MINIMUM ENERGY CONTROL PROBLEM OF POSITIVE FRACTIONAL DISCRETE-TIME SYSTEMS

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ABSTRACT

The minimum energy control problem of positive fractional-discrete time linear systems is addressed. Necessary and sufficient conditions for the reachability of the system are established. Sufficient conditions for the solvability of the minimum energy control of the positive fractional discrete-time systems are given. A procedure for computation of the optimal sequence of inputs minimizing the quadratic performance index is proposed.

INTRODUCTION

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.
Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in standard positive systems is given in the monographs (Farina and Rinaldi 2000; Kaczorek 2002). The realization problem for positive standard and singular continuous-time systems with delays was formulated and solved in (Kaczorek 2007c, 2007d). The reachability, controllability and minimum energy control of positive linear discrete-time systems with time-delays have been considered in (Busłowicz and Kaczorek 2004; Xie and Wang 2003). The realization problem for cone systems has been addressed in (Kaczorek 2006). The reachability and controllability to zero of positive fractional linear systems have been investigated in (Kaczorek 2008a, 2007a, 2007b, 2008b; Klamka 2002; Klamka 2005). Mathematical fundamentals of fractional calculus are given in the monographs (Miller and Ross 1993; Nishimoto 1984; Oldham and Spanier 1974; Oustalup 1993; Podlubny 1999). The fractional order controllers have been developed in (Oustalup 1993). A generalization of the Kalman filter for fractional order systems has been proposed in (Sierociuk and Dzieliński 2006). Some other applications of fractional order systems can be found in (Ferreira and Machado 2003; Moshrefi-Torbati and Hammond 1998; Ortigueira 1997; Ostalczyk 2000, 2004a, 2004b; Podlubny 2002; Samko et al. 1993; Vinagre et al. 2002; Vinagre and Feliu 2002; Gałkowski and Kummert 2005).
The minimum energy control problem has been solved for different classes of linear systems in (Klamka 1991, 1976, 1983; Kaczorek and Klamka 1986).
In this paper the minimum energy control problem will be addressed for positive fractional discrete-time linear systems.
The paper is organized as follows. In section 2 the solution of the state equation and the necessary and sufficient conditions for the positivity of the fractional systems are recalled. Necessary and sufficient conditions for the reachability of the positive fractional systems are established in section 3. The main result of the paper is presented in section 4 in which the minimum energy control problem is formulated and solved. Concluding remarks are given in section 5.
To the best knowledge of the author the minimum energy control problem for the positive fractional discrete-time linear systems have not been considered yet.

POSITIVE FRACTIONAL SYSTEMS

Let $\Re^{n\times m}$ be the set of $n\times m$ real matrices and $\Re^n := \Re^{n\times 1}$. The set of $m\times n$ matrices with nonnegative entries will be denoted by $\Re_+^{m\times n}$ and $\Re_+^n := \Re_+^{n\times 1}$. The set of nonnegative integers will be denoted by Z_+ and the $n\times n$ identity matrix by I_n.
In this paper definition of the fractional difference of the form (Kaczorek 2007a)

$$\Delta^{\alpha} x_k = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x_{k-j}, \tag{1}$$
$$n-1<\alpha<n\in N=\{1,2,...\},\ k\in Z_+$$

will be used, where $\alpha \in R$ is the order of the fractional difference and

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$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j=0 \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & \text{for } j=1,2,... \end{cases} \tag{2}$$

Consider the fractional discrete linear system, described by the state-space equations

$$\Delta^{\alpha} x_{k+1} = Ax_k + Bu_k, \quad k \in Z_+ \tag{3a}$$

$$y_k = Cx_k + Du_k \tag{3b}$$

where $x_k \in \Re^n$, $u_k \in \Re^m$, $y_k \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.
Using definition (1) we may write the equations (3) in the form

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + Bu_k, \quad k \in Z_+ \tag{4a}$$

$$y_k = Cx_k + Du_k \tag{4b}$$

Theorem 1. (Kaczorek 2007a) The solution of equation (4a) with initial condition x_0 is given by

$$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1} B u_i \tag{5}$$

where Φ_k is determined by the equation

$$\Phi_{k+1} = (A + I_n \alpha)\Phi_k + \sum_{i=2}^{k+1} (-1)^{i+1} \binom{\alpha}{i} \Phi_{k-i+1} \tag{6}$$

with $\Phi_0 = I_n$.

Definition1. The system (4) is called the (internally) positive fractional system if and only if $x_k \in \Re_+^N$ and $y_k \in \Re_+^p$, $k \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all input sequences $u_k \in \Re_+^m$, $k \in Z_+$.

