VARIANCE-MEAN MIXTURES AS ASYMPTOTIC APPROXIMATIONS

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ABSTRACT

We present a general transfer theorem for random sequences with independent random indexes in the double array limit setting. We also prove its partial inverse providing necessary and sufficient conditions for the convergence of randomly indexed random sequences. Special attention is paid to the cases of random sums of independent not necessarily identically distributed random variables and statistics constructed from samples with random sizes. Using simple moment-type conditions we prove the theorem on convergence of the distributions of such sums to normal variance-mean mixtures.

INTRODUCTION

Random sequences with independent random indexes play an important role in modeling real processes in many fields. Most popular examples of the application of these models usually deal with insurance and reliability theory (Kalashnikov, 1997; Bening and Korolev, 2002), financial mathematics and queuing theory (Bening and Korolev, 2002; Gnedenko and Korolev, 1996), chaotic processes in plasma physics (Korolev and Skvortsova, 2006) where random sums are principal mathematical models. More general randomly indexed random sequences arrive in the statistics of samples with random sizes. Indeed, very often the data to be analyzed is collected or registered during a certain period of time and the flow of informative events each of which brings a next observation forms a random point process, so that the number of available observations is unknown till the end of the process of their registration and also must be treated as a (random) observation.

The presence of random indexes usually leads to that the limit distributions for the corresponding randomly indexed random sequences are heavy-tailed even in the situations where the distributions of non-randomly indexed random sequences are asymptotically normal see, e. g., (Gnedenko and Korolev, 1996; Bening and Korolev, 2005).

The literature on random sequences with random indexes is extensive, see, e. g., the references above and the references therein. The mathematical theory of random sequences with random indexes and, in particular, random sums, is well-developed. However, there still remain some unsolved problems. For example, necessary and sufficient conditions for the convergence of the distributions of random sums to normal variance-mean mixtures were found only recently for the case of identically distributed summands (see (Korolev, 2013; Grigoryeva and Korolev, 2013)). The case of random sums of *non-identically* distributed random summands and, moreover, more general statistics constructed from samples with random sizes has not been considered yet. At the same time, normal variance-mean mixtures are widely used as mathematical models of statistical regularities in many fields. In particular, in 1977–78 O. Barndorff-Nielsen (Barndorff-Nielsen,

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1977), (Barndorff-Nielsen, 1978) introduced the class of *generalized hyperbolic distributions* as a class of special variance-mean mixtures of normal laws in which the mixing is carried out in one parameter since location and scale parameters of the mixed normal distribution are directly linked. The range of applications of generalized hyperbolic distributions varies from the theory of turbulence or particle size description to financial mathematics, see (Barndorff-Nielsen et al., 1982).

The paper is organized as follows. Basic notation is introduced in section *"Notation. Auxiliary results"*. Here an auxiliary result on the asymptotic rapprochement of the distributions of randomly indexed random sequences with special scale-location mixtures is presented. In section *"General transfer theorem and its inversion. The structure of limit laws"* we present and discuss an improved version of a general transfer theorem for random sequences with independent random indexes in the double array limit setting. We also present its partial inverse providing necessary and sufficient conditions for the convergence of randomly indexed random indexes. Following the lines of (Korolev, 1993), we first formulate a general result improving some results of (Korolev, 1993; Bening and Korolev, 2002) by removing some superfluous assumptions and relaxing some conditions. Special attention is paid to the case where the elements of the basic double array are formed as cumulative sums of independent not necessarily identically distributed random variables. This case is considered in section *"Limit theorems for random sums of independent random variables"*. To prove our results, we use simply tractable moment-type conditions which can be easily interpreted unlike general conditions providing the weak convergence of random sums of non-identically distributed summands in (Szasz, 1972; Kruglov and Korolev, 1990) and (Kruglov and ZhangBo, 2002). In section *"A version of the central limit theorem for random sums with a normal variance-mean mixture as the limiting law"* we present the theorem on convergence of the distributions of such sums to normal variance-mean mixtures. As a simple corollary of this result we can obtain some results of the recent paper by A.A. Toda *"Weak limit of the geometric sum of independent but not identically distributed random variables"* (arXiv:1111.1786v2. 2011). That paper demonstrates that there is still a strong interest to geometric sums of non-identically distributed summands and to the application of the skew Laplace distribution which is a normal variance-mean mixture under exponential mixing distribution (Korolev and Sokolov, 2012). Special attention is paid to the case where the elements of the basic double array are formed as statistics constructed from samples with random sizes. This case is considered in section *"Limit theorems for statistics constructed from samples with random sizes"*.

