ON TRUNCATIONS FOR SZK MODEL

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KEYWORDS

Markovian queueing models; SZK model; inhomogeneous Markov processes; truncations

ABSTRACT

We consider a class of inhomogeneous Markovian queueing models with batch arrivals and group services. Bounds on the truncation errors in weak ergodic case are obtained. Two concrete queueing models are studied as examples.

INTRODUCTION

The problem of existence and construction of limiting characteristics for inhomogeneous continuous-time Markov chains is important for queueing applications, see for instance, (Granovsky and Zeifman 2004, Zeifman et al. 2006). Calculation of the limiting characteristics for birth-death process via truncations was firstly mentioned in (Zeifman 1991) and was considered in details in (Zeifman et al. 2006).

About two decades ago V. Kalashnikov suggested that in some cases one can obtain uniform (in time) error bounds of truncation, and in (Zeifman et al. 2014b) we prove this conjecture for inhomogeneous birth-death processes.

A new class of Markovian time-inhomogeneous queueing models with batch arrivals and group services was introduced and studied in our recent papers (Satin et al. 2013, Zeifman et al. 2014a). Bounds of the rate of convergence, perturbation bounds, and approximations via truncations were studied in these papers.

Here we consider this model and obtain uniform in time error bounds of truncation.

Consider a time-inhomogeneous continuous-time Markovian queueing model ("SZK model") on the state space $E = \{0, 1, \dots\}$ with possible batch arrivals and group services.

Let $X = X(t)$, $t \geq 0$ be a queue-length process for the queue, $p_{ij}(s,t) = Pr\left\{X(t) = j \left| X(s) = i\right.\right\}$, $i, j \geq 0$, $0 \leq s \leq t$, be transition probabilities for $X = X(t)$, and $p_i(t) = Pr\left\{X(t) = i\right\}$ be its state probabilities. Throughout the paper we assume that

$$\Pr\left(X\left(t+h\right) = j/X\left(t\right) = i\right) = \tag{1}$$
$$\begin{cases} q_{ij}\left(t\right)h + \alpha_{ij}\left(t,h\right), \text{ if } j \neq i, \\ 1 - \sum\limits_{k \neq i} q_{ik}\left(t\right)h + \alpha_i\left(t,h\right), \text{ if } j = i, \end{cases}$$

where all $\alpha_i(t,h)$ are $o(h)$ uniformly in i, i. e. $\sup_i |\alpha_i(t,h)| = o(h)$. We also assume $q_{i,i+k}\left(t\right) = \lambda_k(t)$, $q_{i,i-k}\left(t\right) = \mu_k(t)$ for any $k > 0$. In other words, we suppose that the arrival rates $\lambda_k(t)$ and the service rates $\mu_k(t)$ do not depend on the the length of queue. In addition, we assume that $\lambda_{k+1}(t) \leq \lambda_k(t)$ and $\mu_{k+1}(t) \leq \mu_k(t)$ for any k and almost all $t \geq 0$. Applying our standard approach (see details in (Granovsky and Zeifman 2004, Zeifman 1995a, Zeifman et al. 2006, 2008)) we suppose in addition, that all intensity functions are linear combinations of a finite number of locally integrable on $[0, \infty)$ nonnegative functions. Moreover we assume

$$\lambda_k(t) \leq \lambda_k, \quad \mu_k(t) \leq m_k, \tag{2}$$

for any k and almost all $t \geq 0$, and

$$L_\lambda = \sum_{i=1}^{\infty} \lambda_i < \infty, \quad L_\mu = \sum_{i=1}^{\infty} \mu_i < \infty. \tag{3}$$

Then the probabilistic dynamics of the process is represented by the forward Kolmogorov system:

$$\frac{d\mathbf{p}}{dt} = A(t)\mathbf{p}(t), \tag{4}$$

Proceedings 28th European Conference on Modelling and Simulation ©ECMS Flaminio Squazzoni, Fabio Baronio, Claudia Archetti, Marco Castellani (Editors)
ISBN: 978-0-9564944-8-1 / ISBN: 978-0-9564944-9-8 (CD)

where

$$A(t) = \tag{5}$$

$$= \begin{pmatrix} a_{00}(t) & \mu_1(t) & \mu_2(t) & \cdots & \mu_r(t) & \cdots \\ \lambda_1(t) & a_{11}(t) & \mu_1(t) & \cdots & \mu_{r-1}(t) & \cdots \\ \lambda_2(t) & \lambda_1(t) & a_{22}(t) & \cdots & \mu_{r-2}(t) & \cdots \\ \cdots \\ \lambda_r(t) & \lambda_{r-1}(t) & \lambda_{r-2}(t) & \cdots & a_{rr}(t) & \cdots \\ \cdots \end{pmatrix},$$

and $a_{ii}(t) = -\sum_{k=1}^{i} \mu_k(t) - \sum_{k=1}^{\infty} \lambda_k(t)$.

