# SOLUTION OF FEEDBACK REGULATION PROBLEMS BY APPLICATION OF NETWORK OPTIMIZATION ALGORITHMS

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## ABSTRACT

The paper is devoted to studying general features of nonlinear regulation problems. It is shown, that in the case when the optimal control is determined from the solution of a receding horizon optimal control problem, it may be interpreted as a dynamic network routing problem. There is also one-to-one correspondence between the Bellman optimal cost-to-go function in the shortest path problem and the Lyapunov function in the regulation problem. Hence, to calculate the optimal feedback control one may apply well known and very efficient network optimization algorithms. At the end some results of the application of the routing optimization algorithm to an inverted pendulum regulation problem are presented. They show, that the obtained control rules are very accurate and even have some advantages over those calculated in the classical way.

## GENERAL OPTIMAL CONTROL PROBLEM FORMULATION

We consider a deterministic, stationary discrete-time dynamic system described by the state equation:

$$x_{k+1} = f(x_k, u_k), \quad k = 0, 1, 2, ..., \tau \tag{1}$$

where $x_k, u_k$, such that

$$x_k \in S \tag{2}$$

$$u_k \in U \tag{3}$$

are, respectively, the state and control vectors, and

$$f : S \times U \to S \tag{4}$$

By $S$, $C$ we denoted the subsets of some vector spaces of dimensions, respectively, $n$ and $m$.

For this system we would like to find a closed-loop control strategy

$$\pi = \{\mu_0, \mu_1, \ldots, \mu_\tau\} \tag{5}$$

where $\mu_k(.), k = 0, 1, \ldots, \tau$, is the $k-th$ stage control rule, admissible in the sense of state and control constraints, that is

$$u_k = \mu_k(x_k) \in U, \quad \forall x_k \in S, \tag{6}$$

that minimizes the cost functional:

$$J(x_0) = \sum_{k=0}^{\tau} g\left(x_k, u_k\right) \tag{7}$$

with respect to both the policy $\pi$ and the terminal time $\tau$ (i.e., the control horizon is free).

Let us select from the state space $S$ a point $\bar{x}$. We will assume, that for all $x \neq \bar{x}$ and any $u \in U$

$$g(x, u) > 0 \tag{8}$$

and there exists $\bar{u} \in U$ such that:

$$f(\bar{x}, \bar{u}) = \bar{x} \tag{9}$$

with

$$g(\bar{x}, \bar{u}) = 0 \tag{10}$$

For instance $g$ may be a quadratic function:

$$g(x, u) = (x - \bar{x})' Q (x - \bar{x}) + \\ + (u - \bar{u})' R (u - \bar{u}) \tag{11}$$

where the matrix $Q$ is positive semidefinite and the matrix $R$ is positive definite.

Summing up, we consider an optimal control problem with a fixed terminal state, but free terminal time defined by

$$\min_{\pi} \left\{ J(x_0) = \sum_{k=0}^{\tau} g\left(x_k, u_k\right) \right\} \tag{12}$$

$$x_{k+1} = f\left(x_k, u_k\right) \tag{13}$$

$$u_k = \mu_k(x_k) \in U \tag{14}$$

$$x_0 = \mathtt{x} \tag{15}$$

$$x_\tau = \bar{x} \tag{16}$$

where $\forall k \;\; x_k \in S$.

We assume, that the system (13)-(15) is controllable to the point $\bar{x}$ from every point of the state space.

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## ANALYSIS

We will apply an analysis method inspired by Luenberger (1979).

First, let us notice, that in our problem all functions are time-invariant (stationary). This means, that the solution will not depend on time, either. More precisely, the optimal trajectory from a given state $\mathtt{x}$ to the endpoint $\bar{x}$ is independent of the time $k_0$ at which $x_{k_0} = \mathtt{x}$. That is, if $x_0 = \mathtt{x}$ leads to the optimal trajectory $\{\tilde{x}_k\}$ for $k > 0$ with final time $\tau(\mathtt{x})$, then the condition $x_{k_0} = \mathtt{x}$ must lead to the trajectory $\{\tilde{x}_{k+k_0}\}$ with final time $\tau(\mathtt{x}) + k_0$. Delaying the initial time simply delays the whole solution and the terminal time (i.e., the time of reaching the state $\bar{x}$) is simply an unknown function of the initial state only.

