A DISCRETE-TIME QUEUEING SYSTEM WITH DIFFERENT TYPES OF DISPLACEMENT

Ivan Atencia (speaker)
Inmaculada Fortes
Sixto Sánchez
Department of Applied Mathematics
University of Málaga
29071 - Málaga, Spain
Email: iatencia@ctima.uma.es,
ifortes@uma.es, sixto@uma.es

Alexander V. Pechinkin
Institute of Informatics Problems of RAS
119333 - Vavilova, 44-1,
Moscow, Russia
Email: apechinkin@ipiran.ru

KEYWORDS

Discrete-time system, expulsions, triggered customers, loss probability.

ABSTRACT

The performance prediction in communication, jobs processing in computers, etc, are always influenced by the customers behavior and the provision of this additional information will be useful in upgrading the service. Our paper is concerned under a loss and trigger protocol where each customer has a service requirement which may depend on the arrival of a positive or negative customer. In our study we consider customer expulsions and different types of customers displacements taking into account or not its past time of service.

The main purpose of this work is to spread the discrete-time queueing theory about expulsions and displacement. We provide a unified way to handle the combinations of different conditions such as positive arrival, negative arrival, trigger movements, past time in service, etc.

INTRODUCTION

An investigation of discrete-time queueing system is important due to their application to slotted systems such as communication systems and other related areas and therefore it has been found more appropriate than their continuous-time counterpart.

The study of discrete-time queues was initiated by (Meisling 1958; Birdsall et al. 1962; Powell et al. 1967). Reference works and more detailed applications on discrete-time queueing theory include the monographs (Bruneel and Kim 1993; Takagi 1993). Further, a detailed treatment regarding this subject can be found in a two-volume book on applied probability (Hunter 1983).

A rapid increase in the literature on queueing system with negative arrivals are analyzed extensively in continuous-time models but not so much in discrete-time. The arrival of a negative customer to a queueing system causes one ordinary customer to be removed or killed if any is present. The pioneer work on discrete-time considering negative arrivals without retrials can be found in (Atencia and Moreno 2004; Atencia and Moreno 2005) where the authors considered several killing strategies for the negative customers.

For a survey on this topics the authors refer to (Gelenbe and Label 1998) and (Artalejo 2000), for applications on engineering to (Chao et al. 1999) and for application in communication networks we refer to (Harrison et al. 2000) and (Park et al. 2009).

In many real problems it is also interesting to consider the movement of jobs, customers, etc., from one place to another. This mechanism is called a synchronized or triggered motion, see for example (Artalejo 2000) and (Gelenbe and Label 1998) and concerning with inverse order discipline we refer to (Pechinkin and Svischeva 2004), (Pechinkin and Shorgin 2008) and (Cascone et al. 2011). For service interruptions with expulsions we refer to (Atencia and Pechinkin 2012) and (Atencia et al. 2013).

THE MATHEMATICAL MODEL

We consider a discrete-time queueing system where the time axis is segmented into a sequence of equal time intervals (called slots). It is assumed that all queueing activities (arrivals, departures and retrials) occur at the slot boundaries, and therefore, they may take place at the same time. That is why we must detail the order in which the arrivals and departures occur in case of simultaneity in a discrete-time system. Basically, there are two rules: (i) If an arrival takes precedence over a departure, it is identified with Late Arrival System (LAS) (see Figure 1(a)); (ii) if a departure takes precedence over an arrival, it is recognized by Early Arrival System (EAS) (see Figure 1(b)). The former case is also known as Arrival First (AF) policy and the latter as Departure First (DF) policy. For more details on these and related concepts, see (Gravey and Hébuterne 1992) and (Hunter 1983).

Let us note, that for mathematical convenience, we will follow the second policy, that is the departures occur at the moment immediately before the slot boundaries,

Proceedings 27th European Conference on Modelling and Simulation ©ECMS Webjørn Rekdalsbakken, Robin T. Bye, Houxiang Zhang (Editors)
ISBN: 978-0-9564944-6-7 / ISBN: 978-0-9564944-7-4 (CD)

but arrivals occurs at the moment immediately after the slot boundaries.

