with the long-term demand forecasting whereas retail vendors (shops) take care of the short-term (operative) forecast. In these two variants of forecasting the factors effecting on the demand are different; let us agree not to consider them.
Wholesale enterprises, such as UNIFEX Ltd, are the link between a producer and a consumer. In the database of UNIFEX there is the following information, supposedly effecting on sales quantity:

1. Purchasing price of the goods and its alterations;
2. Purchase volumes every week;
3. Total volume of goods dispatched to all shops every week;
3. To what enterprises-consumers and at what prices the goods were delivered;
4. Whether discounts and at what rate were granted for the given goods within the past period of time under consideration;
5. Whether there was deficiency of the given goods in the warehouse and in what volume.

There is a variety of factors affecting sales quantity, but as a rule, it is very difficult to present them in quantitative form. For example, quality of the goods is important only for perishables (the so called sort). In this case one may reduce the price as the shelf life of the given goods expires. As to food goods with long shelf life, such as chocolate and pet fodder, a supplier as a rule suggests shops to organize actions when the shelf life of the given products comes up to the end. As usual advertising allowance and discount for ending up with excessive inventories in the warehouse assumes about 15% reduction of the usual price, whereas at the expiring shelf life discount may reach 30%.

As usual it is necessary for wholesale enterprises to determine the volume of ordered goods not more than for nearest couple of weeks. Therefore, the expected demand for the given goods one should know for the same period of time. We shall forecast the change of the demand quantity for the next week, exactly as for how many times more or less has got the demand of the goods of two groups compared to the last week: A – hygiene goods, 21 items, B – pet fodder, 22 items. As sale analysis for 2004 demonstrates, the demand for the goods of these both groups is about the same during one year.

So, we have information about the sales of goods of the two groups on a weekly basis during 2004. As the demand for the given groups of goods is about the same during a year, we will forecast the change of the demand quantity for each goods from the groups one week in advance, using the data for the last 3 months.

For the suggested model no specially organized structures of data are necessary – all data may be obtained from the enterprise database. The demand estimated with the help of the model further may be used for calculation of the optimal ordered goods batch. It is also important to note that a mere purchase manager as usual does not possess any knowledge in such spheres like inventory management theory and mathematical statistics as well as does not possess specific mathematical software. This is why for him/her it would be preferable to use the already given formulae for ordered batch optimal volume calculation, where he/she realizes them, for example, in Excel. Testing of the regression model soundness has been performed in the Statistica 6.0 package. The suggested approach will help enterprise workers who deal with distribution and logistic to solve the above indicated problem and will allow enterprise to reduce the expenses for purchasing and storing goods.

MULTIPLE REGRESSION MODEL FOR SALE VOLUME FORECAST

Let us consider the standard regression model (Draper and Smith 1998). The dependent variable $Y(t)$ is the proportion of the given goods demand quantity during the current week t to the demand quantity during the past week t-1. Let us pick out the factors effecting on the demand quantity change, basing on the information that we already have:

1. Ordinal week number.
2. Goods realization price.
3. Discount.
4. Stock condition.

The regression model will be written as follows:

$$Y(t) = \beta_0 t^{\beta_1} P^{\beta_2} A^{\beta_3} S^{\beta_4} e^{\varepsilon}, \qquad (1)$$

where t is ordinal week number, t = 1, 2, ... , whereas t =1 means the first week of the year;

$Y(t)$ is the dependent variable, demand quantity change in the week t concerning the previous week:

$$Y(t) = \frac{W(t)}{W(t-1)}, \quad t = 2, 3, \dots ,$$

where $W(t)$ and $W(t\text{-}1)$ – demand in the t-th and (t-1)-th week, accordingly;

P – this variable may be presented in various ways:
1) planned average goods price in the t-th week;
2) difference of prices in the t-th and (t-1)-th weeks; variable equals 0 if in the end of the (t-1)-th week there has not been any change of the price.

In the given model the first way of determination of the variable P has been chosen.

A – variable is equal to the discount portion if during the t-th week any enterprise-consumer had any discount for the given goods; 1 – otherwise;

S – index of goods deficiency, demonstrates the stock of the given goods in the warehouse in the t-th week. This index is calculated on the base of data about arrival, depletion and stock of goods in the warehouse. The index demonstrates the stock of the given goods in the warehouse during a certain period of time after the necessary quantity of the goods has been loaded. It takes values from 0 to 1.0 – situation “out of stock”, stock of goods is equal to zero (or lower than the determined critical level), 1 – maximum possible determined stock of goods. The rest values from 0 to 1 are determined as the proportion of the current stock to the maximum possible one;

ε – odd component, having log-normal distribution.

So, in the regression model we have 4 independent variables: t, P, A, S, and 5 unknown coefficients: β_0, β_1, β_2, β_3, β_4. Our goal is to evaluate these coefficients and check their values at all data that we already have for two groups of the goods under consideration. It should be noted that in the mathematical model (1) goods belonging to one or another group does not get estimated. Nevertheless we keep the classification of the goods to the two groups, following the registration technology accepted at UNIFEX.

As it is obvious, the suggested model is a non-linear one (multiplicative). After having performed logarithms to the left and to the right, we obtain its following parameter-linear form:

$$\ln Y(t) = \ln\beta_0 + \beta_1 \ln t + \beta_2 \ln P + \beta_3 \ln A + \beta_4 \ln S + \varepsilon.$$

To this modified model one may apply standard estimation formulae (the least squares method).

ANALYSIS OF REGRESSION MODEL CORRECTNESS

There has been made a series of experiments to check the correctness of the suggested regression model. With this aim a mode *Multiple Regression* of Statistica 6.0 package has been used. The given procedure allows estimating parameters of parameter-linear regression model, using the least squares method. The check has been made using the data of the sales of each out of 43 types of goods of the two groups during 2004.

To analize the correctness of the suggested model the following criteria was used (Srivastava 2002):

1. Fisher Criterion of the hypothesis check of regression insignificance, level of significance α=5%;
2. Student Criterion of the hypothesis check of the *k*-th independent variable insignificance, level of significance α=5%;
3. Multiple determination coefficient R^2.

Table 1: The Results of the Experiments for the Group A Goods

Goods №	Beta0	Beta1	Beta2	Beta3	Beta4	R^2
1	**4,4**	**-0,1522**	**-19,5**	**22,0**	**1,1**	**0.88**
2	**6,6**	0,0916	**-31,7**	**34,5**	0,6	**0.84**
3	**-5,9**	**-0,0818**	**-17,2**	**16,3**	**0,7**	**0.86**
4	**5,1**	-0,0933	**-20,8**	**20,5**	0,7	0.75
5	**18,0**	-0,0236	**-21,0**	**20,7**	0,2	**0.87**
6	**-3,4**	**-0,1212**	**-11,4**	4,2	**1,1**	**0.93**
7	**13,9**	0,0774	**-60,8**	**60,9**	**1,6**	0.72
8	**15,2**	0,0541	**-69,0**	**68,6**	**0,7**	**0.85**
9	**4,2**	0,2481	**-12,1**	5,3	**0,9**	0.72
10	**0,9**	-0,0876	**-14,2**	**10,5**	**1,6**	**0.93**
11	**0,8**	0,0404	**-20,2**	**19,4**	**1,7**	**0.87**
12	**-2,5**	0,1062	**-22,8**	**23,2**	**1,2**	**0.91**
13	**-3,1**	-0,0497	**-29,8**	**30,9**	**0,8**	**0.93**
14	**5,9**	0,0652	**-29,4**	**28,1**	0,6	0.77
15	**-2,4**	0,0161	**-21,1**	**20,8**	0,4	**0.88**
16	**-2,3**	0,0156	**-22,3**	**22,5**	0,8	**0.85**
17	**5,8**	0,0761	**-30,5**	**30,3**	0,7	0.76
18	**5,4**	0,0124	**-25,4**	**26,9**	**0,7**	**0.85**
19	**0,4**	-0,0995	**-9,1**	**8,3**	**0,8**	**0.85**
20	**-2,1**	-0,1654	**-19,2**	**18,6**	0,3	**0.88**
21	**8,9**	-0,0766	**-45,4**	**44,8**	0,7	**0.86**

Table 2: The Results of the Experiments for the Group B Goods

Goods №	Beta0	Beta1	Beta2	Beta3	Beta4	R^2
1	**55,6**	0,1215	**-26,7**	**26,1**	**0,4**	**0.88**
2	**90,1**	-0,0014	**-37,4**	**33,8**	0,2	**0.84**
3	**160,9**	0,1618	**-67,1**	**50,2**	0,3	**0.91**
4	**61,8**	-0,0744	**-24,6**	**42,2**	0,1	0.78
5	**65,7**	0,0000	**-29,7**	**19,4**	**0,8**	**0.91**
6	**79,7**	0,2065	**-38,4**	**40,7**	0,2	**0.84**
7	**79,9**	0,0668	**-38,4**	4,2	0,2	**0.85**
8	**70,0**	-0,0027	**-27,9**	**28,5**	-0,1	0.75
9	**78,1**	-0,0533	**-35,5**	**24,1**	-0,2	**0.88**
10	**154,6**	-0,1791	**-63,9**	15,4	0,2	0.77
11	**61,2**	-0,1656	-25,0	21,6	**1,3**	0.75
12	**23,4**	0,4181	-9,4	4,1	**2,6**	**0.85**
13	**12,9**	-0,0846	-6,4	3,9	0,7	0.73
14	**10,9**	0,0466	**-5,6**	10,5	**0,9**	**0.82**
15	**6,7**	0,0450	**-3,4**	**3,2**	**1,1**	**0.89**
16	**44,3**	-0,0018	**-20,8**	**20,7**	0,6	0.71
17	**45,1**	-0,0021	**-21,2**	**20,7**	0,3	0.74
18	**29,8**	-0,0300	-13,8	14,1	0,7	0.66
19	**22,3**	0,0505	**-10,5**	2,2	**0,7**	**0.89**
20	**35,4**	0,1507	**-16,8**	1,7	0,6	**0.87**
21	**57,4**	0,0485	**-27,2**	2,7	-0,1	0.78
22	**28,1**	-0,0072	**-15,2**	**14,7**	0,2	**0.80**

The results of the experiments (coefficient estimation values for variables and multiple determination coefficient R^2) for group A goods are shown in the Table 1, for group B goods the results are shown in the Table 2. The calculated values of Fisher Criterion for the goods of the both groups are shown in the Table 3.