The optimal control rule is also a stationary function, that is for every $k$

$$u_k = \mu^* (x_k) \tag{17}$$

It must be so, because the initial control, as we have just stated, depends only on the initial state, not on the initial time, and we can repeat this reasoning at each time instant. Because of the assumptions (8)-(10) there will be:

$$\mu^* (\bar{x}) = \bar{u} \tag{18}$$

If $\mu^*(.)$ is the optimal control rule, then we will obtain the following closed-loop system equation:

$$x_{k+1} = f(x_k, \mu^*(x_k)) \tag{19}$$

Let us notice, that due to (18) and (9) the point $\bar{x}$ is an equilibrium point of the system (19) and according to the construction of the rule $\mu^*(.)$ the system eventually reaches $\bar{x}$. Hence, the system is stable.

Now, let us analyze formally the stability of the system and consider the optimal value function (that is the Bellman function, sometimes called "the optimal cost-to-go") $V_k(x)$ for this problem, expressed as:

$$V_k(x_k) = \sum_{l=k}^{k+\tau(x_k)} g(x_l, \mu^*(x_l))) \tag{20}$$

where the function $g(.,.)$ is defined by (10). This is the optimal (minimal) cost of the passage to $\bar{x}$ at time $k + \tau(x_k)$ when starting from a point $x_k$ at $k$. This function satisfies the following conditions:

**(i)** $V_k(\bar{x}) = 0$

**(ii)** $V_k(x) > 0$ for $x \neq \bar{x}$

**(iii)** $V_{k+1}(x_{k+1}) - V_k(x_k) = -g(x_k, \mu^*(x_k)) < 0$ for $x_k \neq \bar{x}$

Thus $V$ - the Bellman function is a Lyapunov function and we proved the stability of the system.

## DISCRETE-STATE VERSION

In this section we will assume, that the sets $S$ and $U$ are finite and have, respectively, $T+1$ and $V+1$ elements. For the sake of simplicity we denote them by subsequent integers, that is:

$$S = \{0, 1, 2, 3, \ldots, T\} \tag{21}$$

$$U = \{0, 1, 2, 3, \ldots, V\} \tag{22}$$

Consequently we will have:

$$x_k \in S \subset \mathbb{Z}^n \tag{23}$$

$$u_k \in U \subset \mathbb{Z}^m \tag{24}$$

In these circumstances, for any state $x_k = i \in S$, a control $u_k = u \in U$ can be associated with a transition from the state $x_k = i$ to the state $f(i, u) = j \in S$. This passage is characterized by a cost:

$$c_{ij} = \min_{\substack{u \in U \\ f(i,u)=j}} g(i, u) \tag{25}$$

We assumed, that in the case when there are several controls $u \in U$, such that:

$$f(i, u) = j \tag{26}$$

we choose as the passage cost (25) the minimal cost among all costs corresponding to this passage.

Let us define now as a destination state the state $T \in S$. We will assume that the system may remain in this state, that is

$$\exists u^T \in U \;\; f(T, u^T) = T \tag{27}$$

and that the cost of being in this state equals zero that is:

$$g(T, u^T) = 0 \tag{28}$$

In these conditions the state $T$ is absorbing, that is if the system (1) passes to it, it remains in it for ever.

With this notation, we can interpret our deterministic optimal synthesis problem as a shortest path problem from an initial state 0 to the terminal state $T$ (Fig. 1).

Let us denote now by $N(i)$ the set of all direct neighbours of the node $i$. The optimizing dynamic programming algorithm for this problem will have the form:

$$J(i) = \min_{j \in N(i)} \{c_{ij} + J(j)\} \tag{29}$$

with the terminal condition:

$$J(T) = 0 \tag{30}$$

The above shortest path problem may be solved with the help of the Bellman-Ford algorithm (e.g., used in the Internet routing protocols RIP, IGP or Hello (Comer 2000, Karbowski and Niewiadomska-Szynkiewicz 2001)).

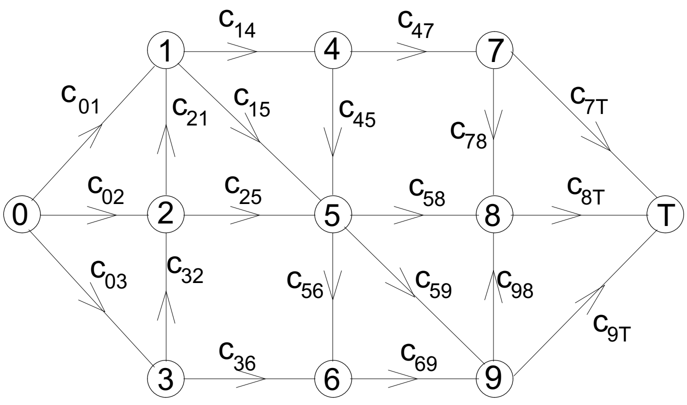


Figure 1: Graph describing deterministic discrete optimal control problem with terminal state

## INTEGRATION

Taking into account conclusions drawn from the previous sections, we can write the following:

1. The optimal control policy in the receding horizon control problem for stationary systems with a Lagrange-type performance index is stationary

2. When the terminal time is free, the optimal closed-loop control problem consists in finding the minimal cost trajectory from any point of the state space to a given point $\bar{x}$.