It is easy to show (Kaczorek 2007a) that for $0 < \alpha < 1$

$$(-1)^{i+1} \binom{\alpha}{i} \Phi_{k-i+1} > 0, \quad i = 1,2,... \tag{7}$$

Theorem 2. (Kaczorek 2007a) The fractional system (4) for $0 < \alpha < 1$ is positive if and only if

$$\begin{aligned} &(A + I_n\alpha) \in \Re_+^{n\times n}, \quad B \in \Re_+^{n\times m}, \\ &C \in \Re_+^{p\times n}, \quad D \in \Re_+^{p\times m}. \end{aligned} \tag{8}$$

From (7), (8) and (5) we have $\Phi_k \in \Re_+^{n\times n}$ for $k = 1,2,...$ and the impulse response matrix

$$g_i = \begin{cases} D & \text{for } i=0 \\ C\Phi_{i-1}B + D & \text{for } i=1,2 \end{cases} \tag{9}$$

of the positive fractional system (4) is nonnegative, $g_i \in \Re_+^{p\times m}$ for $i \in Z_+$.

REACHABILITY

Definition 2. A state $x_f \in \Re_+^n$ of the fractional system (4) is called reachable in q steps if there exists a input sequence $u_k \in \Re_+^m$, $k \in Z_+$ which steers the state from zero initial state $(x_0 = 0)$ to the final state x_f. If every state $x_f \in \Re_+^n$ is reachable in q steps then the system is called reachable in q steps. If for every state $x_f \in \Re_+^n$ there exists a natural number q, such that the state is reachable in k steps then the system is called reachable.

Let e_i, $i = 1,...,n$ be the ith column of the identity matrix I_n. A column ae_i for $a > 0$ is called the monomial column.

Theorem 3. The fractional system (4) is reachable in q steps if and only if the reachability matrix

$$R_q := [B, \Phi_1 B, ..., \Phi_{q-1} B] \tag{10}$$

contains n linearly independent monomial columns.

Proof. Using (5) for $k = q$ and $x_0 = 0$ we obtain

$$x_f = x_q = \sum_{i=0}^{q-1} \Phi_{q-i-1} B u_i = R_q \begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_0 \end{bmatrix} \tag{11}$$

From Definition 2 and (11) it follows that for every $x_f \in \Re_+^n$ there exists a input sequence $u_i \in \Re_+^m$, $i = 0,1,...,q-1$ if and only if the matrix (21) contains n linearly independent monomial columns. ■

Example 1. Consider the fractional system (4) for $0 < \alpha < 1$ with

$$A=\begin{bmatrix}1 & 0\\ 0 & -\alpha\end{bmatrix},\ B=\begin{bmatrix}0\\ 1\end{bmatrix},\ (n=2) \tag{12}$$

The fractional system is positive since

$$A+I_n\alpha=\begin{bmatrix}1+\alpha & 0\\ 0 & 0\end{bmatrix}\in\Re_+^{2\times 2}$$

Using (6) for $k=0,1,...$ we obtain diagonal matrices of the forms

$$\Phi_1=(A+I_n\alpha)\Phi_0=\begin{bmatrix}1+\alpha & 0\\ 0 & 0\end{bmatrix},$$

$$\Phi_2=(A+I_n\alpha)\Phi_1-\binom{\alpha}{2}\Phi_0=$$

$$=\begin{bmatrix}\frac{\alpha^2+5\alpha+2}{2} & 0\\ 0 & \frac{(1-\alpha)\alpha}{2}\end{bmatrix},$$

$$\Phi_3=(A+I_n\alpha)\Phi_2-\binom{\alpha}{2}\Phi_1+\binom{\alpha}{3}\Phi_0=$$

$$=\begin{bmatrix}\frac{(\alpha^2+5\alpha+2)(\alpha+1)-\alpha(\alpha-1)(\alpha+5)}{6} & 0\\ 0 & \frac{\alpha(1-\alpha)(\alpha-2)}{2}\end{bmatrix},...$$

Note that the reachability matrices (10) for $q=2,3,...$ have the forms

$$R_2=[B,\Phi_1 B]=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},$$

$$R_3=[B,\Phi_1 B,\Phi_2 B]=\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & \frac{(1-\alpha)\alpha}{2}\end{bmatrix},$$

$$R_4=\begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 2 & \frac{(1-\alpha)\alpha}{2} & \frac{\alpha(1-\alpha)(\alpha-2)}{6}\end{bmatrix},...$$

and they contain only one linearly independent monomial column. Therefore, by Theorem 3 the system with (12) is unreachable.