Let us comment the obtained results. Firstly, let us note that the signs of the estimated coefficients, as a rule, fit the physical sense of the influence of the considered factors.

So, coefficients *Beta2* and *Beta3* (for the variables P and A accordingly) for all the sets of data have one and the same sign. It means that the chosen model behaves approximately in the same way with all the researched sets of data. The negative coefficient by the variable P that shows the price of the given goods shows that if the prices increase the sales volume decreases.

The positive coefficient by the variable A that shows the value of the discount for the given goods, shows that if the discount increases the sales volume increase too. Coefficient *Beta1* by the variable t, showing the ordinal week number, and coefficient *Beta4* by the variable S, showing the stock state, take positive as well as negative signs. The negative *Beta1* shows that gradually with time the given goods sales volume decreases, and the positive one demonstrates its increasing. The positive *Beta4* shows that the bigger the stock of the given goods in the warehouse, the bigger the given goods sales volume, that means, the demand for given goods is stable; the negative sign of it shows that there has not been any increase in sales volume not withstanding sufficient given goods stock in the warehouse – that means, there has been the so called „over stock", frozen residue – the present stock is much bigger than the present demand. The *Beta0* is the intercept of the presented regression model and is not discussed.

In bold face there are the coefficients of the most important independent variables (by Student criterion). As it has already been supposed, the most important factors affecting the demand changes have occurred to be the price of the given goods along with the given discount. Let us set the variables in the order of decreasing importance: price P, discount A, stock state S, week ordinal number t. Let us consider now the calculated values of Fisher criterion.

Table 3: The Calculated Values of Fisher Criterion

Goods №	The group A goods	The group B goods	Goods №	The group A goods	The group B goods
1	40,92	15,41	12	57,88	23,55
2	28,74	26,95	13	76,78	10,29
3	34,86	36,85	14	15,40	18,74
4	15,09	14,36	15	38,66	34,21
5	37,89	45,86	16	30,69	9,24
6	74,16	21,64	17	15,77	11,42
7	12,36	24,76	18	31,51	6,99
8	29,83	12,13	19	30,98	33,60
9	12,43	31,57	20	38,78	27,59
10	73,41	13,76	21	28,27	14,48
11	34,54	11,70	22	-	16,48

For the experiment of the group A goods data the theoretical value of Fisher criterion (by the freedom degree 4 and 47) equals 2.56, but for the experiment of the group B goods data the theoretical value of Fisher criterion (by the freedom degree 4 and 37) equals 2.62. It is obvious that for all experiments the value of the calculated Fisher criterion is higher than the theoretical one. This shows that the hypothesis of the regression insignificance is not accepted by the significance level 5%. Therefore this shows that our suggestions of the chosen factors' influence on the demand's change cannot be rejected.

Let us consider now the values of the multiple determination coefficient R^2. This coefficient characterizes changeability of the regression and shows what portion of the changeability of the dependent variable is determined by the changeability of the regression (Srivastava 2002). For adequate models its value should be not less than 0.8. For the group A goods in 16 experiments out of 21 this coefficient exceeds the level of 0.8. The situation with group B goods is worse – the coefficient exceeds the level of 0.8 only in 13 experiments out of 22.

The obtained models may be used for forecast of the change of demand value for each goods separately. Further on we shall consider models that are common for all goods of one group. Let us call these models group models.

GROUP MODELS' ANALYSIS

In this case the analyzed massive are the present data of sales of goods of each group. As a result we have the regression model equations for forecast of the change of demand for the both groups of goods.

For group A:

$$Y(t)=0.92t^{-0.05}\,P^{-0.04}\,A^{2.05}\,S^{1.64}. \tag{2}$$

For group B:

$$Y(t)=0.79t^{-0.03}\,P^{-0.02}A^{1.97}\,S^{1.57}. \tag{3}$$

It is apparent that for the both groups the variable coefficients have the same signs and are the values of one and the same magnitude. The variables A and S were declared by Student criterion to be the most important for both experiments. The values of Fisher criterion are 123.01 and 128.79 for the equations (2) and (3) accordingly. The theoretical value of Fisher criterion is 2.38 (as by freedom degree 4 and 715 for the first experiment, so by freedom degree 4 and 914 for the second experiment). It is obvious that also here the calculation values of Fisher criterion exceed the theoretical ones, therefore, the hypothesis of the regression insignificance also is not accepted at the level of significance 5%. The value of the multiple determination coefficient R^2 is equal to 0.64 and 0.60 for equations (2) and (3) accordingly, what is, as it was supposed, much worse than in the case of usage of an individual model.

Thus, with the use of the group model received forecast will be less precise than with the use of the individual model.

As a result the model was acknowledged to be adequate for all experiments and the most significant variables were selected.

CONCLUSIONS

In the future it would be expedient to include a variable into the model that shows the arrival of goods to the warehouse of the enterprise UNIFEX.

It should be noted that the opportunity to build forecast of the demand and to plan purchase volumes, using mathematical methods and software, in fact does not get used in any commercial or logistic enterprise of Latvia. At such enterprises the building of forecast is based on so-called manual approach of average intensity calculation notwithstanding that there are special subprograms within the known management systems (for example, MS Navision). So let us mark the originality of the suggested approach, because building demand forecast using the above considered regression model allows one to mark the tendency of the demand behavior more precisely. The obtained demand values estimations one may use for the further calculations – the volume of the ordered goods batch using, for example, N-staged dynamic model of stock inventory (Taha 2002). It is important that the calculations based on the suggested models may be done in Excel, which is extremely comfortable for distribution and logistic specialists.

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APPLICATION OF THE SUFFICIENT EMPIRICAL AVERAGING METHOD IN ONE STATISTICAL PROBLEM SOLVING

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KEYWORDS

estimation, complete sufficient statistic, inventory control, optimization

ABSTRACT

In this article we discuss the new approach to preparing the input data in system simulation, the so called Sufficient Empirical Averaging (SEA) method. It assumes the existence of the complete sufficient statistics for unknown parameters of input variable distributions. The application of this method allows getting unbiased estimates with minimum variance.

INTRODUCTION

We know that the preparation of the primary data is a very important stage in modeling. In particular, it includes the estimation of the parameters of the random variables used in simulation. In the traditional approach, so called the "*Plug-In*" method, the observations of input variables are used for estimation of probability distributions of these variables. According to these distributions the pseudo-random numbers are produced in the process of simulation, and then they replace the latter variables. Here the estimation of probability distributions leads to making mistakes in choosing the form of a distribution and estimating its parameters. However, the primary statistical samples, on the base of which the estimation of the parameters of distribution is executed, is not sufficient, it leads to the bias of estimated parameters and results of simulation.

The alternative to the "Plug-In" method is the "*Resampling Approach*" method (Andronov and Merkuryev 2002). According to this method well known method the random variables aren't worked out by generators of the random numbers but derived from initial statistical samples according to this or that random mechanism. This method doesn't use preliminary information about the type of the distributions of the random variables, i. e. in fact it is nonparametrical. It is a big advantage of such approach. Nevertheless, if such information is available, it should be also used, and in this case the method of *Sufficient Empirical Averaging method* (*SEA*-method) will be effective (Chepurin 1994, 1995, 1999).

The SEA-method is used when the supposed distributions have the sufficient statistics. It is based on the fact, that conditional distributions of the random variables, calculated with fixed values of sufficient statistics, don't depend on the unknown parameters of distributions. Consequently, the necessary random variables could be produced according to their conditional distributions. The received results of the imitative simulation will be unbiased. Moreover, if the applied sufficient statistics are complete (see Lehman 1983) then they will have the least variance.

The aim of this article is to illustrate the application of the SEA-method for one statistical problem of the inventory control. The SEA-method is described in the following section. An inventory problem setting and solving will be given in the third and forth sections. The results of the problem solution are given in the fifth section. The peculiarities of the discussed approach application are covered in the last section.

SUFFICIENT EMPIRICAL AVERAGING METHOD

Let us suppose that our aim is to estimate the mathematical expectation θ of function f of independent random variables $X_1, X_2, ..., X_n$: $\theta = = Ef(X_1, X_2, ..., X_n)$. This function describes realiability or performance characteristics of considered system. Distributions of random variables $X_1, X_2, ..., X_n$ are unknown, but sample populations $\{H_i\}$ are available for each $\{X_i\}$.

The traditional approach to solving this problem supposes three stages: 1) hypothesis about distributions of random variables are formed; 2) unknown parameters of these distributions are estimated by using samples $\{H_i\}$; 3) the estimated distributions are used to estimate the mathemtical expectation θ by generation values of variables $\{X_i\}$ and simulating the function f on this base.

An alternate approach is the SEA method (Chepurin 1994, 195, 1999; Andronov, Zhukovskaya and Chepurin 2005). It is based on the concept of the sufficient statistics. Let $\boldsymbol{X} = (X_1, X_2, ..., X_n)$ as a sample, which has the corresponding distribution with unknown parameter θ. Then $\boldsymbol{S}$ *statistics* is any function of given sample $\boldsymbol{S} = \boldsymbol{S}(X_1, X_2, ..., X_n) = \boldsymbol{S}(\boldsymbol{X})$. The *statistics* is called *sufficient*, if the conditional distribution of sample $\boldsymbol{X} = (X_1, X_2, ..., X_n)$, which has been calculated on condition that the statistics $\boldsymbol{S}$ has a fixed value $\boldsymbol{s}$, which doesn't depend on the parameter θ (Lehman 1959, Cox and Hinkley 1974).