3. The deterministic closed-loop discrete optimal control problem with a fixed terminal state but free terminal time (i.e. horizon) can be represented as a shortest path problem

Thus, having discretized the problem (12)- (16), connecting all resulting nodes according to the state equation (13) and solving the shortest path problem from all nodes to the node representing the point $\bar{x}$, we can transform the receding horizon optimal control problem into the routing problem.

## APPLICATION OF THE ROUTING ALGORITHM TO THE STABILIZATION OF AN INVERTED PENDULUM

To confirm experimentally the equivalence between routing algorithms and the feedback regulation the presented approach was tested on an example taken from (Kreisselmeier, G. and T. Birkhölzer 1994).

A control law synthesis problem for a simple inverted pendulum was considered. The state variables of this system are the angle $\xi$ and the angular velocity $\dot{\xi}$. The input $u$ is a torque in the shaft, which is bounded to such an amount, that the pendulum cannot directly be turned from the hanging into the upright position. Instead, it is first necessary to "gain enough momentum", which requires a complex trajectory planning, even for this simple system. This nonlinearity poses the main difficulty for the feedback design in this example.

The system is described by the state equation:

$$\dot{x_1}(t) = x_2(t) \tag{31}$$

$$\dot{x_2}(t) = \sin x_1(t) + h(u(t)) \tag{32}$$

where $x_1 = \xi, x_2 = \dot{\xi}$ and $h(.)$ is the linear function with saturation, when the module of its argument exceeds 0.7, that is

$$h(u) = \begin{cases} -0.7 & u \leq -0.7 \\ u & -0.7 < u < 0.7 \\ 0.7 & u \geq 0.7 \end{cases} \tag{33}$$

An interesting feature of the above system is that a continuous state feedback, which asymptotically stabilizes the system for all initial conditions, does not exist! The reason is, that for any continuous feedback there is a different than origin equilibrium point. More precisely, this point has a nonzero first coordinate. It must be so, because the function

$$f(x_1) = \sin x_1 + h(\mu(x_1, 0)) \tag{34}$$

has the positive sign for $x_1 = \pi - arc\sin 0.8$ and the negative sign for $x_1 = \pi + arc\sin 0.8$, which means (from the Darboux theorem) that this function has a root in the interval $[\pi - arc\sin 0.8, \pi + arc\sin 0.8]$. In other words, the dynamic system (31)-(32) has an equilibrium point with a zero second and a nonzero first coordinate.

The system (31)-(32) was discretized under the following conditions:

- the conversion to the discrete-time representation was obtained via the Euler scheme for a sampling interval $T_s = 0.5$
- as the state coordinate $x_1$ space, the interval $[-4, 4]$ was taken; it was discretized into 221 levels
- as the the state coordinate $x_2$ space, the interval $[-1.6, 1.6]$ was taken; it was discretized into 121 levels
- the control space (the interval $[-0.7, 0.7]$) was divided into 20 equal subintervals
- the cost function $g\,(x(t), u(t))$ was assumed to be quadratic, that is

$$g(x, u) = x'Qx + u'Ru \tag{35}$$

with

$$Q = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \tag{36}$$

and $R = 2$.

Several experiments for different initial points were performed. The resulting trajectories of the state and control variables for two simulations starting from hanging freely and horizontal position of the pendulum

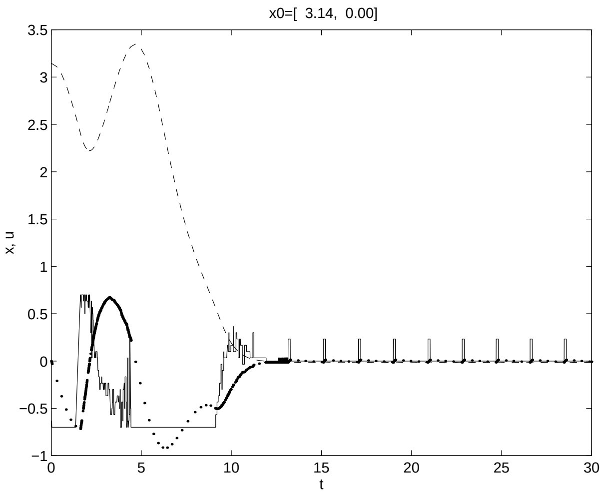


Figure 2: Trajectories $x_1(--), x_2(..), u(-)$ for initial condition $[\pi, 0]$ and RB controller.