Throughout the paper by $\|\cdot\|$ we denote the l_1-norm, i. e. $\|\mathbf{x}\| = \sum |x_i|$, and $\|B\| = \sup_j \sum_i |b_{ij}|$ for $B = (b_{ij})_{i,j=0}^{\infty}$.

Let Ω be a set all stochastic vectors, i. e. l_1 vectors with nonnegative coordinates and unit norm. Hence we have $\|A(t)\| = 2\sum_{k=1}^{\infty}(\lambda_k(t)+\mu_k(t)) \le 2\,(L_\lambda + L_\mu)$ for almost all $t \ge 0$. Hence the operator function $A(t)$ from l_1 into itself is bounded for almost all $t \ge 0$ and locally integrable on $[0;\infty)$. Therefore we can consider (4) as a differential equation in the space l_1 with bounded operator.

It is well known, see (Daleckij and Krein 1974), that the Cauchy problem for differential equation (4) has a unique solutions for an arbitrary initial condition, and $\mathbf{p}(s) \in \Omega$ implies $\mathbf{p}(t) \in \Omega$ for $t \ge s \ge 0$.

Denote by $E(t,k) = E\left\{X(t)\,|X(0) = k\right\}$ the mean (the mathematical expectation) of the process at the moment t under the initial condition $X(0) = k$.

Recall that a Markov chain $X(t)$ is called *weakly ergodic*, if $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $\mathbf{p}^*(0), \mathbf{p}^{**}(0)$, where $\mathbf{p}^*(t)$ and $\mathbf{p}^{**}(t)$ are the corresponding solutions of (4).

A Markov chain $X(t)$ has the *limiting mean* $\varphi(t)$, if $\lim_{t\to\infty}(\varphi(t) - E(t,k)) = 0$ for any k.

Let $\{d_i\}$, $i = 1, 2, \ldots$ be an increasing sequence of positive numbers, $d_1 = 1$. Put

$$W = \inf_{i \ge 1} \frac{d_i}{i}, \quad g_i = \sum_{n=1}^{i} d_n, \tag{6}$$

Denote

$$\alpha_i(t) = -a_{ii}(t) - \sum_{k\ge 1} \lambda_k(t)\frac{d_{k+i}}{d_i} - \sum_{k=1}^{i-1}(\mu_{i-k}(t) - \mu_i(t))\frac{d_k}{d_i}, \tag{7}$$

and

$$\alpha(t) = \inf_{i\ge 1} \alpha_i(t). \tag{8}$$

Assume that for some positive K

$$d_1\lambda_1 + (d_1 + d_2)\lambda_2 + \cdots \le K, \tag{9}$$

By introducing $p_0(t) = 1 - \sum_{i\ge 1} p_i(t)$, from (4) we obtain the equation

$$\frac{d\mathbf{z}}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t), \tag{10}$$

where $\mathbf{f}(t) = (\lambda_1, \lambda_2, \cdots)^\top$, $\mathbf{z}(t) = (p_1, p_2, \cdots)^\top$,

$$B = \left(b_{ij}(t)_{i,j=1}^{\infty}\right) = \tag{11}$$

$$\begin{pmatrix} a_{11} - \lambda_1 & \mu_1 - \lambda_1 & \cdots & \mu_{r-1} - \lambda_1 & \cdots \\ \lambda_1 - \lambda_2 & a_{22} - \lambda_2 & \cdots & \mu_{r-2} - \lambda_2 & \cdots \\ \lambda_2 - \lambda_3 & \lambda_1 - \lambda_3 & \cdots & \mu_{r-3} - \lambda_3 & \cdots \\ \cdots \\ \lambda_{r-1} - \lambda_r & \lambda_{r-2} - \lambda_r & \cdots & a_{rr} - \lambda_r & \cdots \\ \cdots\cdots\cdots \end{pmatrix},$$

see detailed discussion in (Granovsky and Zeifman 2004, Zeifman 1995a, Zeifman et al. 2006, 2008). Let D be upper triangular matrix,