Let us denote $\boldsymbol{X}(\boldsymbol{s}) = \boldsymbol{X} \mid \{\boldsymbol{S} = \boldsymbol{s}\}$ the conditional sequence, that has been calculated on condition $\{\boldsymbol{S} = \boldsymbol{s}\}$. Note that $\boldsymbol{X}(\boldsymbol{s})$ "... is statistically equivalent to the data ($\boldsymbol{X}$), i.e. containing the same amount of information ..." (Chepurin 1999, p.182).

Let us for example estimate the unknown parameter λ of the exponential distribution. The probability density function of this distribution (p.d.f.):

$$f_X(x,\lambda) = \lambda e^{-\lambda x},\ x \ge 0;\ \lambda \in \Omega = (0,\infty) \qquad (1)$$

there $\Omega = (0, \infty)$ – is parametrical space, which consist all possible values of λ.

The sufficient statistics for the parameter λ on the sample $\boldsymbol{X} = (X_1, X_2, ..., X_n)$ is the sum $S_n = X_1 + X_2 + \ldots$ $\ldots + X_n$. It has Erlang distributions with parameters λ and n:

$$f_{S_n}(s,\lambda) = \lambda \frac{(\lambda s)^{n-1}}{(n-1)!} e^{-\lambda s}, \quad s \ge 0. \qquad (2)$$

The conditional p.d.f. of the sample $\boldsymbol{X}(\boldsymbol{s})$ at the point $x = (x_1, x_2, \ldots, x_n)$ is calculated by the formula:

$$f_X(\boldsymbol{x},\lambda \mid S_n = s) = \frac{\prod_{i=1}^{n-1}\left(\lambda e^{-\lambda x_i}\right)\lambda e^{-\lambda(s-\sum_{i=1}^{n-1} x_i)}}{f_{S_n}(x,\lambda)} = \frac{(n-1)!}{s^{n-1}}, \qquad (3)$$

$$\forall x_i \ge 0,\ x_1 + x_2 + \ldots + x_n \le s. \qquad (4)$$

We see that one doesn't depend on the unknown parameterλ.

Now we suppose that we have generated a random value X. Firstly, we must calculate the sufficient statistics $\boldsymbol{S}$, secondly – generate the random value X on condition $\boldsymbol{S} = \boldsymbol{s}$. By this we use the conditional distribution X for the given $\boldsymbol{s}$.

Figure 1 allows us to compare traditional approach and SEA-method.

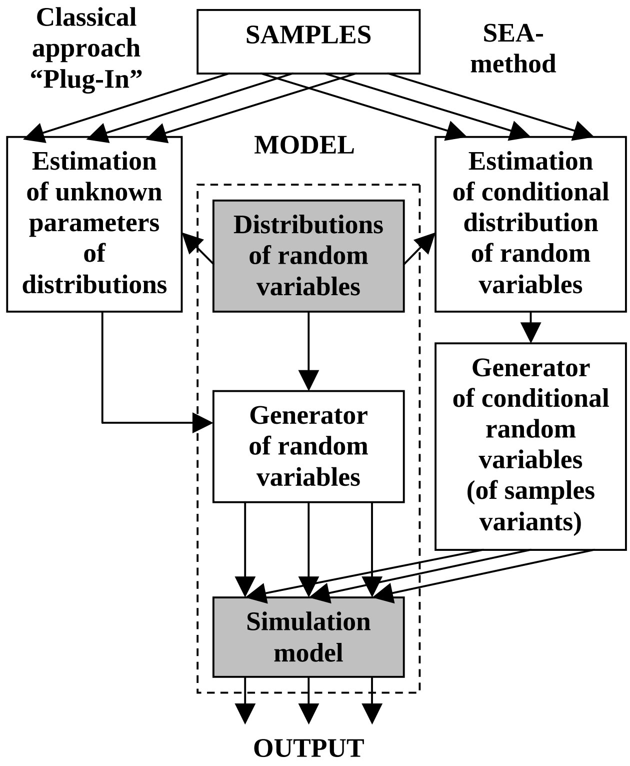

Figure 1. The comparison of classical approach and SEA-method

Note that according (3) the conditional distribution of the sample $X \mid \boldsymbol{S}_n = \boldsymbol{s}$ has a uniform distribution in the n-measure simplex (4). In principle this allows us to generate the conditional distributions of the random values with exponential distribution on condition that the sufficient statistics (sum) is fixed. But this method is inefficient. Often for generation the conditional distributions "special ways" can be used.

For example, for the exponential case (Engen and Lillegard 1997), the following way is supposed. Let the value $\boldsymbol{s}$ of the $\boldsymbol{S}_n$ is fixed. Firstly we generate n exponential distributed random variable $X^0_1, X^0_2, ..., X^0_n$ with parameter $\lambda = 1$. Secondly we calculate their sum: $S^0_n = \sum_{i=1}^{n} X_i^0$. Then random variables of interest X^*_1, X^*_2, ..., X^*_n may be calculated by the formula $X^*_i = \frac{X_i^0 \cdot s}{S_n^0}$, $i = 1, 2, ..., n$.

PROBLEM SETTING

Let's examine the following statistical problem of the inventory control that has been considered earlier (Afanasyeva 2005). Initially there are k articles of a commodity in the warehouse. In the course of time the number of articles can be increased (because of supply some new ones), or decreased (because of selling some of them). The selling of articles are realized according to the Puason flow with the intensity μ. Time intervals between supply of the articles have the exponential distribution with the parameter λ, shifted with the variable δ:

$$f(x) = \begin{cases} \lambda e^{-\lambda(x-\delta)}, & x > \delta, \\ 0, & otherwise. \end{cases}$$

If in the moment of receiving the order for a certain article we have it in the warehouse then this order is performed. Otherwise, the order is rejected and is not examined again.

The value of δ is known while the intensities λ and μ are unknown. We have two samples: the sample $X_1, X_2, ..., X_{n_a}$ of the fixed time intervals between the supply (of the size n_a) and the sample $Y_1, Y_2, ..., Y_{n_d}$ of the time intervals between the sale (of the size n_d). It is necessary to estimate the probability $P(i, k, t)$ that during the given time interval $(0, t)$ i refusals for orders will take place. Besides, it is necessary to estimate the optimal size of the initial stock $k^*(t)$ which maximizes the expected general income. This income is equal to the difference of cost values of the sold articles from the warehouse minus expenses for supply and storage of the initial stock of articles. The income from the sale of one article is c_d, but the expenses per one initially available article are c_a.

PROBLEM SOLVING

We will solve this problem with the help of the SEA-method. In our case the exponential distribution is taking place. The complete sufficient statistics for it are the sum of the sample elements and the sample size. Let's define such sums as A for the articles supply and D for articles sale. Then the simulation of the considered process in the given time t is applied. The time intervals between the supply and selling of the articles are generated according to the appropriate conditional distributions given fixed values (A, n_a) and (D, n_d).

Let's describe the procedure of generation of random variables with respect to supply process. So, as the initial we have the sufficient statistics (A, n_a), calculated with n_a intervals between articles supply. First n_a random variables are generated according to the exponential distribution with the parameter 1. Let's define them as $X_1^0, X_2^0, ..., X_{n_a}^0$. Then let's calculate their sum

$$A^0 = \frac{1}{n_a} \sum_{i=1}^{n_a} X_i^0 . \qquad (5)$$

The needed values of the intervals $X_1^*, X_2^*, ..., X_{n_a}^*$ between articles sale are calculated by the formula

$$X_i^* = X_i^0 (A - \delta n_a) \frac{1}{A^0} + \delta \text{ , } i = 1, 2, \ldots n_a. \qquad (6)$$

In the same way the intervals $Y_1^*, Y_2^*, ..., Y_{n_d}^*$ between the articles supply are generated (for $\delta = 0$). The simulation of the process of articles supply and sale is realized with the m runs.

Let's describe the algorithm of the simulation within one run. For doing this let's introduce the following notations: A_i and D_i – are the moments of the supply of the i-th article and of the i-th demand arise:

$$A_i = \sum_{j=1}^{i} X_j^* , \; D_i = \sum_{j=1}^{i} Y_j^* . \qquad (7)$$

Let's define Na and Nd as the next numbers of supply of the articles and the next number of the articles demands, which take place for the present moment. The factual number of articles in the warehouse at present moment is denoted by S for the current run. Let's R define the number of demands, which were rejected for the current run. The modeling algorithm is such for one run.

Initial date: The sufficient statistics $A = \sum_{i=1}^{n_a} X_i$,

$D = \sum_{i=1}^{n_d} Y_i$, δ, k, t.

Output date: R – number of rejected demands at the current run, $C = (Nd - R) \cdot c_d - k \cdot c_a$ – the value of the income.

Algorithm:

Step 1. To generate exponential distributed with parameter 1 random variables $X_1^0, X_2^0, ..., X_{n_a}^0$ and $Y_1^0, Y_2^0, ..., Y_{n_d}^0$.

Step 2. To form the values $\{A_i\}$ and $\{D_i\}$ with respect to formulas (5) – (7), to take $Na = 1$, $Nd = 1$, $R = 0$, $S = k$.

Step 3. If $A_{Na} < D_{Nd}$, then take $Na = Na + 1$, $S = S + 1$; otherwise to take $Nd = Nd + 1$, $S = S - 1$; if $S < 0$, then take $S = 0$, $R = R + 1$.

Step 4. If $t > min\{A_{Na}, D_{Nd}\}$, then go to the step 3, otherwise – the end.

In the end of this run the number of rejected demands R and the value of the income $C = (N(d\text{-}1) - R) \cdot c_d - k \cdot c_a$ are remembered.

Then we repeat steps 1 – 4 r times. According to the gotten results of the runs, the frequencies of different values of rejected demands are calculated and the average income as well. When changing the initial value of the stock k, we can estimate the optimal value of the initial stock $k^*(t)$ with which the average income will be maximum.