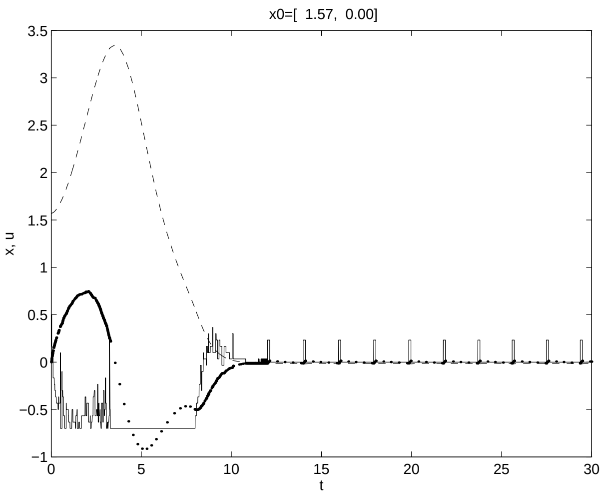


Figure 3: Trajectories $x_1(--), x_2(..), u(-)$ for initial condition $[\frac{\pi}{2}, 0]$ and RB controller.

and are presented in Figures 2,3. The abbreviation RB means Routing Based (controller).

For comparison, next figures (Figs. 4,5) present the same trajectories, obtained with the help of LQ methodology, without saturation of the function $h(.)$ (that is, it was replaced by identity). In those experiments, the system (31)-(32) was linearized in the origin, then the optimal static feedback matrix $K$ (that is $u = K \cdot x$) was calculated, with the help of the Matlab Control Toolbox (procedure 'lqr').

It is seen, that although in both cases the LQ controller was able to stabilize the pendulum, the control $u$ was very big, out of the admissible interval $[-0.7, 0.7]$ of the previous (RB) case.

After the series of experiments it turned out, that in the case when the control constraints are taken into account while implementing the LQ control law, even for

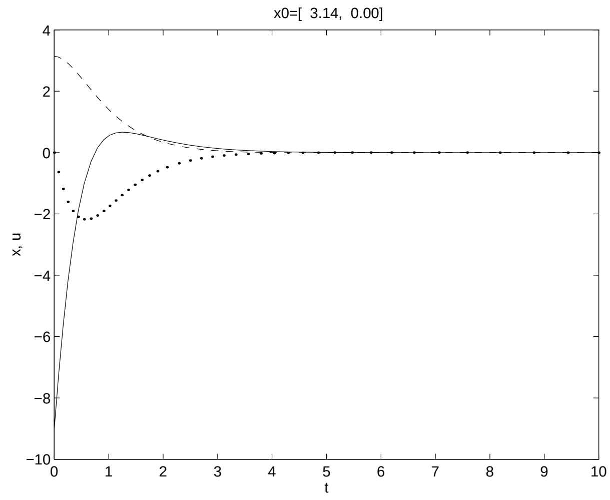


Figure 4: Trajectories $x_1(--), x_2(..), u(-)$ for initial condition $[\pi, 0]$ and LQ controller.

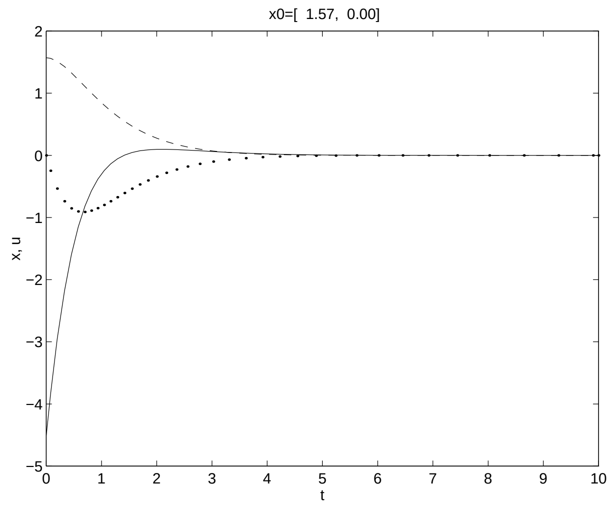


Figure 5: Trajectories $x_1(--), x_2(..), u(-)$ for initial condition $[\frac{\pi}{2}, 0]$ and LQ controller.

much greater values of the coefficient $R$, it is impossible to conduct the pendulum from the free ($[\pi, 0]$) to the upright position (Fig. 6). Let us recall, that it was not a problem for RB controller (Fig. 2). However, after giving the pendulum some momentum, the LQ controller with saturation succeeded in regulating the pendulum to this position (Fig. 7).