Unfortunately, to fit the requirements to the size of the paper we had to omit the details of the proofs which will be published elsewhere.

NOTATION. AUXILIARY RESULTS

Assume that all the random variables considered in this paper are defined on one and the same probability space $(\Omega,\ \mathfrak{F},\ \mathsf{P})$. In what follows the symbols $\stackrel{d}{=}$ and $\Longrightarrow$ will denote coincidence of distributions and weak convergence (convergence in distribution). A family $\{X_j\}_{j\in\mathbb{N}}$ of random variables is said to be *weakly relatively compact*, if each sequence of its elements contains a weakly convergent subsequence. In the finite-dimensional case the weak relative compactness of a family $\{X_j\}_{j\in\mathbb{N}}$ is equivalent to its *tightness*: $\lim_{R\to\infty}\sup_{n\in\mathbb{N}}\mathsf{P}(|X_n|>R)=0$ (see, e. g., (Loeve, 1977)).

Let $\{S_{n,k}\}$, $n,k\in\mathbb{N}$, be a double array of random variables. For $n,k\in\mathbb{N}$ let $a_{n,k}$ and $b_{n,k}$ be real numbers such that $b_{n,k}>0$. The purpose of the constants $a_{n,k}$ and $b_{n,k}$ is to provide weak relative compactness of the family of the random variables $\big\{Y_{n,k}\equiv(S_{n,k}-a_{n,k})/b_{n,k}\big\}_{n,k\in\mathbb{N}}$ in the cases where it is required.

Consider a family $\{N_n\}_{n\in\mathbb{N}}$ of nonnegative integer random variables such that for each $n,k\in\mathbb{N}$ the random variables N_n and $S_{n,k}$ are independent. Especially note that we do not assume the row-wise independence of $\{S_{n,k}\}_{k\geq1}$. Let c_n and d_n be real numbers, $n\in\mathbb{N}$, such that $d_n>0$. Our aim is to study the asymptotic behavior of the random variables $Z_n\equiv(S_{n,N_n}-c_n)/d_n$ as $n\to\infty$ and find rather simple conditions under which the limit laws for Z_n have the form of normal variance-mean mixtures. In order to do so we first formulate a somewhat more general result following the lines of (Korolev, 1993), removing superfluous assumptions, relaxing the conditions and generalizing some of the results of that paper.

The characteristic functions of the random variables $Y_{n,k}$ and Z_n will be denoted $h_{n,k}(t)$ and $f_n(t)$, respectively, $t\in\mathbb{R}$.

Let Y be a random variable whose distribution function and characteristic function will be denoted $H(x)$ and $h(t)$, respectively, $x,t\in\mathbb{R}$. Introduce the random variables $U_n=b_{n,N_n}/d_n$, $V_n=(a_{n,N_n}-c_n)/d_n$. Introduce the function

$$g_n(t)\equiv\mathsf{E}h(tU_n)\exp\{itV_n\}=$$
$$=\sum_{k=1}^{\infty}\exp\Big\{it\frac{a_{n,k}-c_n}{d_n}\Big\}h\Big(\frac{tb_{n,k}}{d_n}\Big),\quad t\in\mathbb{R}.$$

It can be easily seen that $g_n(t)$ is the characteristic function of the random variable $Y\cdot U_n+V_n$ where the random variable Y is independent of the pair $(U_n,\ V_n)$. Therefore, the distribution function

$G_n(x)$ corresponding to the characteristic function $g_n(t)$ is the scale-location mixture of the distribution function $H(x)$:

$$G_n(x) = \mathsf{E}H\big((x - V_n)/U_n\big), \quad x \in \mathbb{R}. \qquad (1)$$

In the double-array limit setting considered in this paper, to obtain non-trivial limit laws for Z_n we require the following additional *coherency condition*: for any $T \in (0, \infty)$

$$\lim_{n\to\infty} \mathsf{E} \sup_{|t|\le T} \big|h_{n,N_n}(t) - h(t)\big| = 0. \qquad (2)$$

To clarify the sense of the coherency condition, note that if we had usual row-wise convergence of $Y_{n,k}$ to Y, then for any $n \in \mathbb{N}$ and $T \in [0, \infty)$

$$\lim_{k\to\infty} \sup_{|t|\le T} |h_{n,k}(t) - h(t)| = 0. \qquad (3)$$

So we can say that coherency condition (2) means that "pure" row-wise convergence (3) takes place "on the average" so that that the "row-wise" convergence as $k \to \infty$ is somehow coherent with the "principal" convergence as $n \to \infty$.