NUMERICAL EXAMPLE

In this article a numerical example is considered. Let input data have the following values $n_a = 9$, $n_d = 12$, $A = 10$, $D = 8$, $\delta = 0.5$, $c_a = 2$, $c_d = 3$, $t = 6$. In first we set $k = 3$ and we have to investigate a dependence of average income C as a function of simulated runs (see *Table 1*). From the data of this table we can conclude that the average value of C^* (as an estimator of C) doesn't change significantly in any way starting from the 200 run. So, we can set the number of runs $r = 200$.

Table 1. The mean values of C as function of the number of runs r

r	$r = 10$	$r = 50$	$r = 100$	$r = 150$
C^*	15,10	13,86	14,27	14,16
r	$r = 200$	$r = 250$	$r = 300$	$r = 400$
C^*	14,14	14,13	14,12	14,12

Initially we have $k = 3$ articles in the warehouse. Next we wish to estimate the probability $P(i, 3, 6)$ that during the given time interval (0, 6) i refusals for orders will take place. The frequencies of rejected demands R are considered in the *Table* 2. The frequency polygon of rejected demands R is shown on the *Figure2*.

Table 2. The frequencies of rejected demands R

i	0	1	2	3
$P^*(i,3,6)$	0,15	0,20	0,21	0,19
i	4	5	6	7
$P^*(i,3,6)$	0,15	0,08	0,02	0,01

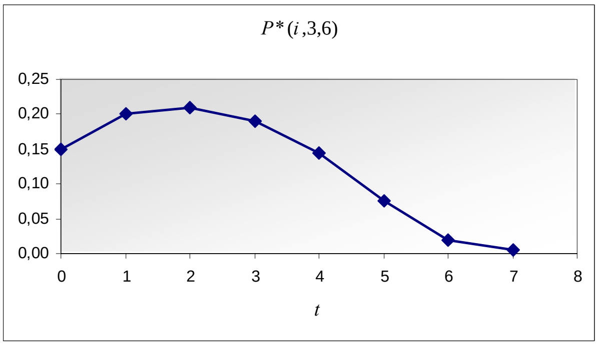

Figure2. The frequency polygon of rejected demands R

Next we wish to determinate the initial stock k, for which the maximum value of C was gotten. The corresponding arithmetical mean $C^*(k)$ (as a function of k) is calculated by formula.

$$C^*(k) = \frac{1}{r}\sum_{i=1}^{r} C^{(i)}(k). \qquad (8)$$

where $C^{(i)}(k)$ is the value of C, that has been gotten in the i-th run.

Table 3. The mean values of C^* as function of the initial stock k

k	0	1	2	3	4
$C^*(k)$	10,67	12,13	13,64	14,23	14,78
k	5	6	7	8	9
$C^*(k)$	14,34	12,44	11,05	9,54	7,57

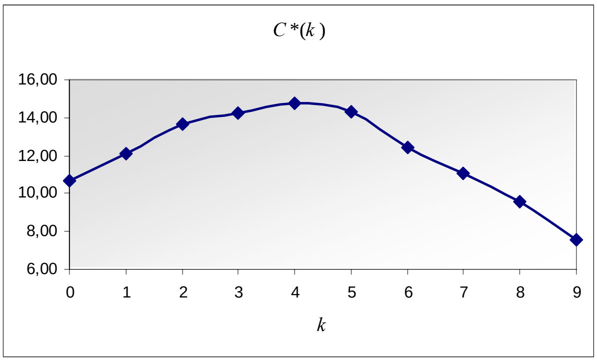

Figure 3. The mean values of C^* as a function of the initial stock k

According to the obtained results we can conclude that $k = 4$ is the optimal value of the initial stock $k^*(t)$ with which the average income will be maximum.

CONCLUSION

Firstly we considered the various approaches to tackle the problem solving: the classical "Plug-In" approach, the "Resampling" approach and the Sufficient Empirical Averaging method. Next we described the *SEA*-method in details. After that we considered some problems of the inventory control and described one method of their solving. Numerical examples end our paper. Gotten results showed that the Sufficient Empirical Averaging method allows solving various practical problems efficiently.

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A MULTI-SERVER QUEUE WITH NEGATIVE CUSTOMERS AND PARTIAL PROTECTION OF SERVICE

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KEYWORDS

Batch Markovian Arrival Process, Phase Type Service Time Distribution, Negative Customers, Partial Service Protection

ABSTRACT

A multi-server queueing system with finite buffer is considered. The input flow is the $BMAP$ (Batch Markovian Arrival Process). The service time has the PH (Phase) type distribution. Customers from the $BMAP$ enter the system according to the discipline of partial admission. Besides ordinary (positive) customers, the Markovian flow (MAP) of negative customers arrives to the system. A negative customer can delete an ordinary customer in service if the state of its PH-service process belongs to some given set. In opposite case the ordinary customer is considered to be protected of the effect of negative customers. The stationary distribution and the main performance measures of the considered queueing system are calculated.

1 Introduction

In real telecommunication and computer systems, customers may leave the system without full service by various reasons such as impatience of customers, deleting files infected by computer viruses, the accidental fails of servers, etc. Such situations can be modeled in terms of queues or networks with negative customers. The notion of a negative customer was introduced by E. Gelenbe in (Gelenbe, 1991). A negative customer has the effect of a signal that induces an ordinary customer (or a batch of customers), if any, to leave the system immediately. Since their introduction, queues and networks with negative customers (so called G-queues and G-networks) were studied by many authors. The detailed overviews of G-queues and G-networks can be found in the surveys by J. Artalejo (Artalejo, 2000) and P. Bocharov and V. Vishnevsky (Bocharov, Vishnevsky, 2003). Among the last years publications we mention the papers (Anisimov, Artalejo, 2001), (Dudin, Kim, Semenova, 2004), (Dudin, Semenova, 2004), (Li, Zhao, 2004), (Shin, 2004), (Shin, 2005), (Shin, Choi, 2003) where G-queues with Markovian flows or/and phase type or general service time distribution are considered. In papers (Dudin, Kim, Semenova, 2004), (Dudin, Semenova, 2004) the controlled systems of the $GI/PH/1$ and the $BMAP/SM/1$ type are investigated. The paper (Li, Zhao, 2004) deals with the $M/G/1$ queue with two types of service and two disciplines of removing positive customers. In papers (Shin, 2004), (Shin, Choi, 2003) the single server G-queues with correlated arrivals of positive and negative customers are considered. As the most close related publications to the present paper, we mention the papers (Anisimov, Artalejo, 2001), (Shin, 2005), (Harrison, 2002) where multi-server queues with negative customers are considered. In (Anisimov, Artalejo, 2001) the $M/M/N$ retrial queue is investigated where a negative customer deletes a batch of secondary customers of random size. The stationary distribution is calculated and the system behavior in overload regime is studied. The paper (Shin, 2005) deals with multi-server retrial queue with finite buffer, exponentially distributed service time and four types of customers (positive customers, two types of negative customers and disasters) which arrive according the $MMAP$ (Marked Markovian Arrival Process). The problem of the approximate calculation of the stationary distribution is solved. The subject of (Harrison, 2002) is the sojourn time distribution in the $MMCPP/GE/c$ queue with negative customers.

In the present paper we consider the model $BMAP/PH/N/R$ with the MAP flow of negative customers. A negative customer can delete an ordinary customer in service if at the moment of the negative customer arrival the state of the PH service process of the ordinary customer belongs to some definite set. In opposite case an ordinary customer is considered to be protected of the effect of negative customers. To resolve conflicts arising at arrival epochs in consequence of finite number of places in the system and unlimited group size in the input flow, we consider the discipline of partial admission to the system. We calculate the stationary distribution and the main performance characteristics of the system such as the probabilities of a positive customer loss in consequence of absence of free places in the system or /and because of the effect of negative customers. We give the simple numerical examples which illustrate the behavior of these probabilities depending on the ordinary and negative customer arrival rates and service rate.

The rest of the paper is organized as follows. In section 2, the mathematical model is described. In section 3, the process of the system states is defined and the stable algorithm for calculating the stationary distribution of the system states is presented. The formulas for unsuccessful service probabilities are derived in section 4. Section 5 contains the numerical examples.

2 Model description

We consider an N-server queueing system with a buffer of size $R < \infty$. Ordinary (positive) customers arrive to the system according to the $BMAP$. To the present day, the $BMAP$ (Batch Markovian Arrival Process)(see, e.g., (Lucantoni, 1991)) is the most popular mathematical model for the telecommunication networks traffic because it catches the typical features of this traffic such as correlation and burstiness. In the $BMAP$, the customers arrival is directed by an irreducible continuous time Markov chain ν_t, $t \geq 0$, (directing process) with the state space $\{0, 1, \ldots, W\}$. The intensities of transitions of the Markov chain ν_t, $t \geq 0$, which are accompanied by arrival of a batch consisting of k customers, are described by the matrices D_k, $k \geq 0$, with the generating function $D(z) = \sum\limits_{k=0}^{\infty} D_k z^k, |z| < 1$. For more information about the $BMAP$, its properties, partial cases, usefulness in telecommunication networks modeling and related research, see (Chakravarthy, 2001).

The system under consideration has finite waiting space. Due to possibility of group arrivals, it can occur that there are free places in the system at arrival epoch, however the number of these places is less than the number of customers in a group. In such situation the acceptance of customers to the system is realized according to the partial admission discipline (only a part of the group corresponding to the number of free places is allowed to enter the system while the rest of the group is lost).