## CONCLUSIONS

The paper presented connections between a nonlinear stabilization problem and a network routing problem. The may idea lies in the formulation of the original regulation problem as a set of discrete-time receding horizon control problems, solved for all initial states. The optimal control rule may then be calculated (after

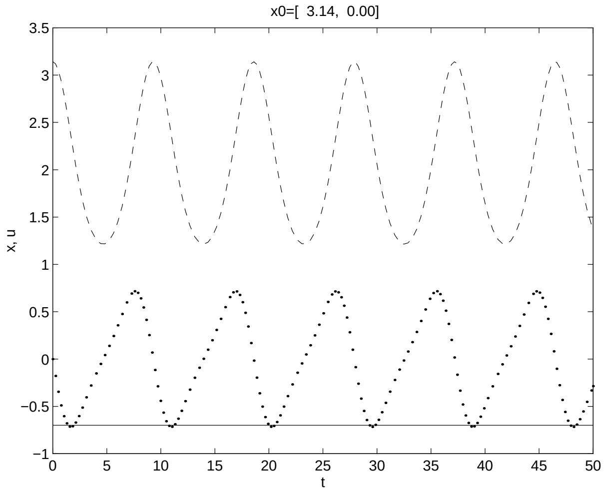


Figure 6: Trajectories $x_1(--), x_2(..), u(-)$ for moving pendulum and LQ controller with saturation for initial condition $[\pi, 0]$.

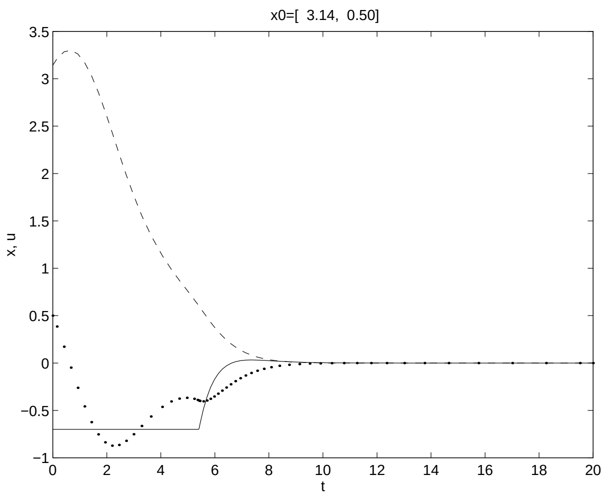


Figure 7: Trajectories $x_1(--), x_2(..), u(-)$ for moving pendulum and LQ controller with saturation for initial condition $[\pi, 0.5]$.

state discretization) by the application of the Bellman-Ford algorithm, which is an elementary method for calculation of the shortest paths in networks.

An inverted pendulum case study results showed, that the regulator obtained in this simple way has some advantages over classical LQ approach: it requires much smaller controls to move the state of the system to the equilibrium point neighbourhood, and it can successfully control the system even for initial conditions lying very far from the equilibrium point (that is, it is global). The drawbacks of this regulator are small oscillations around the terminal point, caused by discretization, and the longer time of regulation. Because of that, the best solution in the case of continuous nonlinear systems would be probably a hybrid regulator: discrete - routing based for points lying far from the terminal point and continuous - LQ methodology based, in its neighbourhood.

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## BIOGRAPHY

**Andrzej Karbowski** received M.Sc. degree in electronic engineering, specialization automatic control from Warsaw University of Technology, Faculty of Electronics, in 1983. He received Ph.D. in 1990 in automatic control and robotics. His scientific interests concentrate on nonlinear feedback control, optimal control in risk conditions, data networks management, decomposition and parallel implementation of numerical algorithms, global optimization. He works as adjunct both at the Faculty of Electronics and Information Technologies of Warsaw University of Technology and at Research and Academic Computer Network (NASK). His scientific results were described in about 15 articles published in international journals and about 40 conference papers.