COMMODITIES PRICES MODELING USING GAUSSIAN POISSON-EXPONENTIAL STOCHASTIC PROCESSES, A PRACTICAL IMPLEMENTATION IN THE CASE OF COPPER

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ABSTRACT

Due to an assignment, received from a Chilean mining company, to value a copper mine with an estimated life span of several decades, we implemented a model of copper prices using mean reversion with Gaussian Poisson exponential jumps.

The parameters of the model are extracted from the copper prices series. The exponential distributions of the jumps are estimated via a standard simulation program using best likelihood methods.

Until the model was implemented, the company had been using a long term mean price to estimate mining projects' cash flows. This approach had worked satisfactorily given that, as shown in the Chart 1, the average price of copper had ranged around 100 cents of USD per pound between 1996 and 2004.

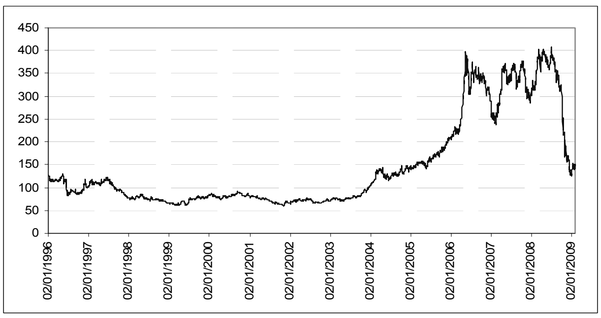

Chart 1. Daily copper prices from January 1996 to January 2009. Source: Cochilco

However, a turnaround in the cycle occurred in 2004, with prices going up between that year and 2008 to reach a mean value of 325 ç/pound. At the end of 2008 the price dropped down up to the level of 150 ç/pound. The prices series analysis suggests the existence of mean reversion with stochastic jumps especially from 2006.

INTRODUCTION

Mean reversion models originally proposed by (Ornstein & Uhlenbeck 1930) are predicated on the notion that the long-term behavior of prices gravitates towards a mean price.

The price adjustment mechanism is accounted for by market forces, in that –leaving aside speculative movements in financial markets, which in specific cases can be the cause of price volatility- the market adjusts itself in periods of strong demand and rising prices through an increase in the offer, e.g. by opening up closed mines, by extending the life span of mines scheduled to be shut down, etc. Following this increase in the offer, prices tend to fall. In periods when prices are low, the opposite occurs, so that the offer decreases and prices tend to go up.

(Dixit & Pindyck 1994) suggest the use of the mean reversion model to model the behavior of commodity prices and propose a model that will be used in the paper.

The model of Poisson jumps for the securities prices was proposed by (Press 1967). (Merton 1976) proposes a model of geometric Brownian motion with jumps which characterizes the jumps as the arriving of new important information. Merton makes an important improvement to the model adjusting the Poisson process to convert it to a Martingale.

(Das 1998) uses a model of mean reversion with exponentially distributed Bernoulli Poisson jumps for modeling interest rates developing an analytical formula for the parameterization of the model.

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(Dias & Rocha 2001) applies the (Dixit & Pindyck 1994) model with Gaussian distributed Poisson jumps for modeling petroleum prices. The important contribution of (Dias & Rocha 2001) is the specification of the discrete model for the simulation although they make a simplification because they apply the same probability of jumps up or down using a Gaussian distribution for the size of the jump.

In our case we will use the (Dixit & Pindyck 1994) model adapted by Dias & Rocha, but deriving ourselves the discrete simulation formula of exponentially distributed Bernoulli Poisson jumps.

MEAN REVERSION MODELS OF PRICES

A given random variable, X_T, responds to the following stochastic differential equation:

$$dx = \eta\left(\overline{x} - x\right)dt + \sigma dz \tag{1.1}$$

With $\overline{x}$ being the expected mean price, x the current price, and η a reversion speed parameter that will be explained below.

Formula (1.1) tells us that *dx* tends to be negative when $\overline{x} < x$ -that is, when the mean price is smaller than the current price-, thus meaning that the change tends to push prices down. However, when $\overline{x} > x$, the mean price is higher than the current price, and *dx* tends to represent a positive change that pushes the price upward.

The speed of reversion is given by parameter η, a reversion process's mean life is the time it takes for the price to cover half the distance that separates it from the mean price. In other words, a mean life of one year means that it takes two years for the price to reach its mean value.