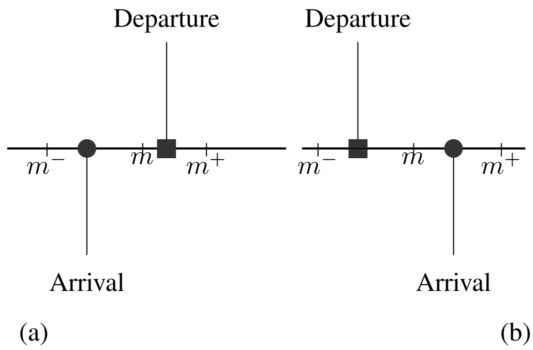

Figure 1: Options of the arrival models

Customers arrive according to a geometrical arrival process with rate a, i.e., a is the probability that an arrival occurs in a slot. Service times are independent and identically distributed with general distribution $\{s_i\}_{i=1}^{\infty}$ and generating function (GF) $S(z) = \sum_{i=1}^{\infty} z^i s_i$. We will denote by $S_k = \sum_{i=k}^{\infty} s_i;\ k \geq 1$, the probability that the service lasts not less than k slots.

If, upon arrival, the server is idle, the service of the arriving customer begins immediately, otherwise, the arriving customer has several options depending on its nature. That is, with probability θ_0 it will join the last place of the queue and with probability θ_1 expels the customer that is currently in the server and starts immediately its service. With probability θ_2 it becomes a triggered customer moving the customer from the server to the first place at the queue but considering for the future its cumulative service and with probability θ_3 it is moved to the last place of the queue without considering its cumulative service. Finally with probability θ_4 it becomes a negative customer expelling out of the system the customer in service. Obviously it verifies that $\sum_{i=0}^{4} \theta_i = 1$. Let us note that the customers arriving from outside has priority on others. In order to avoid trivial cases, we assume $0 < a < 1$.

THE STEADY STATE PROBABILITIES

Let π_0 be the stationary probability that in the moment immediately after a potential arrival the system is empty and $\pi_{i,k};\ i \geq 1,\ k \geq 1$, the stationary probability that there are k customers in the system and that the customer in service has passed exactly i slots of service. Therefore we have

$$\pi_{i,1} = \overline{a}\pi_{i-1,1}\frac{S_{i+1}}{S_i} + a\theta_2\pi_{i-1,1}\frac{S_{i+1}}{S_i};\ \ i \geq 1, \quad (1)$$

$$\begin{aligned}\pi_{i,k} &= \overline{a}\pi_{i-1,k}\frac{S_{i+1}}{S_i} + a\theta_2\pi_{i-1,k}\frac{S_{i+1}}{S_i} + \\ &\quad + a\theta_0\pi_{i-1,k-1}\frac{S_{i+1}}{S_i};\ \ i \geq 1,\ \ k \geq 2, \qquad (2)\end{aligned}$$

$$\begin{aligned}\pi_{0,1} &= a(1-\theta_4)\pi_0 + a\sum_{j=1}^{\infty}\theta_0\pi_{j-1,1}\frac{s_j}{S_j} + \\ &\quad + a\sum_{j=1}^{\infty}\theta_1\pi_{j-1,1} + a\sum_{j=1}^{\infty}\theta_2\pi_{j-1,1}\frac{s_j}{S_j} + \\ &\quad + a\sum_{j=1}^{\infty}\theta_3\pi_{j-1,1}; \qquad (3)\end{aligned}$$

$$\begin{aligned}\pi_{0,k} &= a\sum_{j=1}^{\infty}\theta_0\pi_{j-1,k}\frac{s_j}{S_j} + a\sum_{j=1}^{\infty}\theta_1\pi_{j-1,k} + \\ &\quad + a\sum_{j=1}^{\infty}\theta_2\pi_{j-1,k}\frac{s_j}{S_j} + a\sum_{j=1}^{\infty}\theta_3\pi_{j-1,k} + \\ &\quad + a\sum_{j=1}^{\infty}\theta_0\pi_{j-1,k-1}\frac{S_{j+1}}{S_j} + \\ &\quad + a\sum_{j=1}^{\infty}\theta_2\pi_{j-1,k-1}\frac{S_{j+1}}{S_j} + \\ &\quad + a\sum_{j=1}^{\infty}\theta_3\pi_{j-1,k-1}\frac{S_{j+1}}{S_j};\ \ k \geq 2. \qquad (4)\end{aligned}$$

Let us define

$$\begin{aligned}q_{i,k} &= \pi_{i,k}/S_{i+1};\ \ i \geq 0,\ \ k \geq 1, \qquad (5)\\ Q_k &= \sum_{i=0}^{\infty} s_{i+1}q_{i,k};\ \ k \geq 1,\\ \pi_k &= \sum_{i=0}^{\infty}\pi_{i,k} = \sum_{i=0}^{\infty} S_{i+1}q_{i,k};\ \ k \geq 1.\end{aligned}$$