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots \\ 0 & d_2 & d_2 & \cdots \\ 0 & 0 & d_3 & \cdots \\ & \ddots & \ddots & \ddots \end{pmatrix}, \tag{12}$$

and let l_{1D} be the corresponding space of sequences

$$l_{1D} = \left\{\mathbf{z} = (p_1, p_2, \cdots)^\top \,|\, \|\mathbf{z}\|_{1D} \equiv \|D\mathbf{z}\|_1 < \infty\right\}.$$

Consider equation (10) in the space l_{1D}, where $B(t)$ and $\mathbf{f}(t)$ are locally integrable on $[0, +\infty)$. Consider the logarithmic norm of operator function $B(t)$, see the respective motivation in (Van Doorn et al. 2010, Granovsky and Zeifman 2004, Zeifman et al. 2008) and detailed proofs in (Zeifman 1995b). Recall that the logarithmic norm of operator function $B(t)$ is defined as

$$\gamma(B(t)) = \lim_{h\to+0} h^{-1}\left(\|I + hB(t)\| - 1\right).$$

The important inequality

$$\|V(t,s)\| \le \exp\int_s^t \gamma(B(u))\,du$$

holds, where $V(t,s) = V(t)V^{-1}(s)$ is the Cauchy operator of equation (10). Further, for an operator function from l_1 to itself we have the simple formula

$$\gamma(B(t)) = \sup_j \left(b_{jj}(t) + \sum_{i\ne j} |b_{ij}(t)|\right).$$

Hence we obtain the following bound for the logarithmic norm of the operator function $B(t)$:

$$\gamma(B(t))_{1D} = \gamma(DB(t)D^{-1}) = \sup_{i\ge 1}\{-\alpha_i(t)\} = -\alpha(t), \tag{13}$$

where

$$DBD^{-1} = \tag{14}$$

$$\begin{pmatrix} a_{11} & (\mu_1-\mu_2)\frac{d_1}{d_2} & (\mu_2-\mu_3)\frac{d_1}{d_3} & \cdots & (\mu_{r-1}-\mu_r)\frac{d_1}{d_r} & \cdots \\ \lambda_1\frac{d_2}{d_1} & a_{22} & (\mu_1-\mu_3)\frac{d_2}{d_3} & \cdots & (\mu_{r-2}-\mu_r)\frac{d_2}{d_r} & \cdots \\ \lambda_2\frac{d_3}{d_1} & \lambda_1\frac{d_3}{d_2} & a_{33} & \cdots & (\mu_{r-3}-\mu_r)\frac{d_3}{d_r} & \cdots \\ \cdots \\ \lambda_{r-1}\frac{d_r}{d_1} & \lambda_{r-2}\frac{d_r}{d_2} & \lambda_{r-3}\frac{d_r}{d_3} & \cdots & a_{rr} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Therefore

$$\|V(t,s)\|_{1D} \le e^{-\int_s^t \alpha(u)\,du}. \tag{15}$$

TRUNCATIONS

Consider the "truncated" process $X_{N-1}(t)$ on the state space $E_{N-1} = \{0, 1, \ldots, N-1\}$ with the corresponding intensity matrix

$$A_{N-1} = \tag{16}$$

$$= \begin{pmatrix} b_{00} & \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots & \mu_{N-1} \\ \lambda_1 & b_{11} & \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{N-2} \\ \lambda_2 & \lambda_1 & b_{22} & \mu_1 & \mu_2 & \cdots & \mu_{N-3} \\ \cdots \\ \lambda_{N-1} & \lambda_{N-2} & \lambda_{N-3} & \cdots & \lambda_2 & \lambda_1 & b_{N-1,N-1} \end{pmatrix}$$

where $b_{ii}(t) = -\sum_{k=1}^{i} \mu_k(t) - \sum_{k=1}^{N-1-i} \lambda_{N-1-k}(t)$.