The relationship between mean life, H, and reversion speed, η, is:

$$H = \frac{\log(2)}{\eta}; \qquad \eta = \frac{\log(2)}{H} \tag{1.2}$$

The solution of (Kloeden & Platen 1992) for the stochastic differential equation (1.1) in terms of Ito's stochastic integral is as follows:

$$x_T = x_0 e^{-\eta T} + \left(1 - e^{-\eta T}\right)\overline{x} + \sigma e^{-\eta T}\int_0^T e^{\eta t} dz \tag{1.3}$$

As pointed out by (Dixit & Pindyck 1994), the mean and the variance of the X_T variable are given by the following expressions:

$$E\left(X_T\right) = x_0 e^{-\eta T} + \overline{x}\left(1 - e^{-\eta T}\right)$$
$$Var\left(X_T\right) = \left(1 - e^{-2\eta}\right)\frac{\sigma^2}{2\eta} \tag{1.4}$$

Since X_T follows a distribution which may take negative values, we will define $x = \log(S)$ and $S = e^x$, with *S* being the asset price.

The variance needs to be adjusted by ½ in order to obtain the exact formula for the simulation, which will thus be:

$$S_t = e^{\left(\log\left(S_{t-1}\right)e^{-\eta\Delta t} + \log\left(\overline{S}\right)\left(1 - e^{-\eta\Delta t}\right) - \left(1 - e^{-2\eta\Delta t}\right)\frac{\sigma^2}{4\eta} + \sigma\sqrt{\frac{1 - e^{-2\eta\Delta t}}{2\eta}}dz\right)} \tag{1.5}$$
$$dz \sim N\left(0,1\right)$$

where the first two terms on the right-hand side are the process's drift terms that weigh the initial value and the mean value of equilibrium. The third term is an adjustment term required by Jensen's inequality, and the fourth term is the variance term multiplied by a Wiener stochastic process that draws values from a *N* (0,1) distribution.

MEAN REVERSION MODEL WITH POISSON EXPONENTIAL JUMPS

Mathematically, a mean reversion process with jumps is formulated by adding a new term to the mean reversion's stochastic differential equation:

$$dx = \eta\left(\overline{x} - x\right)dt + \sigma dz + dq \tag{1.6}$$

The *dq* term is a Poisson term whose value is zero most of the time; yet λ often takes a value that prompts a jump in the variable $x = \log(S)$.

$$dq \begin{cases} 0 & \text{with probability} \quad 1 - \lambda dt \\ 1 & \text{with probability} \quad \lambda dt \end{cases} \tag{1.7}$$

Should there be up jumps with a frequency λ_u and down jumps with a frequency λ_d, then:

$$\lambda = \lambda_u + \lambda_d \tag{1.8}$$

So that *dq* is the jump-size distribution when such jumps are up or down (ϕ_u, ϕ_d).

We start from the assumption that the mean reverting process, *dz*, and the Poisson process, *dq*, are not correlated.

The jump's size and direction are random, and $k = E\left[\phi\right]$ is the expected value of the joint distribution (ϕ) for up jumps and down jumps.

In order for the expected value of the dx process to be independent of the jump parameters, it needs to be compensated by subtracting from it the term λk. We thus use a compensated Poisson process and eliminate the trend that might incorporate into the process the effect of a higher number of up or down jumps. Hence, the modified formula is:

$$dx = \left[\eta\left(\overline{x} - x\right) - \lambda k\right]dt + \sigma dz + dq \quad (1.9)$$

The mean of X_T now becomes:

$$E\left[X_T\right] = xe^{-\eta} + \left(\overline{x} + \lambda k\eta\right)\left(1 - e^{-\eta}\right) \quad (1.10)$$

Note that when the expectation of the jumps is $k = 0$, this last expression is identical to the expectation of X_T in a mean reversion process without jumps, confirming that the possible trend introduced by the jumps has indeed been eliminated.