It is clear, that π_k is the probability that there are k; $k \geq 1$, customers in the system. The system (1)–(4) can be transformed, taking into account the previous definitions, in the following equations:

$$\begin{aligned}q_{i,1} &= (\overline{a} + a\theta_2)q_{i-1,1};\ \ i \geq 1, \qquad (6)\\ q_{i,k} &= (\overline{a} + a\theta_2)q_{i-1,k} + a\theta_0 q_{i-1,k-1};\\ &\quad i \geq 1,\ \ k \geq 2, \qquad (7)\end{aligned}$$

$$\begin{aligned}q_{0,1} &= a(1-\theta_4)\pi_0 + a\theta_0\sum_{j=1}^{\infty} q_{j-1,1}s_j + a\theta_1\pi_1 + \\ &\quad + a\theta_2\sum_{j=1}^{\infty} q_{j-1,1}s_j + a\theta_3\pi_1 = \\ &= a[(1-\theta_4)\pi_0 + (\theta_1+\theta_3)\pi_1 + (\theta_0+\theta_2)Q_1], \qquad (8)\end{aligned}$$

$$\begin{aligned}q_{0,k} &= a\theta_0\sum_{j=1}^{\infty} q_{j-1,k}s_j + a\theta_1\pi_k + \\ &\quad + a\theta_2\sum_{j=1}^{\infty} q_{j-1,k}s_j + a\theta_3\pi_k + a\theta_0\pi_{k-1} -\end{aligned}$$

$$
\begin{aligned}
&- a\theta_0 \sum_{j=1}^{\infty} q_{j-1,k-1} s_j + a\theta_2 \pi_{k-1} - \\
&- a\theta_2 \sum_{j=1}^{\infty} q_{j-1,k-1} s_j + \\
&+ a\theta_3 \pi_{k-1} - a\theta_3 \sum_{j=1}^{\infty} q_{j-1,k-1} s_j = \\
&= a[(\theta_1 + \theta_3)\pi_k + (\theta_0 + \theta_2) Q_k + \\
&+ (\theta_0 + \theta_2 + \theta_3)\pi_{k-1} - \\
&- (\theta_0 + \theta_2 + \theta_3) Q_{k-1}]; \quad k \geq 2. \qquad (9)
\end{aligned}
$$

Solving eq. (6) and (7) in k, we obtain

$$
q_{i,k} = \sum_{j=0}^{\min\{i,k-1\}} \binom{i}{j} (a\theta_0)^j (\overline{a} + a\theta_2)^{i-j} q_{0,k-j}; \quad i \geq 1, \;\; k \geq 1. \qquad (10)
$$

Let us determine π_k and Q_k for $k \geq 1$. In order to have a compact formulae, we denote by $\hat{S}(z) = [1 - S(z)]/(1 - z)$ and $\tilde{S}(z) = S(z)/z$.

Firstly for $k = 1$, that is from (10), we obtain:

$$
\begin{aligned}
\pi_1 &= q_{0,1} \sum_{i=0}^{\infty} (\overline{a} + a\theta_2)^i \sum_{j=i+1}^{\infty} s_j = \\
&= q_{0,1} \sum_{j=1}^{\infty} s_j \sum_{i=0}^{j-1} (\overline{a} + a\theta_2)^i = \\
&= q_{0,1} \sum_{j=1}^{\infty} s_j \frac{1 - (\overline{a} + a\theta_2)^j}{1 - (\overline{a} + a\theta_2)} = q_{0,1} \hat{S}(\overline{a} + a\theta_2), \qquad (11)
\end{aligned}
$$

$$
Q_1 = q_{0,1} \sum_{i=0}^{\infty} s_{i+1} (\overline{a} + a\theta_2)^i = q_{0,1} \tilde{S}(\overline{a} + a\theta_2), \qquad (12)
$$

where $q_{0,1}$ can be determined by substituting (11) and (12) into (8) and after some algebra we have

$$
q_{0,1} = \frac{a(1-\theta_4)\pi_0}{1 - a(\theta_1+\theta_3)\hat{S}(\overline{a}+a\theta_2) - a(\theta_0+\theta_2)\tilde{S}(\overline{a}+a\theta_2)}. \qquad (13)
$$