It is assumed that all servers are identical and independent of each other. Service time of a customer by a server has PH type distribution with irreducible representation $(\boldsymbol{\beta}, S)$. It means the following. Service time is interpreted as the time until the continuous time Markov chain m_t, $t \geq 0$, with the state space (set of phases) $\{1, \ldots, M+1\}$ reaches the absorbing state (phase) $M+1$. Transitions of the chain m_t, $t \geq 0$, within the state space $\{1, \ldots, M\}$ are defined by the sub-generator S while the intensities of transitions into the absorbing state are defined by the vector $S_0 = -S\mathbf{e}$. At the service beginning epoch, the state of the process m_t, $t \geq 0$, is chosen within the state space $\{1, \ldots, M\}$ according to the probabilistic row vector $\boldsymbol{\beta}$. It is assumed that the matrix $S + S_0\boldsymbol{\beta}$ is an irreducible one.

For more information about the PH type distribution, see, e.g., (Neuts, 1981).

Negative customers arrive to the system according to the MAP. The MAP is defined by the state space $\{0, 1, \ldots, V\}$ of underlying process $\eta_t, t \geq 0$, and by the matrix generating function $H(z) = H_0 + H_1 z, |z| \leq 1$. We suppose that there exist a set of phases of service process which are protected of the effect of negative customers. Without loss of generality we define this set as $\{j+1, \ldots, M\}$. A negative customer with equal probability goes to any busy server. If the state of the PH service process of an ordinary customer, which occupies the server-target, belongs to the set $\{1, \ldots, j\}$ the negative customer deletes the ordinary customer being in service. In the opposite case, the ordinary customer is considered to be protected of the effect of negative customers. In such case the negative customer leaves the system without any effect.

3 Stationary distribution

The process of the system states is described in terms of the irreducible continuous-time Markov chain $\xi_t = \{i_t, \nu_t, \eta_t, m_t^{(1)}, \ldots, m_t^{(min\{i_t, N\})}\}$, $t \geq 0$, where i_t is the number of customers in the system, ν_t and η_t are the states of the $BMAP$ and the MAP underlying processes respectively, $m_t^{(r)}$ is the state of the PH service process on the r-th busy server at time t (the busy servers are numerated in order of their occupying, i.e., the server, which begins service, is appointed the maximal number among all busy servers; when some server finishes the service or becomes free as a result of a negative customer effect, the rest busy severs are enumerated in the above manner), $\nu_t = \overline{0, W}$, $\eta_t = \overline{0, V}, m_t^{(r)} = \overline{1, M}, r = \overline{1, min\{i_t, N\}}, i_t = \overline{0, N+R}$. We suppose that the states of the ξ_t, $t \geq 0$, are enumerated in lexicographic order.

In the sequel we will use the following denotations:

I_a is an identity matrix of size a, $I_0 = 1$; I is an identity matrix of appropriate dimension; $\bar{I}_M$ is a diagonal matrix of size M having zeroes as the first j diagonal entries and 1's as the rest of diagonal; $\boldsymbol{e}_a$ ($\boldsymbol{0}_a$) is a column vector of size a, consisting of 1's (zeroes); $\hat{\boldsymbol{e}}_M$ is a column vector of size M, having 1's as the first j entries and zeroes as the rest entries; $\otimes$ and $\oplus$ are the symbols of Kronecker product and sum of matrices, see, e.g., (Graham, 1981); $A^{\otimes l} = \underbrace{A \otimes \ldots \otimes A}_{l}$, $l \geq 1$, $A^{\otimes 0} = 1$; $A^{\oplus l} = \sum\limits_{m=0}^{l-1} I_{M^m} \otimes A \otimes I_{M^{l-m-1}}$, $l \geq 1$; $\mathcal{H}^{(i)} = i^{-1} I_{\bar{W}} \otimes H_1 \otimes \hat{\boldsymbol{e}}_M^{\oplus i}$, $i = \overline{1, N}$; $\mathcal{H} = N^{-1} I_{\bar{W}} \otimes H_1 \otimes (\hat{\boldsymbol{e}}_M \boldsymbol{\beta})^{\oplus N}$; $\bar{\mathcal{H}}^{(i)} = i^{-1} H_1 \otimes \bar{I}_M^{\oplus i}$, $i = \overline{1, N}$; $\bar{W} = W + 1$; $\bar{V} = V + 1$.

Lemma 1. *Infinitesimal generator A of the Markov chain ξ_t, $t \geq 0$, has the block structure $A = (A_{i,j})_{i,j=\overline{0,N+R}}$, where the non-zero blocks have form*

$$A_{i,i-1} = \begin{cases} I_{\bar{W}\bar{V}} \otimes \boldsymbol{S}_0^{\oplus i} + \mathcal{H}^{(i)}, \ i = \overline{1, N}, \\ I_{\bar{W}\bar{V}} \otimes (\boldsymbol{S}_0 \boldsymbol{\beta})^{\oplus N} + \mathcal{H}, \\ i = \overline{N+1, N+R}, \end{cases}$$

$$A_{i,i} = \begin{cases} D_0 \oplus H(1), \ i = 0, \\ D_0 \oplus (H_0 \oplus S^{\oplus \min\{i,N\}} + \bar{\mathcal{H}}^{(\min\{i,N\})}), \\ i = \overline{1, N+R-1}, \\ D(1) \oplus (H_0 \oplus S^{\oplus N} + \bar{\mathcal{H}}^{(N)}), \ i = N+R, \end{cases}$$

$$A_{i,i+k} = \begin{cases} D_k \otimes I_{\bar{V}M^i} \otimes \boldsymbol{\beta}^{\otimes \min\{k, N-i\}}, \\ k = \overline{1, N+R-i-1}, \ i = \overline{0, N}, \\ D_k \otimes I_{\bar{V}M^N}, \ k = \overline{1, N+R-i-1}, \\ i = \overline{N+1, N+R-1}, \\ \left(D(1) - \sum\limits_{l=0}^{k-1} D_l \right) \otimes I_{\bar{V}M^i} \otimes \boldsymbol{\beta}^{\otimes (N-i)}, \\ k = N+R-i, \ i = \overline{0, N}, \\ \left(D(1) - \sum\limits_{l=0}^{k-1} D_l \right) \otimes I_{\bar{V}M^N}, \\ k = N+R-i, \ i = \overline{N+1, N+R-1}. \end{cases}$$

The lemma is proved by analyzing the intensities of the Markov chain ξ_t, $t \geq 0$, transitions.

Let $p(i, \nu, \eta, m^{(1)}, \ldots, m^{(min\{i,N\})})$, $\nu = \overline{0, W}$, $\eta = \overline{0, V}$, $m_t^{(r)} = \overline{1, M}$, $r = \overline{1, min\{i, N\}}$, $i = \overline{0, N+R}$ be the steady state probabilities of the Markov chain ξ_t, $t \geq 0$, and $\boldsymbol{p}_i$ be the row vector of these probabilities corresponding to the state i of the first component of the chain, $i = \overline{0, N+R}$. The vectors $\boldsymbol{p}_i$, $i = \overline{0, N+R}$, satisfy Chapman-Kolmogorov's equations (equilibrium equations)

$$\sum_{j=0}^{N+R} \boldsymbol{p}_i A_{i,j} = 0, \ j = \overline{0, N+R}, \ \sum_{i=0}^{N+R} \boldsymbol{p}_i \boldsymbol{e} = 1. \quad (1)$$

The rank of the system is equal to $\bar{W}\bar{V}(\frac{M^{N+1}-1}{M-1} + RM^N)$ and can be very large. E.g., in case $\bar{W} = 2$, $\bar{V} = 2$, $M = 2$, $N = 4$, $R = 4$ the rank is equal to 380, in case $N = 5$ it is equal to 768, etc. Thus, the direct solution of system (1) can be time and resource consuming. Fortunately, the matrix A has special, upper-block hessenbergian structure. It allows to develop the stable effective procedure for solving the system. Such a procedure is presented in the following statement.

Theorem 1. *The stationary probability vectors $\boldsymbol{p}_i$, $i = \overline{0, N+R}$, are calculated as follows:*

$$\boldsymbol{p}_l = \boldsymbol{p}_0 F_l, \ l = \overline{1, N+R},$$

where the matrices F_l are calculated recurrently:

$$F_l = (\bar{A}_{0,l} + \sum_{i=1}^{l-1} F_i \bar{A}_{i,l})(-\bar{A}_{l,l})^{-1}, \ l = \overline{1, N+R-1},$$

$$F_{N+R} = (A_{0,N+R} + \sum_{i=1}^{N+R-1} F_i A_{i,N+R})(-A_{N+R,N+R})^{-1},$$

the matrices $\bar{A}_{i,l}$ are calculated from the backward recursion

$$\bar{A}_{i,N+R} = A_{i,N+R}, \ i = \overline{0, N+R},$$

$$\bar{A}_{i,l} = A_{i,l} + \bar{A}_{i,l+1} G_l,$$

$$i = \overline{0, l}, \ l = N+R-1, N+R-2, \ldots, 0,$$

the matrices G_i, $i = \overline{0, N+R-1}$ are calculated from the backward recursion

$$G_i = (-A_{i+1,i+1} - \sum_{l=1}^{N+R-i-1} A_{i+1,i+1+l} G_{i+l} G_{i+l-1} \ldots G_{i+1})^{-1} A_{i+1,i},$$

$$i = N+R-1, N+R-2, \ldots, 0,$$

the vector $\boldsymbol{p}_0$ is calculated as the unique solution of the system:

$$\boldsymbol{p}_0 \bar{A}_{0,0} = 0, \qquad \boldsymbol{p}_0 (\sum_{l=1}^{N+R} F_l \boldsymbol{e} + \boldsymbol{e}) = 1.$$

The proof of the theorem is analogous to the proof of theorem 1 in (Klimenok, Kim, Orlovsky, Dudin, 2005).

The given algorithm operates with the matrices of which size does not exceed the value $\bar{W}\bar{V}M^N$. The stability of the algorithm is explained by the fact that all matrices included into recursions are nonnegative.