Instead of (4), for $X_{N-1}(t)$ we have the following forward Kolmogorov system:

$$\frac{d\mathbf{p}^*}{dt} = A_{N-1}(t)\mathbf{p}^*. \tag{17}$$

Similarly, instead of (10), we obtain the correspondent reduced system for the truncated process in the form:

$$\frac{d\mathbf{z}^*}{dt} = B_{N-1}(t)\mathbf{z}^*(t) + \mathbf{f}_{N-1}(t), \tag{18}$$

where $\mathbf{f}_{N-1}(t) = (\lambda_1, \cdots, \lambda_{N-1}, 0, \cdots)^\top$, $\mathbf{z}^*(t) = (p_1, p_2, \cdots, p_{N-1})^\top$,

$$B_{N-1} = \left(b_{ij}{}^*(t)\right)_{i,j=1}^{N-1} = \tag{19}$$

$$\begin{pmatrix} b_{11} - \lambda_1 & \mu_1 - \lambda_1 & \cdots & \mu_{N-1} - \lambda_1 \\ \lambda_1 - \lambda_2 & b_{22} - \lambda_2 & \cdots & \mu_{N-2} - \lambda_2 \\ \lambda_2 - \lambda_3 & \lambda_1 - \lambda_3 & \cdots & \mu_{N-3} - \lambda_3 \\ \cdots \\ \lambda_{N-2} - \lambda_{N-1} & \lambda_{N-3} - \lambda_{N-1} & \cdots & b_{N-1,N-1} - \lambda_{N-1} \end{pmatrix}$$

Below we will identify the finite vector with entries, say, $(a_1, \ldots, a_{N-1})^\top$ and the infinite vector with the same first $N-1$ coordinates and the others equal to zero.

For bounding of the truncation error we rewrite (18) in the following form:

$$\frac{d\mathbf{z}^*}{dt} = B(t)\mathbf{z}^*(t) + \mathbf{f}(t) - \hat{B}(t)\mathbf{z}^*(t) - \hat{\mathbf{f}}(t), \tag{20}$$

where $\hat{B}(t) = B(t) - B_{N-1}(t)$, and $\hat{\mathbf{f}}(t) = \mathbf{f}(t) - \mathbf{f}_{N-1}(t)$.

Then we have

$$\mathbf{z}^*(t) = V(t)\mathbf{z}^*(0) + \int_0^t V(t,\tau)\mathbf{f}(\tau)\,d\tau - \tag{21}$$
$$\int_0^t V(t,\tau)\hat{B}(\tau)\mathbf{z}^*(\tau)\,d\tau - \int_0^t V(t,\tau)\hat{\mathbf{f}}(\tau)\,d\tau,$$

Hence, if $\mathbf{z}(0) = \mathbf{z}^*(0)$, then the sum of first and second summands gives us $\mathbf{z}(t)$, and we obtain in *any* norm

$$\|\mathbf{z}(t) - \mathbf{z}^*(t)\| \le \|\int_0^t V(t,\tau)\hat{B}(\tau)\mathbf{z}^*(\tau)\,d\tau\| + \|\int_0^t V(t,\tau)\hat{\mathbf{f}}(\tau)\,d\tau\| \le$$
$$\int_0^t \|V(t,\tau)\|\|\hat{B}(\tau)\mathbf{z}^*(\tau)\|\,d\tau + \int_0^t \|V(t,\tau)\|\|\hat{\mathbf{f}}(\tau)\|\,d\tau. \tag{22}$$

We have

$$\hat{\mathbf{f}}(t) = \mathbf{f}(t) - \mathbf{f}_{N-1}(t) = (0, \cdots, 0, \lambda_N(t), \lambda_{N+1}(t), \cdots)^\top, \tag{23}$$

and

$$\|\hat{\mathbf{f}}(t)\|_{1D} = \|D\hat{\mathbf{f}}(t)\|_1 = (d_1 + \cdots + d_N)\,\lambda_N(t) +$$
$$(d_1 + \cdots + d_{N+1})\,\lambda_{N+1}(t) + \cdots \le \tag{24}$$
$$g_N\lambda_N + g_{N+1}\lambda_{N+1} + \cdots \to 0 \text{ as } N \to \infty,$$

in accordance with assumption (9).