With the same procedure but involving equation (9) we have

$$
\pi_k = \sum_{i=0}^{k-1} \frac{(a\theta_0)^i}{i!} q_{0,k-i} \hat{S}^{(i)}(\overline{a} + a\theta_2); \; k \geq 2, \qquad (14)
$$

$$
Q_k = \sum_{i=0}^{k-1} \frac{(a\theta_0)^i}{i!} q_{0,k-i} \tilde{S}^{(i)}(\overline{a} + a\theta_2); \;\; k \geq 2, \qquad (15)
$$

where $q_{0,k}$ can be determined by substituting (12), (13) and (10) into (9) and after some algebra we have

$$
\begin{aligned}
q_{0,k} = a\big[&1 - a(\theta_1 + \theta_3)\hat{S}(\overline{a} + a\theta_2) - \\
&- a(\theta_0 + \theta_2)\tilde{S}(\overline{a} + a\theta_2)\big]^{-1} \times \\
&\times \Bigg[\sum_{i=1}^{k-1} \frac{(a\theta_0)^i}{i!} \big((\theta_1 + \theta_3)\hat{S}^{(i)}(\overline{a} + a\theta_2) + \\
&+ (\theta_0 + \theta_2)\tilde{S}^{(i)}(\overline{a} + a\theta_2)\big) q_{0,k-i} + \\
&+ (\theta_0 + \theta_2 + \theta_3)\pi_{k-1} - (\theta_0 + \theta_2 + \theta_3) Q_{k-1}\Bigg]; \\
&k \geq 2. \qquad (16)
\end{aligned}
$$

We can summarize the above results in the following:

Theorem 1 *The stationary probability π_k; $k \geq 1$, that there are k customers in the system is given in the formula (11) or (14). This formula can be determined by $q_{0,k}$ and Q_k from the formulae (13) and (16) or (12) and (15).*

The stationary probability $\pi_{i,k}$; $k \geq 1$, $i \geq 0$, that there are k customers in the system and the one in the server has already i slots of service is obtained by formula (5) where $q_{i,k}$; $k \geq 1$, $i \geq 1$, can be determined by formula (10).

It should be pointed out that Theorem 1 entails an algorithm that enables us, in a convenient way, to calculate the stationary probabilities π_k and $\pi_{i,k}$.

The probability π_0, included in the obtained formulae, can be calculated under the normalization condition which will be shown in a clear form in the following section. The condition for the existence of the stationary regime will be calculated further on.

GENERATING FUNCTION

In this section we obtain the stationary distribution of the number of customers in terms of its generating functions. For this aim we define the following GF's:

$$
\begin{aligned}
q_0(z) &= \sum_{k=1}^{\infty} z^k q_{0,k}, \\
Q(z) &= \sum_{k=1}^{\infty} z^k Q_k, \\
\pi(z) &= \sum_{k=1}^{\infty} z^k \pi_k.
\end{aligned}
$$

Multiplying (13) and (16) by z^k and summing over k we obtain

$$
\begin{aligned}
q_0(z) = a\Big(&[1 - \theta_4] z\pi_0 + [(\theta_1 + \theta_3) + \\
&+ (\theta_0 + \theta_2 + \theta_3) z]\pi(z) + \\
&+ [(\theta_0 + \theta_2) - (\theta_0 + \theta_2 + \theta_3) z] Q(z)\Big). \qquad (17)
\end{aligned}
$$

From eqs. (14)–(17) we have

$$
\begin{aligned}
\pi(z) &= \sum_{i=0}^{\infty} \frac{(a\theta_0 z)^i}{i!} q_0(z) \hat{S}^{(i)}(\overline{a} + a\theta_2) = \\
&= \hat{S}(\overline{a} + a\theta_2 + a\theta_0 z) q_0(z), \qquad (18) \\
Q(z) &= \sum_{i=0}^{\infty} \frac{(a\theta_0 z)^i}{i!} q_0(z) \tilde{S}^{(i)}(\overline{a} + a\theta_2) = \\
&= \tilde{S}(\overline{a} + a\theta_2 + a\theta_0 z) q_0(z), \qquad (19)
\end{aligned}
$$

where $q_0(z)$ has the following expression:

$$q_0(z) = \frac{(1-\theta_4)az}{D(z)}\pi_0,$$

and

$$\begin{aligned} D(z) = 1 - a\big(&[\theta_1 + \theta_3 + (\theta_0+\theta_2+\theta_3)z] \times \\ &\times \hat{S}(\overline{a} + a\theta_2 + a\theta_0 z) + \\ &+ [\theta_0 + \theta_2 - (\theta_0+\theta_2+\theta_3)z] \times \\ &\times \tilde{S}(\overline{a} + a\theta_2 + a\theta_0 z)\big). \end{aligned} \tag{20}$$