Example 2. Consider the fractional system (4) for $0\le\alpha\le 1$ with

$$A=\begin{bmatrix}-\alpha & 0\\ 1 & 2\end{bmatrix},\ B=\begin{bmatrix}1\\ 0\end{bmatrix},\ (n=2) \tag{13}$$

The system is positive since

$$A+I_n\alpha=\begin{bmatrix}0 & 0\\ 1 & 2+\alpha\end{bmatrix}\in\Re_+^{2\times 2}$$

Using (6) for $k=0$ we obtain

$$\Phi_1=(A+I_n\alpha)\Phi_0=\begin{bmatrix}0 & 0\\ 1 & 2+\alpha\end{bmatrix}$$

The reachability matrix (10) for $q=2$ has the form

$$R_q=[B,\Phi_1 B]=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

It contains two linearly independent monomial columns. Therefore, the fractional system with (13) is reachable in two steps.

Remark 1. From (6) and (10) it follows that the positive fractional system (4) is reachable only if the matrix

$$[B,\ A+I_n\alpha] \tag{14}$$

contains *n* linearly independent columns.

MINIMUM ENERGY CONTROL

Problem formulation

Consider the positive fractional system (4). If the system is reachable in *q* steps then there exist many input sequences that steer the state of the system from $x_0=0$ to the final state $x_f\in\Re_+^n$. Among these input sequences we are looking for the sequence $u_i\in\Re_+^m$, $i=0,1,...,q-1$, $i\in Z_+$ that minimizes the performance index

$$I(u)=\sum_{j=0}^{q-1}u_j^T Q u_j \tag{15}$$

where $Q\in\Re^{m\times m}$ is a symmetric positive define matrix and *q* is a given number of steps in which the state of the system is transferred from $x_0=0$ to $x_f\in\Re_+^n$.

The minimum energy control problem for the positive fractional system (4) can be stated as follows. Given matrices *A*, *B* and the order α of the system (4), the number of steps *q*, $x_f\in\Re_+^n$ and the matrix *Q* of the performance index (15), find a sequence of inputs

$u_i \in \Re_+^m$, $i = 0,1,...,q-1$, that steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (15).

Problem solution

To solve the problem we define the matrix

$$W = W(q,Q) = R_q \bar{Q} R_q^T \in \Re^{n\times n} \tag{16}$$

where R_q is defined by (10) and

$$\bar{Q} = \text{block diag}\left[Q^{-1},...,Q^{-1}\right] \in \Re^{qm\times qm} \tag{17}$$

From (16) it follows that the matrix W is nonsingular if and only if $\operatorname{rank} R_q = n$. If the condition is met then the system is reachable in q steps (as standard but not as a positive systems). In this case we may define for a given $x_f \in \Re_+^n$ the following sequence of inputs

$$\hat{u}_{0q} = \begin{bmatrix} \hat{u}_{q-1} \\ \hat{u}_{q-2} \\ \vdots \\ \hat{u}_0 \end{bmatrix} := \bar{Q} R_q^T W^{-1} x_f \tag{18}$$

From (18) it follows that $\hat{u}_i \in \Re_+^m$, $i = 0,1,...,q-1$ if

$$\bar{Q} R_q^T W^{-1} \in \Re_+^{m\times n} \tag{19}$$

and this holds if

$$Q^{-1} \in \Re_+^{m\times m} \tag{20a}$$

and

$$W^{-1} \in \Re_+^{n\times n} \tag{20b}$$

Theorem 4. Let the positive fractional system (4) be reachable in q steps and the conditions (20) are satisfied. Moreover let $\bar{u}_i \in \Re_+^m$, $i = 0,1,...,q-1$ be a sequence of inputs that steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$. Then the sequence of inputs $\hat{u}_i \in \Re_+^m$, $i = 0,1,...,q-1$ defined by (18) also steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (15), i.e.

$$I(\hat{u}) \le I(\bar{u}) \tag{21}$$

The minimal value of (15) for (18) is given by

$$I(\hat{u}) = x_f^T W^{-1} x_f \tag{22}$$

Proof. If the fractional system (4) is positive and reachable in q steps and the assumptions (20) are satisfied then for $x_f \in \Re_+^n$ we have $\hat{u}_i \in \Re_+^m$ for $i = 0,1,...,q-1$. We shall show that the sequence (18) steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$. Using (5) for $k = q, x_0 = 0$ and (18), (16) we obtain

$$x_q = R_q \hat{u}_{0q} = R_q \bar{Q} R^T W^{-1} x_f = x_f \tag{23}$$

since $R_q \bar{Q} R^T W^{-1} = I_n$.