On the other hand,

$$\hat{B}(t)\mathbf{z}^*(t) = (B(t) - B_{N-1}(t))\,\mathbf{z}^*(t) = \tag{25}$$
$$((a_{11}(t) - b_{11}(t))p_1^*(t), \cdots, (a_{N-1,N-1}(t) - b_{N-1,N-1}(t))p_{N-1}^*(t))^\top,$$

and

$$\|\hat{B}(t)\mathbf{z}^*(t)\|_{1D} = \|D\,(B(t) - B_{N-1}(t))\,\mathbf{z}^*(t)\|_1 =$$
$$d_1(b_{11}(t) - a_{11}(t))p_1^*(t) + (d_1 + d_2)(b_{22}(t) - a_{22}(t))p_2^*(t) + \cdots +$$
$$(d_1 + \cdots + d_{N-1})(b_{N-1,N-1}(t) - a_{N-1,N-1}(t))p_{N-1}^*(t) = \tag{26}$$
$$d_1 \sum_{k \ge N-1} \lambda_k(t)p_1^*(t) + (d_1 + d_2)\sum_{k \ge N-2} \lambda_k(t)p_2^*(t) + \cdots +$$
$$(d_1 + \cdots + d_{N-1})\sum_{k \ge 1} \lambda_k(t)p_{N-1}^*(t).$$

Let now a sequence $\{d_i\}$ be such that $1 = d_1 \le d_2 \le \cdots$, and the following two assumptions hold:

$$\|V(t,s)\|_{1D} \le Me^{-a(t-s)} \tag{27}$$

for any $0 \le s \le t$, and some positive numbers M and a,

and

$$\|V(t,s)\|_{1D^*} \le M^*e^{-a^*(t-s)} \tag{28}$$

for any $0 \le s \le t$, some positive numbers M^* and a^*, where

$$D^* = \begin{pmatrix} d_1^2 & d_1^2 & d_1^2 & \cdots \\ 0 & d_2^2 & d_2^2 & \cdots \\ 0 & 0 & d_3^2 & \cdots \\ & \ddots & \ddots & \ddots \end{pmatrix}. \tag{29}$$

Let, in addition, there exist a positive number K^* such that

$$d_1^2\lambda_1 + (d_1^2 + d_2^2)\lambda_2 + \cdots \le K^*. \tag{30}$$

Now we try to estimate $\|\hat{B}(t)\mathbf{z}^*(t)\|_{1D}$.

Firstly,

$$\|\mathbf{z}^*(t)\|_{1D^*} \le \|V(t)\|_{1D^*}\|\mathbf{z}^*(0)\|_{1D^*} + \int_0^t \|V(t,\tau)\|_{1D^*}\|\mathbf{f}(\tau)\|_{1D^*}\,d\tau \le M^* e^{-a^* t}\|\mathbf{z}^*(0)\|_{1D^*} + \frac{K^*M^*}{a^*}, \tag{31}$$

because $\|\mathbf{f}(t)\|_{1D^*} \le K^*$ for almost all $t \ge 0$.

Put $X(0) = X_{N-1}(0) = 0$, then $\mathbf{z}^*(0) = \mathbf{0}$, hence

$$\|\mathbf{z}^*(t)\|_{1D^*} \le \frac{K^*M^*}{a^*}, \tag{32}$$

for any $t \ge 0$.

Suppose for definiteness that N is odd. All $p_i^*(t) \ge 0$, then

$$\|\mathbf{z}^*(t)\|_{1D^*} = \sum_{i\ge 1} p_i^*(t)\sum_{k=1}^{i} d_k^2 \ge \sum_{i\ge \frac{N-1}{2}} d_i^2 p_i{}^*(t) \ge \sum_{i=\frac{N-1}{2}}^{N-1} d_i^2 p_i{}^*(t), \tag{33}$$

On the other hand we have the following bound:

$$\begin{gathered} d_1 \sum_{k\ge N-1} \lambda_k(t) p_1^*(t) + (d_1+d_2)\sum_{k\ge N-2}\lambda_k(t)p_2^*(t) + \cdots + (d_1+\cdots+d_{N-1})\sum_{k\ge 1}\lambda_k(t)p_{N-1}^*(t) \le \\ (d_1+\cdots+d_{\frac{N-1}{2}})\sum_{k\ge\frac{N-1}{2}}\lambda_k(t)\sum_{k=1}^{\frac{N-1}{2}} p_k^*(t) + \\ \sum_{k\ge 1}\lambda_k(t)\left((d_1+\cdots+d_{\frac{N-1}{2}})p_{\frac{N-1}{2}}(t)+\cdots+ (d_1+\cdots+d_{N-1})p_{N-1}^*(t)\right). \end{gathered} \tag{34}$$

Denote by $\Lambda_K = \sum_{k\ge K}\lambda_k$.