In order to find π_0 we use the following normalization condition

$$\sum_{k=0}^{\infty} \pi_k = \pi_0 + \pi(1) = 1, \tag{21}$$

which entails to

$$\begin{aligned} \pi_0 = \big(&1 - a\big[(1+\theta_3-\theta_4)\hat{S}(\overline{a} + a\theta_2 + a\theta_0) - \\ &- \theta_3\tilde{S}(\overline{a} + a\theta_2 + a\theta_0)\big]\big) \times \\ &\times \big(1 - a\theta_3\big[\hat{S}(\overline{a} + a\theta_2 + a\theta_0) - \\ &- \tilde{S}(\overline{a} + a\theta_2 + a\theta_0)\big]\big)^{-1}. \end{aligned} \tag{22}$$

Therefore the necessary condition for the stability of the system is

$$S(\overline{a} + a\theta_2 + a\theta_0) > \frac{(\overline{a} + a\theta_2 + a\theta_0)(1-\theta_1-2\theta_4)}{(\overline{a} + a\theta_2 + a\theta_0)(1-\theta_4) + \theta_3}.$$

It can be proved that the above condition is also sufficient.

Theorem 2 *The following GF's of the number of customers in the system and in the queue are respectively obtained under simple algebraical transformations, that is:*

$$\begin{aligned} \Phi(z) &= \pi_0 + \pi(z) = \\ &= \Big[1 + \frac{1-S(A(z))}{1-A(z)}(1-\theta_4)z \times \\ &\quad \times \Big\{\frac{1}{a} - (1-\theta_4)\frac{S(A(z))}{A(z)} - \\ &\quad - [(\theta_1+\theta_3) + (\theta_0+\theta_2+\theta_3)z] \times \\ &\quad \times \frac{A(z) - S(A(z))}{A(z)(1-A(z))}\Big\}^{-1}\Big]\pi_0, \end{aligned} \tag{23}$$

$$\Psi(z) = \pi_0 + \frac{\pi(z)}{z}. \tag{24}$$

where $A(z) = \overline{a} + a\theta_2 + a\theta_0 z$.

Differentiating the above GF's in the point $z = 1$, we can obtain the moments for any stationary characteristic of the number of customers in the system and in the queue. Even more, in certain cases, for a determined rational-linear function $S(z)$, Theorem 2 can be used to find directly the stationary distribution of the number of customers in the system and in the queue.

STATIONARY CHARACTERISTICS

In this section we present some performance measures for the system at the stationary regime.

The mean number of customers in the system

In order to find N we differentiate formula (23) in the point $z = 1$, obtaining

$$\begin{aligned} N &= \Phi'(1) = \pi'(1) = \\ &= \frac{[1 - S(A(1)) - S'(A(1))(1-A(1))]}{(1-A(1))^2} \times \\ &\quad \times \frac{1-A(1)}{1-S(A(1))}a\theta_0(1-\pi_0) + \\ &\quad + \frac{1-S(A(1))}{1-A(1)} \cdot \frac{D(1) - D'(1)}{D^2(1)}a(1-\theta_4)\pi_0, \end{aligned} \tag{25}$$

where $D(z)$ it arises from formula (20), that is:

$$\begin{aligned} D(z) = 1 - a\Big(&(1-\theta_4)\frac{S(A(z))}{A(z)} + [(\theta_1+\theta_3) + \\ &+ (\theta_0+\theta_2+\theta_3)z]\frac{A(z) - S(A(z))}{A(z)[1-A(z)]}\Big). \end{aligned}$$

The mean number of customers in the queue

In order to find Q we differentiate formula (24) in the point $z = 1$, obtaining

$$Q = \Psi'(1) = N - (1 - \pi_0). \tag{26}$$

Loss probability of customers

Let us define ω as the loss probability of the pure customers, that is, only the customers that have being expelled by a positive customer and not by a negative customer. In order to find this probability we need to give some previous definitions:

Let us define r_i as the stationary probability that the system is not empty on condition that the customer that is currently at service has already spent i slots of service. Therefore:

$$r_i = \sum_{k=1}^{\infty} \pi_{i,k}, \ \ i \geq 0. \tag{27}$$

The stationary loss intensity of customers in the system due to negative or positive customers is given by

$$\mu_{\,\mathrm{loss}} = a(\theta_4 + \theta_1)\sum_{i=1}^{\infty} r_{i-1}\frac{S_{i+1}}{S_i}.$$

Therefore we have:

$$\omega = \frac{\mu_{\,\mathrm{loss}}}{l_{\,\mathrm{pos}}}, \tag{28}$$

where $l_{\,\mathrm{pos}} = a(1-\theta_4)$.