The both sequences of inputs $\bar{u}_{0q}$ and $\hat{u}_{0q}$ steer the state of the system from $x_0 = 0$ to x_f. Hence $x_f = R_q \hat{u}_{0q} = R_q \bar{u}_{0q}$ and

$$R_q[\hat{u}_{0q} - \bar{u}_{0q}] = 0 \tag{24}$$

Using (24) and (18) we shall show that

$$[\hat{u}_{0q} - \bar{u}_{0q}]^T \hat{Q} \hat{u}_{0q} = 0 \tag{25}$$

where $\hat{Q} = \text{block diag}[Q,...,Q]$.

Transposition of (24) yields $[\hat{u}_{0q} - \bar{u}_{0q}]^T R_q^T = 0.$

Postmultiplying the equality by $W^{-1}x_f$ we obtain

$$[\hat{u}_{0q} - \bar{u}_{0q}]^T R_q^T W^{-1} x_f = 0. \tag{26}$$

Using (18) and (26) we obtain (25) since

$$\begin{aligned}[\hat{u}_{0q} - \bar{u}_{0q}]^T \hat{Q} \hat{u}_{0q} &= [\hat{u}_{0q} - \bar{u}_{0q}]^T \hat{Q} \bar{Q} R^T W^{-1} x_f = \\ &= [\hat{u}_{0q} - \bar{u}_{0q}]^T R^T W^{-1} x_f = 0\end{aligned}$$

and $\hat{Q}\bar{Q} = I_{qm}$.

Using (25) it is easy to verify that

$$\bar{u}_{0q}{}^T \bar{Q} \hat{u}_{0q} = \hat{u}_{0q}^T \hat{Q} \hat{u}_{0q} + [\bar{u}_{0q} - \hat{u}_{0q}]^T \hat{Q} [\bar{u}_{0q} - \hat{u}_{0q}] \tag{27}$$

From (27) it follows that the inequality (21) holds, since

$$[\bar{u}_{0q} - \hat{u}_{0q}]^T \hat{Q} [\bar{u}_{0q} - \hat{u}_{0q}] \ge 0$$

To find the minimal value of the performance index we substitute (18) into (15) and we use (16). Then we obtain

$$I(\hat{u}) = \hat{u}_{0q}^T \hat{Q} \hat{u}_{0q} = (\bar{Q} R_q^T W^{-1} x_f)^T \hat{Q} (\bar{Q} R_q^T W^{-1} x_f) =$$
$$= x_f^T W^{-1} R_q \bar{Q} R_q^T W^{-1} x_f = x_f^T W^{-1} x_f$$

since $\hat{Q}\bar{Q} = I_{qm}$ and $W^{-1} R_q \bar{Q} R_q^T = I_n$. ■

If the conditions of Theorem 4 are met then the minimal energy control problem can be solved by the use of the following procedure.

Procedure

Step 1. Knowing A, B, α, q, Q and using (10) and (17) find R_q and $\bar{Q}$

Step 2. Knowing R_q, $\bar{Q}$ and using (16) find the matrix W

Step 3. Using (18) compute the sequence of inputs $\hat{u}_0, \hat{u}_1, ..., \hat{u}_{q-1}$

Step 4. Using (22) compute $I(\hat{u})$

Example 3. Given the positive fractional system (4) for $0 < \alpha < 1$ with (13). Find an optimal sequence of inputs that steers the state of the system from $x_0 = 0$ to $x_f = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in two steps ($q = 2$) and minimizes the performance index (15) for $Q = [2]$.

In Example 2 it was shown that the system is reachable in two steps. It is easy to see that the conditions of Theorem 4 are met. Using Procedure we obtain the following.

Step 1. In this case

$$R_2 = [B, \Phi_1 B] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$Q = \mathrm{diag}[Q^{-1} \quad Q^{-1}] = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step 2. Using (16) we obtain

$$W = R_q \bar{Q} R_q^T = \bar{Q} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step 3. Using (18) we obtain

$$\hat{u}_{02} = \begin{bmatrix} \hat{u}_1 \\ 0 \end{bmatrix} = \bar{Q} R_2^T W^{-1} x_f = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{28}$$

It is easy to verify that the sequence (28) steers the state of the system in two steps from $x_0 = 0$ to $x_f = [1 \quad 1]^T$.

Step 4. The minimal value of the performance index in this case is equal to

$$I(\hat{u}) = x_f^T W^{-1} x_f = [1 \quad 1]\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4$$

CONCLUDING REMARKS

The minimum energy control problem of positive fractional discrete linear systems has been addressed. Necessary and sufficient conditions for the reachability of the systems have been established. Sufficient conditions for the solvability of the minimum energy control of the positive fractional discrete-time systems have been given and a procedure for computation of the optimal sequence of inputs minimizing the performance index (15) has been proposed. The considerations can be extended for positive fractional discrete-time systems with delays and continuous-time linear systems.
An extension of these considerations for positive 2D linear systems is an open problem.

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