Then we obtain from (26), (33) and (34):

$$\begin{gathered} \|\hat{B}(t)\mathbf{z}^*(t)\|_{1D} \le g_{\frac{N-1}{2}}\Lambda_{\frac{N-1}{2}}\sum_{k=1}^{\frac{N-1}{2}} p_k^*(t) + L_\lambda\left(g_{\frac{N-1}{2}}p^*_{\frac{N-1}{2}}(t)+\cdots+g_{N-1}p^*_{N-1}(t)\right) \le \\ g_{\frac{N-1}{2}}\Lambda_{\frac{N-1}{2}} + L_\lambda \frac{g_{N-1}}{d^2_{\frac{N-1}{2}}}\left(d^2_{\frac{N-1}{2}}p^*_{\frac{N-1}{2}}(t)+\cdots+d^2_{N-1}p^*_{N-1}(t)\right) \le g_{\frac{N-1}{2}}\Lambda_{\frac{N-1}{2}} + \\ L_\lambda\frac{g_{N-1}}{d^2_{\frac{N-1}{2}}}\|\mathbf{z}^*(t)\|_{1D^*} \le g_{\frac{N-1}{2}}\Lambda_{\frac{N-1}{2}} + L_\lambda \frac{g_{N-1}}{d^2_{\frac{N-1}{2}}}\frac{K^*M^*}{a^*}, \end{gathered} \tag{35}$$

for any $t \ge 0$.

Finally, we have from (22), (24), and (35) the following bound of truncation error:

$$\begin{gathered} \|\mathbf{z}(t)-\mathbf{z}^*(t)\| \le \int_0^t Me^{-a(t-\tau)}\left(g_{\frac{N-1}{2}}\Lambda_{\frac{N-1}{2}} + L_\lambda\frac{g_{N-1}}{d^2_{\frac{N-1}{2}}}\frac{K^*M^*}{a^*}\right)d\tau + \\ \int_0^t Me^{-a(t-\tau)}\left(g_N\lambda_N + g_{N+1}\lambda_{N+1}+\cdots\right)d\tau \le \\ \frac{M}{a}\left(g_{\frac{N-1}{2}}\Lambda_{\frac{N-1}{2}} + L_\lambda\frac{g_{N-1}}{d^2_{\frac{N-1}{2}}}\frac{K^*M^*}{a^*} + g_N\lambda_N + g_{N+1}\lambda_{N+1}+\cdots\right). \end{gathered} \tag{36}$$

Now let l_{1E} be the space of sequences,

$$l_{1E} = \left\{z = (p_1, p_2, \cdots)^\top \,|\, \|z\|_{1E} \equiv \sum n|p_n| < \infty\right\}.$$

Then (36) and well-known inequality $\|\mathbf{z}\|_{1E} \le W^{-1}\|\mathbf{z}\|_{1D}$ (see, for instance, Zeifman et al 2006), imply the following statement.

Theorem 1. Let (9), (27), (28), (30) be fulfilled. Then $X(t)$ is exponentially weakly ergodic, has the limiting mean, say, $E(t,0)$, and the following bound of truncation error holds:

$$|E(t,0) - E_{N-1}(t,0)| \le \frac{M}{aW}\Big(g_{\frac{N-1}{2}}\Lambda_{\frac{N-1}{2}} + L_\lambda\frac{g_{N-1}}{d^2_{\frac{N-1}{2}}}\frac{K^*M^*}{a^*} + g_N\lambda_N + g_{N+1}\lambda_{N+1}+\cdots\Big). \tag{37}$$

for any $t \ge 0$, where $E_{N-1}(t,k) = E\left\{X_{N-1}(t)\,|X_{N-1}(0)=k\right\}$ is the mean (the mathematical expectation) of the truncated process at the moment t under initial condition $X_{N-1}(0) = k$.

EXAMPLES

1. Consider the simplest analogue of $M_t/M_t/S$ queue for a queueing system with group services, see Section 4 in (Zeifman et al. 2014a).

Namely, we suppose that the arrival intensity of a customer to the queue is $\lambda_1(t) = \lambda(t) = 1 + \sin 2\pi t$, $\lambda_i(t) = 0$ for any $i > 1$ and any $t \ge 0$. Let the intensity of departure (servicing) of a group of k customers be $\mu_k(t) = \frac{\mu(t)}{k}$ if $k \le S < \infty$, and $\mu_i(t) = 0$ for any $i > S$ and any $t \ge 0$, where $\mu(t) = 3 + \cos 2\pi t$.