The mean sojourn time of a customer in the system

Let T be the mean sojourn time of a positive customer in the system, therefore following Little's formula we have:

$$T = \frac{N}{l_{\,\mathrm{pos}}}. \tag{29}$$

The mean sojourn time of a customer in the queue
Let S be the mean sojourn time of a positive customer in the queue, therefore following Little's formula we have:

$$S = \frac{Q}{l_{\text{pos}}}. \tag{30}$$

NUMERICAL RESULTS

In this section, we present some numerical examples of the performance measures obtained in the previous section in relation with the most specific parameters of our model. For the numerical examples we have considered the values of the following probabilities $\theta_0 = 0.2,\ \ \theta_1 = 0.15,\ \ \theta_2 = 0.3,\ \ \theta_3 = 0.05,\ \ \theta_4 = 0.3$ and three different probability distributions for the service time, that is, its GF's are of the following type for $0 < p < 1,\ q = 1 - p$ and $0 < \alpha < 1,\ 0 < p_1, p_2 < 1,$ $q_1 = 1 - p_1,\ q_2 = 1 - p_2$:

$$S_1(z) = \frac{zp}{1 - qz},$$
$$S_2(z) = \frac{zp^2}{(1 - qz)^2},$$
$$S_3(z) = \frac{\alpha p_1 z}{1 - q_1 z} + \frac{(1 - \alpha)p_2 z}{1 - q_2 z}.$$

Let us note that the position of the three graphics are in concordance to its mean service time which is closely related to the performance measures.

In Figure 1, we show the effect of the probability that the system is empty against the parameter a, for different types of Gf's for the service time distribution.

The effect of these curves is that π_0 decreases as functions of the parameter a. Due to the fact that the parameter has a direct influence on the number of customers in the system, obviously we obtain that the probability that the system is idle decreases. In this case, for each curve the intersections with the abscissas axes correspond to the limiting condition for the stability of our system.

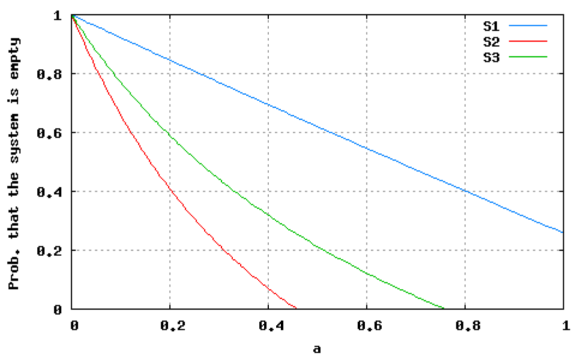

Figure 2: π_0 vs. a

In Figure 2, the mean number of customers in the system, N, is plotted against the parameter a. As it was to be expected, N, is increasing as a function of a which also agrees with the intuitive expectations. As we can observe, all the curves increase asymptotically as they tend to the stability condition of the system.

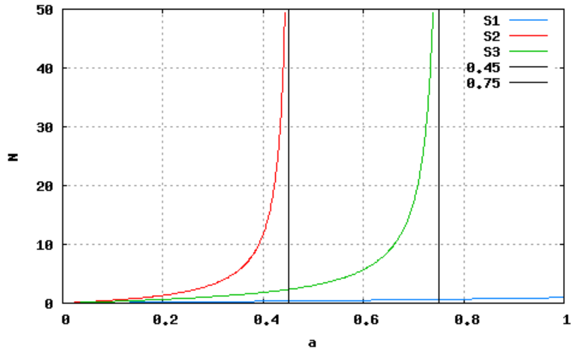

Figure 3: N vs. a

In Figure 3, The mean number of customers in the queue, Q, is plotted against the parameter a. By the expression (26) we can observe that the graphic looks very similar to the previous one. Although we observe that Q increases as a function of a and asymptotically tends to the stability condition of the system.

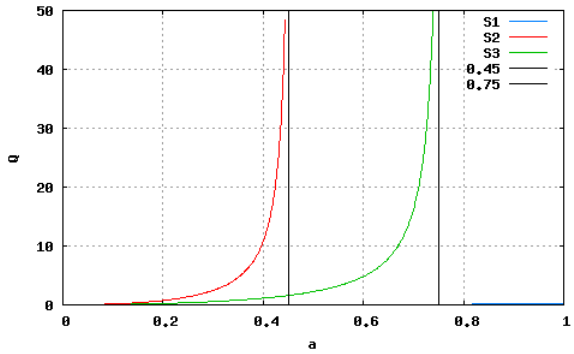

Figure 4: Q vs. a

In Figures 4 and 5, that is, the mean sojourn time of a customer in the system, T, and in the queue S respectively are plotted against the incoming parameter a. We observe that they have the same upward tendency asymptotically to the stability condition of the system.