4 Performance measures

Having the stationary distribution $\boldsymbol{p}_i$, $i = \overline{0, N+R}$, been calculated we can find a number of stationary performance characteristics of the considered system. The most important of them are the loss probability and the probability of breaking a service by negative customers. We call these probabilities as unsuccessful service probabilities and define them in the next statement.

Theorem 2. *A collection of unsuccessful service probabilities is evaluated as follows.*

- *Probability that an arbitrary customer will be lost because of an absence of free places in the system*

$$P_{loss} = \tag{2}$$

$$\frac{1}{\lambda}\sum_{i=0}^{N+R} \boldsymbol{p}_i \sum_{k=N+R-i+1}^{\infty} kD_k \otimes I_{\bar{V}M^{min\{i,N\}}}\boldsymbol{e}.$$

- *Probability that an arbitrary customer will be deleted by a negative customer*

$$P_{break} = \tag{3}$$

$$\frac{1}{\lambda}\sum_{i=1}^{N+R} \boldsymbol{p}_i (I_{\bar{W}} \otimes \frac{1}{min\{i,N\}} H_1 \otimes \hat{\mathbf{e}}_M^{\oplus min\{i,N\}})\boldsymbol{e}.$$

- *Probability that an arbitrary customer will be lost or deleted by a negative customer*

$$P_{failure} = P_{loss} + P_{loss}. \tag{4}$$

- *Probability that an arbitrary customer entering the system, will be deleted by a negative customer*

$$\hat{P}_{break} = (1 - P_{loss})^{-1} P_{break}.$$

Proof. According to the formula of the total probability, the probability P_{loss} is calculated as

$$P_{loss} = 1 - \sum_{i=0}^{N+R-1}\sum_{k=1}^{\infty} P_k P_i^{(k)} R^{(i,k)}, \tag{5}$$

where P_k is a probability that an arbitrary customer arrives in a batch consisting of k customers; $P_i^{(k)}$ is a probability to see i customers in the system at the epoch of the $k-$ size batch arrival; $R^{(i,k)}$ is a probability that an arbitrary customer will not be loss conditional it arrives in a batch consisting of k customers and see i customers in the system.

It can be shown that

$$P_i^{(k)} = \frac{\boldsymbol{p}_i D_k \otimes I_{\bar{V}M^{min\{i,N\}}}\mathbf{e}}{\boldsymbol{\theta} D_k \mathbf{e}}, \tag{6}$$

$$i = \overline{0, N+R-1}, k \geq 1,$$

$$P_k = \frac{k\boldsymbol{\theta} D_k \mathbf{e}}{\boldsymbol{\theta}\sum\limits_{l=1}^{\infty} lD_l \mathbf{e}} = k\frac{\boldsymbol{\theta} D_k \mathbf{e}}{\lambda}, k \geq 1, \tag{7}$$

$$R^{(i,k)} = \tag{8}$$

$$\begin{cases} 1, \ k \leq N+R-i, \\ \frac{N+R-i}{k}, \ k > N+R-i, \ i = \overline{0, N+R-1}. \end{cases}$$

By substituting (6)-(8) into (5) after some algebra we get (2).

Expression (3) for P_{break} stems from the following formula:

$$P_{break} = \tag{9}$$

$$\frac{1}{\lambda}\sum_{i=1}^{N+R} \boldsymbol{p}_i (I_{\bar{W}} \otimes \frac{1}{min\{i,N\}} H_1 \otimes (I - \bar{I}_M)^{\oplus min\{i,N\}})\boldsymbol{e}.$$

The sum in (9) is the average intensity of the flow of negative customers which meet the system non empty and go to a busy server with service process being on phases $1, 2, ..., j$. Every of such negative customer deletes an ordinary customer in a service. So, the sum under consideration gives the intensity of ordinary customers which leave the system being deleted by negative customers. The ratio of this intensity to the average intensity λ of input flow gives the probability P_{break}. Using the relation $(I - \bar{I}_M)^{\oplus min\{i,N\}}\boldsymbol{e} = \hat{\mathbf{e}}_M^{\oplus min\{i,N\}}\boldsymbol{e}$ we reduce formula (9) to form (3).

Formula (4) for the probability $P_{failure}$ is evident.

And, finally, the expressions for conditional probability $\hat{P}_{break}$ is derived in trivial way by using the formula for conditional probability.

5 Numerical examples

Present the results of four experiments. The goal of the first experiment is to analyze the influence of the $BMAP$ intensity on the probabilities of unsuccessful service. In the second experiment we are interested in how the service rate impacts on the probabilities of the unsuccessful service. In the third experiment the dependence of unsuccessful service probabilities of the negative customer arrival rate is shown. And in the fourth experiment unsuccessful service probabilities for the case of unprotected service are compared with the corresponding probabilities for the case when one of two phases of a service is assumed to be protected of the effect of negative customers.

We suppose that the PH service process is characterized by the vector $\boldsymbol{\beta} = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and the matrix $S = \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix}$. It means that the service time has Erlangian distribution of order 2 with the service rate $\mu = 1$ and squared variation coefficient $(c_{var})^2 = 0.5$.

We consider $BMAP$ having the fundamental rate $\lambda = 5$ and geometric distribution (with parameter $q = 0.8$) of a number of customers in a batch. The matrices D_k are defined as $D_k = Dq^{k-1}(1-q)/(1-q^{10})$, $k = \overline{1,10}$, and then normalized to get the rate $\lambda = 5$,

$$D_0 = \begin{pmatrix} -25.53984 & 0.393329 & 0.361199 \\ 0.14515 & -2.2322 & 0.200007 \\ 0.295961 & 0.3874445 & -1.752618 \end{pmatrix},$$

$$D = \begin{pmatrix} 24.24212 & 0.466868 & 0.076323 \\ 0.034097 & 1.666864 & 0.186082 \\ 0.009046 & 0.255481 & 0.804685 \end{pmatrix}.$$

The $BMAP$ has the squared variation coefficient $(c_{var})^2 = 4$ and the correlation coefficient $c_{cor} = 0.3$.

The MAP of the negative customers is described by the matrices

$$H_0 = \begin{pmatrix} -15.732675 & 0.606178 & 0.592394 \\ 0.517817 & -2.289675 & 0.467885 \\ 0.597058 & 0.565264 & -1.959664 \end{pmatrix},$$

$$H_1 = \begin{pmatrix} 14.15020 & 0.302098 & 0.081805 \\ 0.107066 & 1.032280 & 0.164627 \\ 0.085830 & 0.197946 & 0.513566 \end{pmatrix}.$$

The MAP has the rate $h = 5$, the squared variation coefficient $(c_{var})^2 = 2$ and the correlation coefficient $c_{cor} = 0.2$.

We investigate the systems with $N = 4$ servers having $R = 4$ waiting positions.

In the first experiment we vary the $BMAP$ fundamental rate by multiplying the matrices D_k, $k = \overline{0,10}$, by some positive number. In this way any desired value of the system load $\rho = \lambda/(N\mu)$ can be obtained. The dependence of unsuccessful service probabilities of the system load is shown on figure 1.

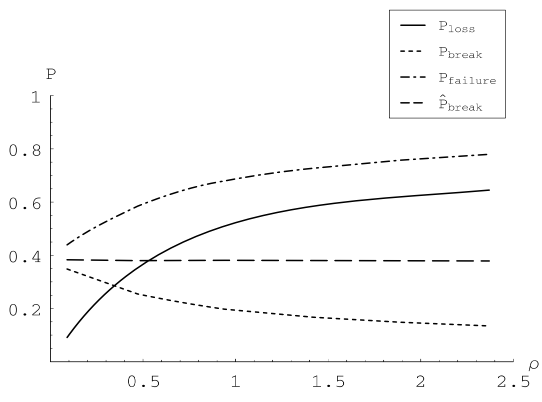

Figure 1: Dependence of unsuccessful service probabilities of the system load ($BMAP$ intensity is changed)

In the second experiment we vary the service rate by multiplying the matrix S by some positive number. In this case the dependence of unsuccessful service probabilities of the system load is shown on figure 2.

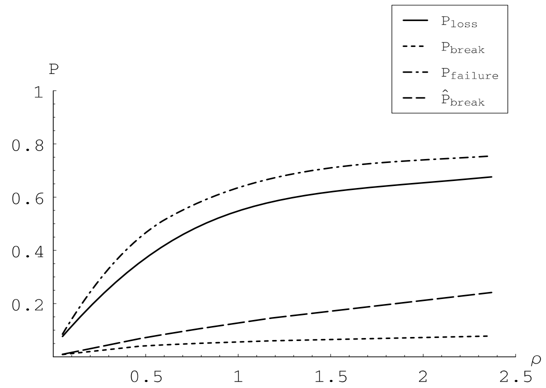

Figure 2: Dependence of unsuccessful service probabilities of the system load (PH intensity is changed)

In the third experiment we vary the negative customers arrival rate by multiplying the matrices H_0, H_1 by some positive number. The dependence of unsuccessful service probabilities of the negative customer arrival rate h is shown on figure 3.

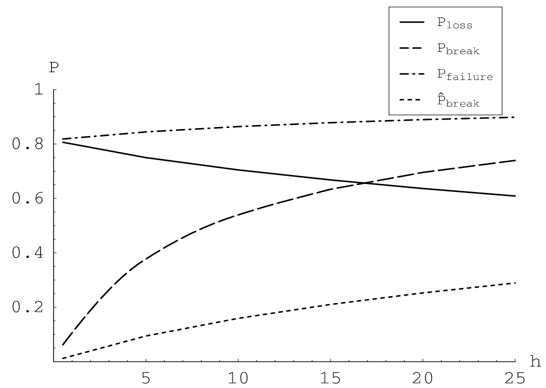

Figure 3: Dependence of unsuccessful service probabilities of the negative MAP intensity

In the last experiment we compare unsuccessful service probabilities for the case $j = 1$ (the second phase is protected of the effects of negative customers) and for the case $j = 2$ (the service process has no protected phases). The behavior of these probabilities depending on the system load is shown on figure 4 - figure 6. Here the variation of the system load is realized by means of the $BMAP$ intensity variation.