Let $X = X(t)$, $t \ge 0$ be a queue-length process for the queue. For definiteness, put $S = 10^{12}$. For this queue bound (37) looks essentially simpler:

$$|E(t,0) - E_{N-1}(t,0)| \le \frac{L_\lambda M K^* M^*}{aa^*W}\frac{g_{N-1}}{d^2_{\frac{N-1}{2}}}. \tag{38}$$

Here $L_\lambda = 2$.

Put $d = \sqrt{2}$, and $d_{k+1} = d^k$. Then $K = K^* = 2$, $W = \sqrt{2}$. Now, in (8) we have

$$\alpha(t) \geq \mu(t) - (d-1)\lambda(t), \quad \alpha^*(t) \geq \mu(t) - (d^2-1)\lambda(t), \tag{39}$$

therefore we can apply bound (15) and obtain $a \geq 2$, $a^* \geq 2$, $M \leq 2$, $M^* \leq 2$.

Finally, we have $g_{N-1} \leq 2^{\frac{N}{2}}$, $d^2_{\frac{N-1}{2}} = 2^{N-2}$, and the following estimate for the error of truncation:

$$|E(t,0) - E_{N-1}(t,0)| \leq 16 \cdot 2^{-\frac{N}{2}}, \tag{40}$$

which does not depend on t in contrast to the estimate (50) in (Zeifman et al. 2014a).

Applying the approach of (Zeifman et al. 2006) we can find the approximation of the mathematical expectation of the length of queue, see the following figures.

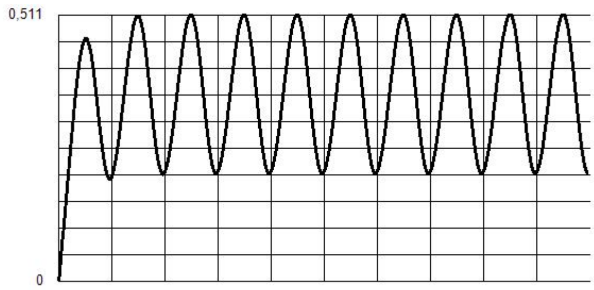

Figure 1: First example, approximation of the mean $E(t,0)$ on $[0,10]$ with an error 10^{-3}.

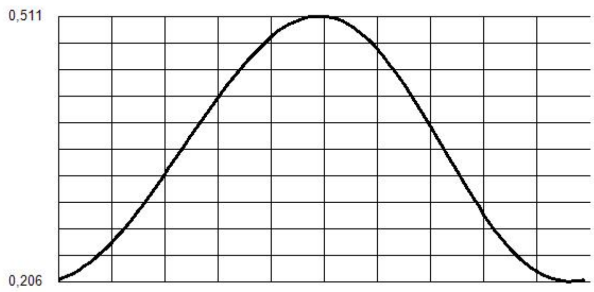

Figure 2: First example, approximation of the mean $E(t,0)$ on $[9,10]$ with an error 10^{-3}.

2. Consider a simplest queuing system with one server and batch arrivals, see Section 5 in (Zeifman et al. 2014a). Namely, let $\mu_1(t) = \mu(t) = 3 + \cos 2\pi t$ be the service rate of a customer, $\mu_i(t) = 0$ for any $i > 1$ and any $t \geq 0$. Let $\lambda_k(t) = \frac{\lambda(t)}{4^k}$, where $\lambda(t) = 1 + \sin 2\pi t$ be the arrival intensity of k customers to the queue. We have $L_\lambda < 1$. Put also $d = \sqrt{2}$, and $d_{k+1} = d^k$. Then $K \leq 2$, $K^* = 2$, $W = \sqrt{2}$. Now, in (8) we have

$$\alpha(t) \geq \mu(t) - (d-1)\lambda(t), \quad \alpha^*(t) \geq \mu(t) - (d^2-1)\lambda(t), \tag{41}$$

therefore we can apply bound (15) and obtain $a \geq \frac{1}{2}$, $a^* \geq \frac{1}{2}$, $M \leq 2$, $M^* \leq 2$, $\Lambda_K \leq 4^{-K}$. We have also $g_{N-1} \leq 2^{\frac{N}{2}}$, $d^2_{\frac{N-1}{2}} = 2^{N-2}$ For this queue we have instead of (37) the following estimate for the error of truncation:

$$|E(t,0) - E_{N-1}(t,0)| \leq 10^2 \cdot 2^{-\frac{N}{2}}, \tag{42}$$

which does not depend on t in contrast to the estimate (52) in (Zeifman et al. 2014a).