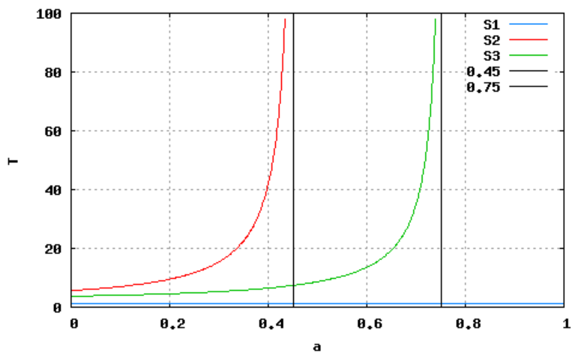

Figure 5: T vs. a

In Figure 6, we represent the distribution function of the number of customers in the system for $a = 0.4$.

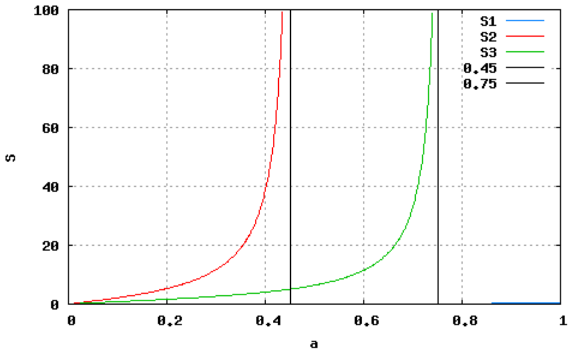

Figure 6: S vs. a

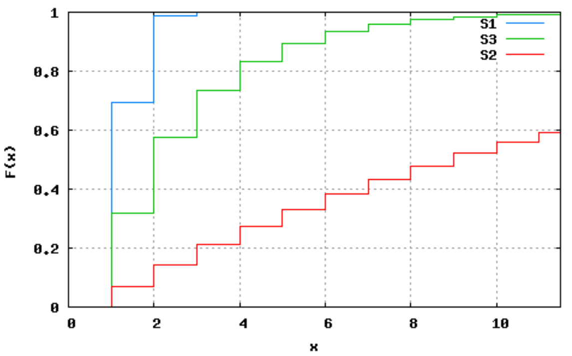

Figure 7: $F(x)$ for $a = 0.4$

SUMMARY

In the foregoing study, a discrete-time queueing system under a loss and trigger protocol has been analyzed to obtain analytical expressions for various performance measures of interest. It is important to select models that can be efficiently solved and for whose tools and solution methodologies are developed and available. Let us note that our system includes many particular cases that can be applied in several practical situations depending on the different types of displacement. To this end, we have managed not a single technique but a combination of different mathematical tools. In this paper, we establish some important results via two theorems in which theorem 1 enables us to calculate the stationary probabilities of the number of customers in the system without considering and considering the passed service slots. Theorem 2 is concerned with the calculation of the generating functions of the number of customers in the system and in the queue obtaining some important stationary characteristics. We also study the loss probability of customers and the mean sojourn time of a customer in the system and in the queue. Several performance characteristics of the model have been calculated and numerical results are given for different distributions.

Notes and Comments. The research of A. Pechinkin was supported in part by the Russian Foundation for Basic Research (grants 11-07-00112, 12-07-00108 and 13-07-00223). The research of I. Fortes and S. Sánchez is partially supported by Junta de Andalucía grant P09-FQM-5233.

REFERENCES

Artalejo, J. R. 2000. "G-networks: A versatile approach for work removal in queueing networks". *European Journal of Operational Research*, 126, Pp. 233–249.

Atencia, I. and Moreno, P. 2004. "The discrete-time Geo/Geo/1 queue with negative customers and disasters". *Computers and Operations Research* 31 (9), Pp. 1537–1548.

Atencia, I. and Moreno, P. 2005. "A single-server G-queue in discrete-time with geometrical arrival and service process." *Performance Evaluation*, 59, Pp. 85–97.

Atencia, I., Fortes, I. and Sánchez, S. 2013. "Discrete-time queueing system with expulsions." *Communications in Computer and Information Science*, 356, Pp. 20–25.

Atencia, I. and Pechinkin, A.V. 2012. "A discrete-time queueing system with optional LCFS discipline". *Annals Operation Research.* 202, Pp. 3–17.