6 Conclusion

We have considered the $BMAP/PH/N/R$ G-

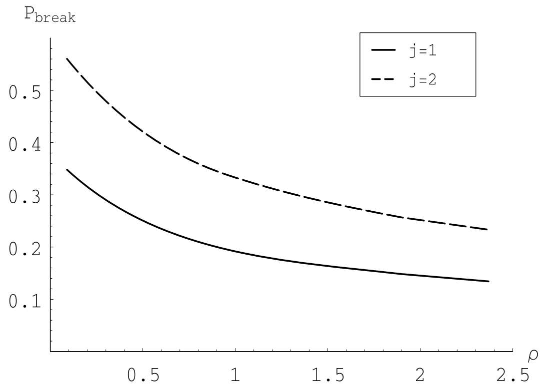

Figure 4: Dependence of the probability P_{break} of the system load in the cases $j = 1$ and $j = 2$

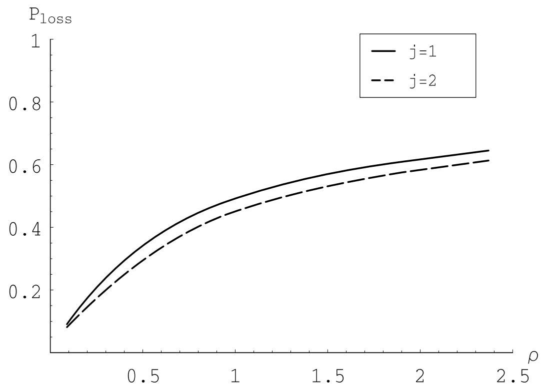

Figure 5: Dependence of the probability P_{loss} of the system load in the cases $j = 1$ and $j = 2$

queue with partial admission discipline and partial protection of service. We calculated the stationary distribution of the system and the collection of unsuccessful service probabilities including the probability to lose a customer due to the absence of free places in the system and the probability of service interruption because of the negative customer effect. The numerical examples illustrating the behavior of these probabilities depending on the system parameters are presented.

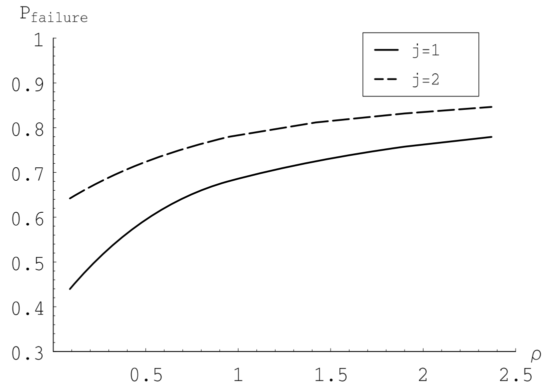

Figure 6: Dependence of the probability $P_{failure}$ of the system load in the cases $j = 1$ and $j = 2$

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BROWNIAN RATCHET WITH TIME-DELAY ON PERIODIC SURFACE

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KEYWORDS

Brownian Ratchet, Stochastic Differential Delayed Equations, Mass Separation.

ABSTRACT

The motion of Brownian particles in ratchet-like potentials (Magnasco 1993) has attracted great interest due to its wide applications in connection with transport processes in many fields including nanotechnologies.

In many applications, a delay time elapses between the cause of a certain phenomenon and its subsequent effect. For example, in Entropy Trap Arrays (Tessier 2002) transport occurs when the configuration of a flexible polymer is deformed in order to pass through medium spatial constraints. The number of accessible configurations depends on the present as well as on the past position of the polymer. Differential equations with time delay are often used to take into consideration such phenomena.
We consider a model of single particle moving in one dimensional periodic potential subject to thermal noise, additional forcing which is independent of particle position, and time-delayed feedback. We introduce feedback in such a way that when the delay vanishes, the model reduces to the well-studied case of Brownian motion in the periodic potential (Risken 1996).

INTRODUCTION

Recently, there has been a growing interest in the effect of noise on dynamical system with delays. In many natural and physical situations memory effects arise as a consequence of delay mechanism, see references in (Guillouzic 2000) for examples in physics, biology, psychology, etc.

One interesting application area is transport of particles in ratchet devices. One of the properties of deterministic ratchets is the presence of parameter regions where dynamics of the system is chaotic. Introduction of delayed coupling to two intertial deterministic ratchets driven by a periodic time-dependent external force in an asymmetric period may lead to anticipated synchronization (Kostur 2005).

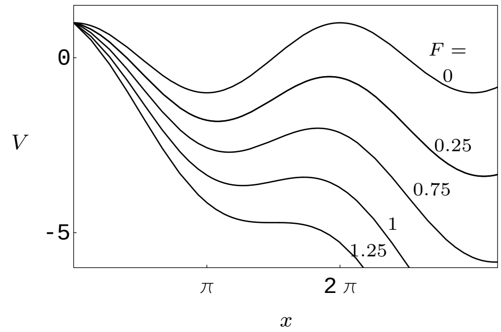

Figure 1: The total potential $V(x) = f(x) - Fx$ for potential $f(x) = \cos x$ and forcing values $F = 0$, 0.25, 0.5, 0.75, 1 and 1.25.

Restricting here to the one-dimensional case, we deal with particles which are kicked randomly around by Langevin forces and move in a one-dimensional periodic potential. Because of the excitation due to the presence of noise, the particles may leave the well and go either to the neighboring left or right well or they may move in the course of time to the other wells which are further away. If we apply an additional force F, (this could be electric current in the case of electrophoresis for example), the particles will preferably diffuse in the direction of this force. Figure 1 shows The total potential $V(x) = f(x) - Fx$ for forcing values $F = 0$, 0.25, 0.5, 0.75, 1 and 1.25 and $A = 1$ when the potential is $f(x) = \cos x$.

The novel aspect of the current work is the feedback in the system introduced by a delay term in the model. In Entropy Trap Arrays (Tessier 2002), which are nanoscale devices consisting of shallow and flat regions, a flexible polymer is deformed in order to pass through medium spatial constraints as it is transported. To mimic the geometry of the device, we choose a cosine potential as a simplest possible model, and include a delay term to account for non-local constraints on the trajectory.

The equation of motion for the coordinate $x(t)$ of a representative particle is a Langevin equation with additive noise:

$$\gamma\, dx(t) = [F - (1-\varepsilon) f'(x(t)) - \varepsilon f'(x(t-\tau))]\, dt$$

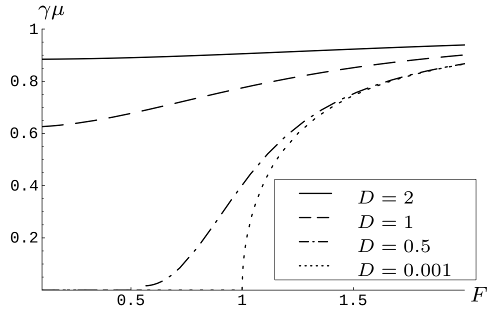

Figure 2: The mobility times damping constant, as a function of force, F, for various noise intensities, D.

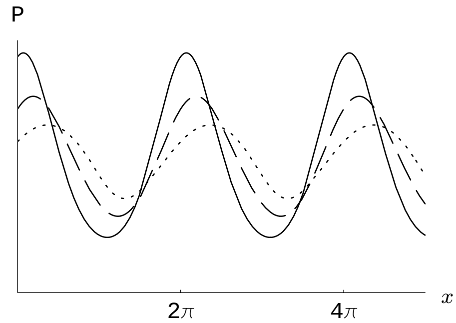

Figure 3: Pseudo steady-state probability density for different values of ε, 0.25 (solid line), 0.5 (dashed line) and 0.75 (dotted line). Parameters are $F = 0.5$, $\tau = 0.5$ and $D = 1$.

$$+\sqrt{2D}dW(t), \qquad (1)$$

with potential

$$f(x) = -A\cos(x). \qquad (2)$$

Here γ is a constant friction coefficient, $x(t)$ is the position of the particle, A is the amplitude of potential (we take $A = 1$ throughout this paper), D is the diffusion coefficient, and $W(t)$ denotes white noise with zero mean and correlation given by $\langle W(t)W(s)\rangle = \delta(t-s)$. The delay effect involves two parameters, the strength ε and delay time τ. In the limit of vanishing ε, our system dynamics reduce to Brownian motion in periodic potential (Risken 1996).

One of quantities that characterizes transport in ratchet-like devices is current:

$$< j >= \lim_{T\to\infty} \frac{\langle x(T) - x(t_0)\rangle}{T}. \qquad (3)$$

(Angle brackets denote averaging over an ensemble of experiments). A related quantity is the mobility μ, which is the current divided by the applied external force:

$$\mu = \frac{< j >}{F}. \qquad (4)$$

This paper is organized as follows. First, we describe properties of model without delay, which helps to identify certain regimes of the model. Later we present simulation results for the Stochastic Deleyed Differential Equation (SDDE) and draw conclusions.

PROPERTIES OF MODEL WITHOUT DELAY

In the limit of vanishing ε, our system dynamics reduces to Brownian motion in periodic potential, given by

$$\gamma\, dx(t) = [F - f'(x(t))]\, dt + \sqrt{2D}dW(t) \qquad (5)$$

with periodic potential

$$f(x) = -A\cos(x). \qquad (6)$$

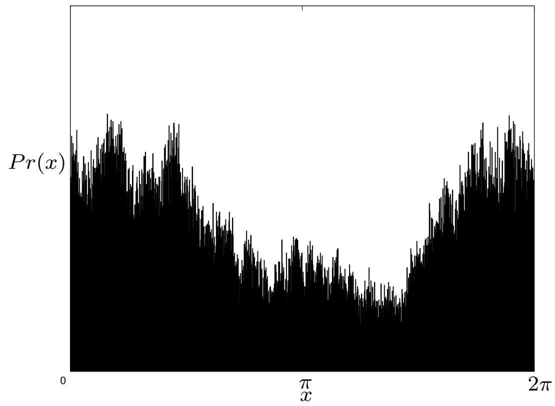

Figure 4: Pseudo steady-state probability (numerical simulations) for $\varepsilon = 0.75$. Parameters are $F = 0.5$, $\tau = 0.5$ and $D = 1$.