The variance of the mean reversion process with jumps is:

$$Var\left(X_T\right) = \frac{\sigma^2 + \lambda Var(\phi)}{2\eta}\left(1 - e^{-2\eta}\right) \quad (1.11)$$

We adjust the solution of the process by including the variance term of the Poisson distribution for up and down jumps:

$$S_t = e^{\left(\begin{array}{l}\log(S_{t-1})e^{-\eta\Delta t} \\ +\log\left(\overline{S} + \lambda k\eta\right)\left(1 - e^{-\eta\Delta t}\right) \\ +dq - \left(1 - e^{-2\eta\Delta t}\right)\frac{\sigma^2 + (\lambda_u + \lambda_d)Var(\phi)}{4\eta} \\ +\sigma\sqrt{\frac{1 - e^{-2\eta\Delta t}}{2\eta}}dz\end{array}\right)} \quad (1.12)$$

For the simulation of the S_t variable, which follows a mean reversion process with jumps, we use the following formulas:

$$\begin{aligned} x_t &= \log\left(S_{t-1}\right)e^{-\eta\Delta t} \\ &+ \log\left(\overline{S} + \lambda k\eta\right)\left(1 - e^{-\eta\Delta t}\right) \\ &+ \sigma\sqrt{\frac{1 - e^{-2\eta\Delta t}}{2\eta}}dz + dq \end{aligned} \quad (1.13)$$

and:

$$S_t = e^{\left(x_t - \left(1 - e^{-2\eta\Delta t}\right)\frac{\sigma^2 + \lambda Var(\phi)}{4\eta}\right)} \quad (1.14)$$

$\lambda = \lambda_u + \lambda_d$ is the sum of the frequencies of up jumps and down jumps, respectively, and $Var(\phi)$ is the variance of the jump process that we will be obtained directly from the market data.

To simulate the occurrence of a jump, the expected frequency of up jumps and down jumps per time unit is compared, for each period, with a random value obtained from a [0,1] uniform distribution. If the value drawn in the random extraction is $u < \lambda_u dt$, then a jump up is assumed, and dq takes its value. If, on the contrary, $u < \lambda_d dt$, then a jump down is assumed, and dq takes its value.

Calculating the mean reversion parameters

(Dixit & Pindyck 1994) described how to estimate the mean reversion parameters, building on the idea that the mean reversion's partial differential equation

$$dx = \eta\left(\overline{x} - x\right)dt + \sigma dz \quad (1.15)$$

matches the following AR (1) discrete-time autoregressive process:

$$x_t - x_{t-1} = \overline{x}\left(1 - e^{-\eta}\right) + \left(e^{-\eta} - 1\right)x_{t-1} + \varepsilon_t \quad (1.16)$$

Where ε_t follows a normal distribution with standard deviation σ_ε, and:

$$\sigma_\varepsilon^2 = \frac{\sigma^2}{2\eta}\left(1 - e^{-2\eta}\right) \quad (1.17)$$

Hence, the parameters of equation (1.17) may be estimated using market price data by means of the following regression:

$$x_t - x_{t-1} = a + bx_{t-1} + \varepsilon_t \quad (1.18)$$

Which allows us to determine the mean reversion parameters:

$$\begin{aligned} \overline{x} &= \frac{-\hat{a}}{\hat{b}}, \\ \hat{\eta} &= -\log\left(1 + \hat{b}\right) \end{aligned} \quad (1.19)$$

We take (1.17) as a measure of the reversion process volatility, with σ^2 being the variance of the price series return figures.

The Data

We use copper price data from January 1996 to January 2009. The series consists of 3305 daily price figures. These observed data span a period of 13 years.

Following the Dixit & Pindyck model (1994), we assure that the mean reversion model is adequate and make the Unit Root Test accepting the hypothesis that the series has a unit root and is not a random walk, in other words, the x_{t-1} term is significantly different from 0 as can be seen in Table 1:

Null Hypothesis: COPPER_PRICE has a unit root
Exogenous: Constant
Lag Length: 1 (Automatic based on SIC, MAXLAG=28)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-0.980970	0.7620
Test critical values:	1% level	-3.432142	
	5% level	-2.862217	
	10% level	-2.567174	

*MacKinnon (1996) one-sided p-values.

Table 1. Unit Root Test of the prices series.