Birdsall, T., Ristenbatt, M. and Weinstein, S. 1962. "Analysis of asynchronous time multiplexing of speech sources". *IRE Transactions on Communication Systems.* 2. vol. 10, Pp. 390–397.

Bruneel, H. and Kim, B.G. 1993. *Discrete-time models for communication systems including ATM. Kluwer Academic Publishers*, Boston.

Chao, X., Miyazawa, M. and Pinedo, M. 1999. *Queueing networks: Customers, signals and product form solutions.* Wiley.

Cascone, A., Manzo, P., Pechinkin, A. and Shorgin, S. 2011. "A $Geom/G/1/n$ system with a LIFO discipline without interruptions in the service and with a limitation for the total capacity for the customers". *Avtomatika i Telemejanika* vol. 1, Pp. 107–120. (In russian).

Gelenbe, E. and Label, A. 1998. "G-networks with multiple classes of signals and positive customers." *European Journal of Operation Research* 108, Pp. 293–305.

Gravey, A. and Hébuterne, G. 1992. "Simultaneity in discrete-time single server queues with bernoulli inputs." *Performance Evaluation* 14, Pp. 123–131.

Harrison, P. G., Patel, N. M. and Pitel, E. 2000. "Reliability modelling using G-queues". *European Journal of Operational Research*, 126, Pp. 273–287.

Hunter, J. 1983. *Mathematical Techniques of Applied Probability.* Operations Research and Industrial Engineering. Academic Press, New York. Vol. 1,2.

Meisling, T. 1958. "Discrete time queueing theory". *Operation Research*, vol. 6, Pp. 96–105.

Park, H. M., Yang, W. S. and Chae, K. C. 2009. "The Geo/G/1 queue with negative customers and disasters". *Stochastic Models*, 25, Pp. 673–688.

Pechinkin, A. and Svischeva, T. 2004. "The stationary state probability in the BMAP/G/1/r queueing system with inverse discipline and probabilistic priority". In *Transactions of XXIV International Seminar on Stability Problems for Stochastic Models* Pp. 141–174, Jurmala, Latvia.

Pechinkin, A. and Shorgin, S. 2008. “A $Geo/G/1/\infty$ system with a one non-standard discipline for the service”. *Informatics and it applications*, vol. 2, Pp. 55–62. (In russian).

Pechinkin, A. and Shorgin, S. 2008. “The discrete-time queueing system with inversive service order and probabilistic priority”. *Proceedings of the 3rd International Conference on Performance Evaluation Methodologies and Tools*, 20.

Powell, B. and Avi-Itzhak, B. 1967. “Queueing systems with enforced idle times”. *Operations Research.* vol. 15(6), Pp. 1145–1156.

Takagi, H. 1993. *Queueing analysis: A foundation of performance evaluation. Discrete-Time Systems.* North-Holland, Amsterdam vol. 3.

AUTHORS BIOGRAPHIES

IVAN ATENCIA graduated in 1997 and obtained his Ph.D. in Mathematics in 2000 from Málaga University, Spain. During this period he collaborated with the Department of Probability Theory and Mathematical Statistics of The Peoples' Friendship University of Russia, Moscow. At present, he is Associate Professor in the Department of Applied Mathematics at Málaga University and actively collaborates with the Institute of Informatics Problems of RAS, Russia (Moscow). His research interests include queueing theory and stochastic modeling of communication systems. His e-mail address is: `iatencia@ctima.uma.es`

INMACULADA FORTES graduated in 1996 and obtained her Ph.D. in Mathematics in 2002 from Málaga University, Spain. She is Associate Professor in the Department of Applied Mathematics at Málaga University. Her research interests are queueing theory, stochastic modelling of communication systems as well as data mining and mathematical logic. Her e-mail address is: `ifortes@uma.es`

ALEXANDER V. PECHINKIN is a Doctor of Sciences in Physics and Mathematics and principal scientist at the Institute of Informatics Problems of the Russian Academy of Sciences, and a professor at the Peoples' Friendship University of Russia. He is the author of more than 150 papers in the field of applied probability theory. His e-mail address is: `apechinkin@ipiran.ru`

SIXTO SÁNCHEZ graduated in 1992 and obtained his Ph.D. in Mathematics in 2002 from Málaga University, Spain. He is Associate Professor in the Department of Applied Mathematics at Málaga University. His research interests are queueing theory, stochastic modelling of communication systems as well as mathematical logic. His e-mail address is: `sixto@uma.es`