For the cosine potential (6) it is possible to obtain an analytical formula for the mobility by using an expansion in terms of continued fractions (Risken 1996):

$$\gamma\,\mu = 1 - \langle\sin x\rangle A/F, \qquad (7)$$

where $\langle\sin x\rangle$ is the imaginary part of continued fraction

$$\langle\cos x\rangle - i\langle\sin x\rangle = 2\frac{0.25}{D/A + iF/A + \frac{0.25}{2D/A+iF/A+\frac{0.25}{3D/A+iF/A+\ldots}}}.$$

The dependence of mobility μ on strength of force, F, for various noise intensities, $D = 10^{-3}, 10^{-1}, 1, 2$ is shown in Figure 2. These results indicate three scenarios for noise-assisted transport in the periodic potential: (i) at low intensities of noise, nonzero transport occurs only for loading $F/A > 1$, (ii) for intermediate intensities of noise, current shows aa nonlinear dependency on force strength, and (iii) for high intensities of noise, the mobility is almost independent of the forcing.

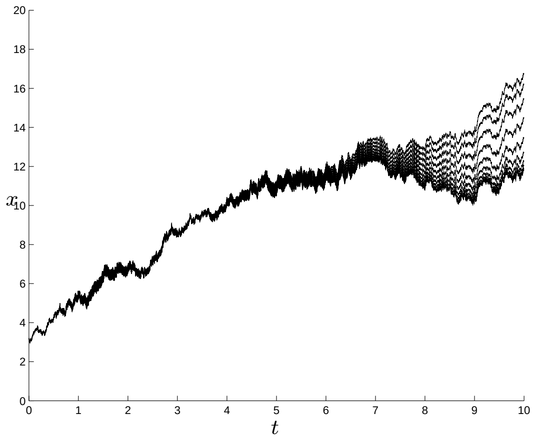

Figure 5: Trajectories of particle for different values of time delay, from $\tau = 0$ (bottom trajectory), to $\tau = 1$ (top trajectory). The parameters are $F = 1.2$, $D = 0.5$ and $\varepsilon = 0.5$.

APPROXIMATE FOKKER-PLANCK EQUATION FOR DELAYED CASE

An expansion in powers of τ can be used to obtain an approximate nondelayed stochastic equation from a SDDE (Guillouzic 2000). Consider the general SDDE:

$$dx = \alpha(x(t), x(t-\tau))dt + \sqrt{2D}dW. \tag{8}$$

Applying a Taylor expansion in terms of τ to Eq. (8) yields the approximate SDE (see Guillouzic 2000)

$$dx = \alpha_a(x)dt + \sqrt{2D}\beta_a(x)dW, \tag{9}$$

where

$$\alpha_a(x) \equiv \alpha(x,x)\big(1 - \tau\frac{\partial}{\partial y}\alpha(x,y)|_{y=x}\big) \tag{10}$$

and

$$\beta_a(x) \equiv 1 - \tau\frac{\partial}{\partial y}\alpha(x,y)|_{y=x}. \tag{11}$$

The Fokker-Planck equation for probability $p_a(x,t|x',t')dx$ that $x(t) \in [x, x+dx]$ then takes the form

$$\begin{aligned}\frac{\partial}{\partial t}p_a(x,t|x',t') = &-\frac{\partial}{\partial x}\big(\alpha_a(x)p_a(x,t|x',t')\big)\\ &+D\frac{\partial^2}{\partial x^2}\big(\beta_a^2(x)p_a(x,t|x',t')\big).\end{aligned} \tag{12}$$

For the SDDE (1), we have the coefficients

$$\begin{aligned}\alpha_a(x) &= (1 + A\varepsilon\tau\cos x)(F - A\sin x),\\ \beta_a(x) &= 1 + A\varepsilon\tau\cos x.\end{aligned} \tag{13}$$

Using a Fourier series expansion for the PDF in (12) with diffusion and drift coefficients given by (13), and with periodic boundary conditions, we can obtain the steady-state

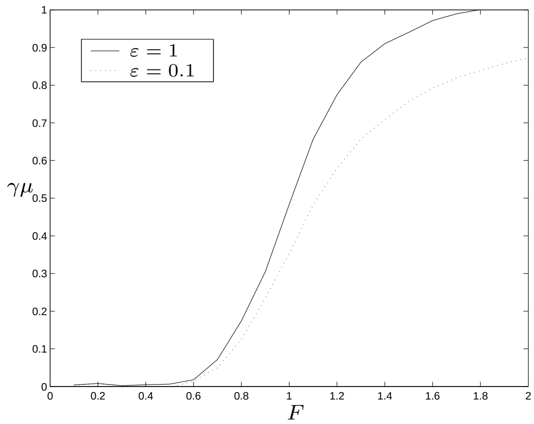

Figure 6: . The mobility times damping constant, as a function of force, F, for $D = 0.1$ and $\tau = 1$.

PDF for the Fokker-Planck equation with periodic boundary conditions. This periodic solution allows us to calculate the current for the SDE approximating the original SDDE. Figure 3 illustrates the influence of strength of delay term ε on the dynamics of the model. As the memory effect is increased, in our case ε takes values from 0.25 to 0.75, the trajectories are more likely to be scattered around minimums of potential. Figure 4 shows the PDF obtained from numerical simulations for $\varepsilon = 0.75$.

RESULTS OF NUMERICAL SIMULATIONS

In the following, we numerically analyze the dynamics of ratchet with time delay Eq. (1). The Langevin equation is solved by employing the second order Runge-Kutta method (Kloeden 1992) for stochastic differential equations, which is modified to incorporate delay terms. We use constant initial conditions to define x value prior the delay, and we consider that the particle is at a minimum of the potential for time interval τ prior to the initial time t_0: $x(t) = \pi$ for $t \in [t_0 - \tau, t_0]$.

Typical transporting trajectories, possessing positive-value transport velocities, are depicted in Figure 5, for parameter values $F = 1.2$, $D = 0.5$ and $\varepsilon = 0.5$. Each trajectory represents simulation for a different value of the delay (range of τ is from 0 to 1) but for identical noise realizations. As a time delay increases, higher values of the single-realization current are observed.

The dependence of mobility μ on the strength of force F, for noise intensity $D = 10^{-1}$ and strength of delay $\varepsilon = 0.1$ and $\varepsilon = 1$ is shown in Figure 6. These results indicate that the presence of delay in ratchet device has significantly increased the mobility for high values of forcing F.

Figures 7 and 8 show numerical results for the absolute change in current, $\frac{|j - j_{\tau=0}|}{|j_{\tau=0}|} \cdot 100\%$ as a function of both τ and ε for noise intensities $D = 0.5$ and $D = 0.1$. Here $j_{\tau=0}$ represents the current for model without delay, and its value

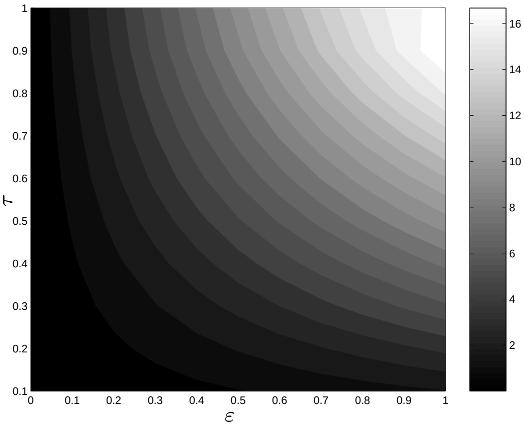

Figure 7: Absolute change in current, $\frac{|j-j_{\tau=0}|}{|j_{\tau=0}|} \cdot 100\%$ for $F = 1.2$ and $D = 0.5$.

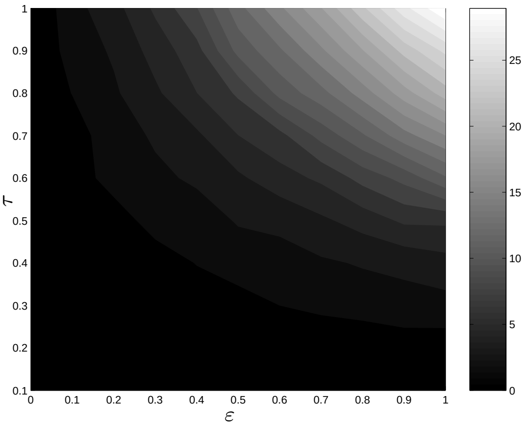

Figure 8: Absolute change in current, $\frac{|j-j_{\tau=0}|}{|j_{\tau=0}|} \cdot 100\%$ for $F = 1.2$ and $D = 0.1$.

can be determined from Fig. 2: $j = 0.78$ for $D = 0.5$ and $j = 0.58$ for $D = 0.1$ at fixed value of $F = 1.2$. It can be observed that the average current is monotonically increasing with both ε (strength of deal) and with τ (delay time). For $\varepsilon = 1$ and $\tau = 1$, the increase in average current is about 16% for $D = 0.5$ (Fig. 7). It is interesting to observe that as the noise intensities decreases, the delay effect becomes more crucial and there is a larger absolute changed in the current compared to higher intensities, about 30% for $D = 0.1$ (Fig. 8).

CONCLUSIONS

We have reported preliminary results on the influence of delay on Brownian motion in periodic potentials. Results of numerical simulation indicate that the presence of the delay in stochastic system can significantly increase average current. Further work will examine the utility of the approximating SDE by comparing the numerical values for SDDE current with the value obtained from the Fokker-Planck equation.

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