We use the log of the prices series to estimate the regression parameters $\hat{a} = 0.002214$, $\hat{b} = -0.000455$ and $\varepsilon = 0.0177112$, using these parameters to determine the reversion values given by Dixit and Pindyck's (1.19) formula, and we arrive to the results of Table 2.

$$\bar{x} = -\frac{0.002214}{-0.000455} = 4.86508; \quad \bar{P} = e^{(4.86508)} = 129.68$$

$$\hat{\eta} = -\log(1 - 0.0004551) = 0.000452 \Rightarrow \quad Mean\ Life = \frac{\log(2)}{0.000452} = 1522\, days$$

$$\hat{\sigma} = \sqrt{\frac{(0.017711)^2}{2(0.0004552)}\left(1 - e^{-2(0.0004552)}\right)} = 0.0125236$$

Table 2. Mean reversion formulas and parameters.

We obtain a mean price of 129.68 ç/pound. The estimated mean life of 1522 days correspond to a mean life of 6 years entails that the series goes back to its mean approximately every 12 years.

As for the standard deviation, we notice that it tends to 0.01252, which is lower than its value in the original series (0.01771).

Estimating the jump distribution

We will estimate the parameters of the jump distribution on the basis of (Clewlow & Strickland's 2000) work considering that values of more than 3 standard deviation are jumps in the series as can be seen in Chart 2.

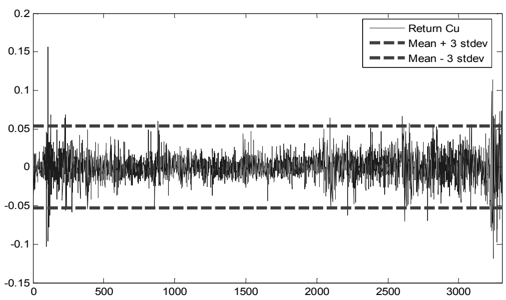

Chart 2. Returns of the daily copper prices and the ±3 standard deviation limits.

In order to isolate the jump component, we subject the returns to a recurrent filtering. The methodology is as follows:

1. We analyze the series and take out the returns located above or below the 3 standard deviation limits.
2. We analyze the series again and similarly take out the returns above or below 3 standard deviations.
3. We keep on doing this until there are no more returns left above or below the 3 standard deviation limits.

This way we obtain three different series, the first one with the up jumps, the second one with the down jumps, and the third the filtered returns.

Parametrizing the jumps

The outcome of the filtering iteration reveals that there are a total of 43 up jumps and 51 down jumps. These results allow us to obtain the frequency and distribution of the up and down jump functions.

The up jumps' expected frequency is 43 jumps every 3304 days -i.e. a frequency of 1.3014%. The down jumps' expected frequency is 51 jumps every 3304 days - i.e. a frequency of 1.5435%. The overall frequency is 94 jumps every 3304 days -that is, a frequency of 2.845%.

In order to characterize the exponential distribution function, the data gathered from both the sample of up jumps and the sample of down jumps are introduced into an optimization program, which estimates through the maximum verisimilitude method using the Chi Squared Test the parameters of the exponential distributions.

The output from the program is the mean of the exponential distribution $\bar{x}$; to calculate its parameter λ, we thus need to apply the formula of the exponential distribution's mean:

$$\bar{x} = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{\bar{x}} \qquad (1.20)$$

In the case of up jumps, the exponential distribution is on Table 3:

$$\phi_u(x)\begin{cases} 61.92e^{-61.92x} & , x \geq 0.0422 \\ 0 & , x < 0.0422 \end{cases}$$

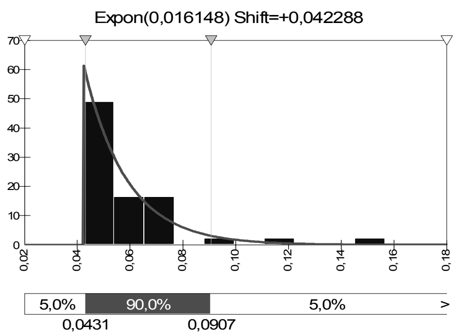

Table 3. Jumps Up adjusted Exponential Distribution

Evidences a 0.0422 shift of the function to the right, since this is the size of the minimum jump. Therefore, we have to add this figure to the mean, which then becomes: mean = 0.01614 + 0.0422 = 0.05834. The variance obtained is: variance = 0.00026107.

$$\phi_d(x)\begin{cases} 55,16e^{-55,16x} & ,x \geq 0.04229 \\ 0 & ,x < 0.04229 \end{cases}$$