Now, applying the approach of (Zeifman et al. 2006) we can find the approximation of the mathematical expectation of the length of queue, see the following figures.

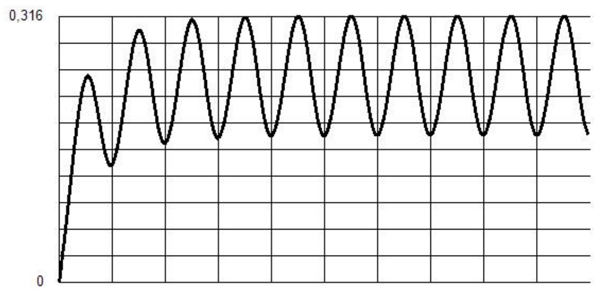

Figure 3: Second example, approximation of the mean $E(t,0)$ on $[0,10]$ with an error 10^{-3}.

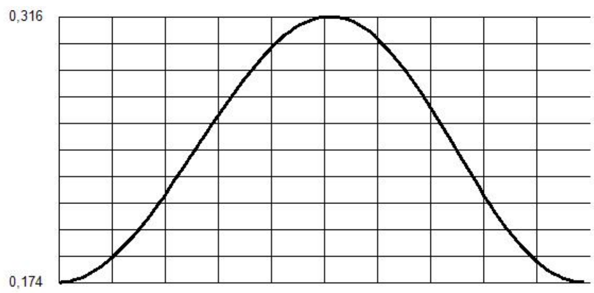

Figure 4: Second example, approximation of the mean $E(t,0)$ on $[9,10]$ with an error 10^{-3}.

Acknowledgement. This work was supported by the Russian Foundation for Basic Research, projects no. 14-07-00041, 13-07-00223, 12-07-00115, 12-07-00109.

REFERENCES

Daleckij, Ju.L. and M.G. Krein. 1974. *Stability of solutions of differential equations in Banach space.* Amer. Math. Soc. Transl. 43.

Van Doorn, E.A., A.I. Zeifman and T.L. Panfilova. 2010. ”Bounds and asymptotics for the rate of convergence of birth-death processes.” *Theory Probab. Appl.* 54, 97–113.

Granovsky, B.L. and A.I. Zeifman. 2004. ”Nonstationary Queues: Estimation of the Rate of Convergence.” *Queueing Syst.* 46, 363–388.

Satin, Ya. A.; A. I. Zeifman and A. V. Korotysheva. 2013. ”On the rate of convergence and truncations for a class of Markovian queueing systems.” *Theory. Prob. Appl.* 57, 529-539.

Zeifman, A. I. 1991. ”Qualitative properties of inhomogeneous birth and death processes.” *J. Math. Sci.* 57, 3217-3224 (The Russian original paper was published in 1988).

Zeifman, A. I. 1995a. ”Upper and lower bounds on the rate of convergence for nonhomogeneous birth and death processes.” *Stoch. Proc. Appl.* 59, 157–173.

Zeifman, A. I. 1995b. ”On the estimation of probabilities for birth and death processes.” *J. App. Probab.* 32, 623–634.

Zeifman, A.; S. Leorato; E. Orsingher; Ya. Satin and G. Shilova. 2006. ”Some universal limits for nonhomogeneous birth and death processes.” *Queueing Syst.* 52, 139–151.

Zeifman, A. I.; V.E. Bening and I.A. Sokolov. 2008. *Continuous-time Markov chains and models.* Elex-KM, Moscow (in Russian).

Zeifman, A.; V. Korolev; A. Korotysheva, Y. Satin and V.Bening. 2014a. ”Perturbation bounds and truncations for a class of Markovian queues.” *Queueing Syst.* 76, 205–221.

Zeifman, A.I.; Y. Satin; V. Korolev and S. Shorgin. 2014b. ”On truncations for weakly ergodic non-stationary birth and death processes.” *Int. J. Appl. Math. Comp. Sci.*

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