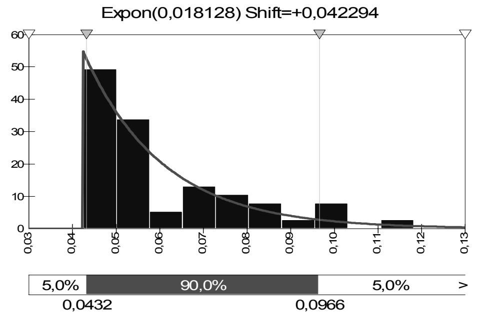

Table 4. Jumps Down adjusted Exponential Distribution

In the case of down jumps, we cannot generate an exponential distribution, as returns are now negative. We therefore need to convert the negative return distribution into a positive one in order to derive our distribution. When using it for the simulation, we thus turn to negative the result for the down jumps yielded by the distribution obtained. The actual result for the down jumps is shown.

The distribution derived from the optimization process is described on Table 4.

Having a mean = - 0.018128 - 0.042294 = -0.06042, and variance = 0.00032861. Graphically, the adjusted curve is in the Chart 8.

Simulating the mean reversion motion with jumps

We use the above-stated formulas (1.13) and (1.14) to simulate the mean reversion process with the preceding parameters

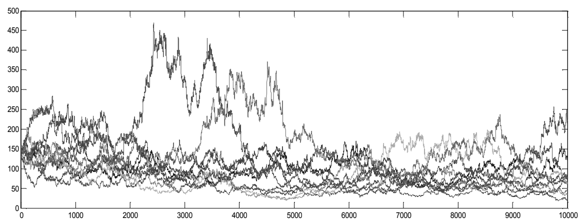

Chart 3. Graphical output of the simulation of the mean reversion with exponential jumps model with 10,000 steps corresponding to 40 years.

The MATLAB code needed to simulate the process is set out in the appendix as MATLAB Code 1.

Conclusions

The model laid out above allows us to derive the parameters directly from market data, and makes it possible to adjust them according to executives' expectations. All parameters used in the model can be directly and intuitively understood, and its comprehension requires just average mathematical and statistical know-how.

The model is capable of coherently replicating the shifts in commodity markets, allowing for situations such as the one we have analyzed here, which involves a jump up in prices, to fit in.

The process of feeding and updating the model with new market data is simple and can be implemented both on MATLAB-type programs and on simulation-based spreadsheet programs.

APPENDIX

MATLAB Code 1

```
T=10000; %Number of periods
dt=1; %Time interval
N=T/dt;
Speed=.000452; %Reversion Speed
Sigma=.0125236; %Volatility of the process
paths=10; %Number of paths
S0=142; %Initial Value
S = 129.68; %Expected mean value
LambdaU = 0.013014; %Jump up frequency
LambdaD = 0.015435; %Jump down frequency
Lambda = 0.02845; %Total jump frequency
MeanPhiU = 0.016148; %Mean jump up
ShiftPhiU = 0.042288; %Jump up scaling
VarPhiU = 0.00026107; %Jump up variance
MeanPhiD = -0.018128; %Mean jump down
ShiftPhiD = -0.042294; %Jump down scaling
VarPhiD = 0.00032861; %Jump down variance
MeanPhi = MeanPhiU + ShiftPhiU + MeanPhiD + ShiftPhiD; %Mean of Phi
VarPhi = VarPhiU + VarPhiD; %Variance of Phi
UniformU = find(unifrnd(0,1,N,paths)>LambdaU);
UniformD = find(unifrnd(0,1,N,paths)>LambdaD);
dQUp = exprnd(MeanPhiU, N,paths)+ ShiftPhiU;
dQUp(UniformU) = 0;
dQDown = -exprnd(-MeanPhiD, N,paths)+ ShiftPhiD;
dQDown(UniformD) = 0;
dQ = dQUp + dQDown;
Wiener=randn(N,paths);
SpeedDt=Speed*dt;
St=S0*ones(N,paths);
G1=exp(-SpeedDt);
G2=log(S+Lambda*MeanPhi*Speed)*(1-exp(-SpeedDt));
G3=Sigma*sqrt((1-exp(-2*SpeedDt))/(2*Speed));
G4=(1-exp(-2*SpeedDt))*(((Sigma^2)+Lambda*VarPhi)/(4*Speed));
for i=1:N
    for j=1:paths
    St(i+1,j)=exp(log(St(i,j))*G1+G2+dQ(i,j)+G3*Wiener(i,j)-G4);
    end
end
```

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