REMARK 1. It can be easily verified that, since the values under the expectation sign in (2) are nonnegative and bounded (by two), then coherency condition (2) is equivalent to that $\sup_{|t|\le T} \big|h_{n,N_n}(t) - h(t)\big| \longrightarrow 0$ in probability as $n \to \infty$.

LEMMA 1. *Let the family of random variables $\{U_n\}_{n\in\mathbb{N}}$ be weakly relatively compact. Assume that coherency condition* (2) *holds. Then for any $t \in \mathbb{R}$ we have*

$$\lim_{n\to\infty} |f_n(t) - g_n(t)| = 0.$$

Lemma 1 makes it possible to use the distribution function $G_n(x)$ (see (1)) as an *accompanying asymptotic* approximation to $F_n(x) \equiv \mathsf{P}(Z_n < x)$. In order to obtain a *limit* approximation, in the next section we formulate and prove the transfer theorem.

GENERAL TRANSFER THEOREM AND ITS INVERSION. THE STRUCTURE OF LIMIT LAWS

THEOREM 1. *Assume that coherency condition* (2) *holds. If there exist random variables U and V such that the joint distributions of the pairs (U_n, V_n) converge to that of the pair (U, V) :*

$$(U_n, V_n) \Longrightarrow (U, V) \quad (n \to \infty), \qquad (4)$$

then

$$Z_n \Longrightarrow Z \stackrel{d}{=} Y \cdot U + V \quad (n \to \infty). \qquad (5)$$

where the random variable Y is independent of the pair (U, V).

It is easy to see that relation (5) is equivalent to the following relation between the distribution functions $F(x)$ and $H(x)$ of the random variables Z and Y:

$$F(x) = \mathsf{E}H\big((x - V)/U\big), \quad x \in \mathbb{R}, \qquad (6)$$

that is, the limit law for normalized randomly indexed random variables Z_n is a scale-location mixture of the distributions which are limiting for normalized non-randomly indexed random variables $Y_{n,k}$. Among all scale-location mixtures, *variance-mean mixtures* attract a special interest (to be more precise, we should speak of *normal variance-mean mixtures*). Let us see how these mixture can appear in the double-array setting under consideration.

Assume that the centering constants $a_{n,k}$ and c_n are in some sense proportional to the scaling constants $b_{n,k}$ and d_n. Namely, assume that there exist $\rho > 0$, $\alpha_n \in \mathbb{R}$ and $\beta_n \in \mathbb{R}$ such that for all $n, k \in \mathbb{N}$ we have

$$a_{n,k} = \frac{b_{n,k}^{\rho+1}\alpha_n}{d_n^{\rho}}, \quad c_n = d_n\beta_n,$$

and there exist finite limits

$$\alpha = \lim_{n\to\infty} \alpha_n, \quad \beta = \lim_{n\to\infty} \beta_n.$$

Then under condition (4) and $n \to \infty$

$$(U_n, V_n) = \Big(\frac{b_{n,N_n}}{d_n}, \frac{a_{n,N_n} - c_n}{d_n}\Big) = \\ = \big(U_n, \alpha_n U_n^{\rho+1} + \beta_n\big) \Longrightarrow \big(U, \alpha U^{\rho+1} + \beta\big),$$

so that in accordance with theorem 2 the limit law for Z_n takes the form

$$\mathsf{P}(Z < x) = \mathsf{E}H\Big(\frac{x - \beta - \alpha U^{\rho+1}}{U}\Big), \quad x \in \mathbb{R}.$$

If $\rho = 1$, then we obtain the “pure” variance-mean mixture

$$\mathsf{P}(Z < x) = \mathsf{E}H\Big(\frac{x - \beta - \alpha U^2}{U}\Big), \quad x \in \mathbb{R}.$$

We will return to the discussion of convergence of randomly indexed sequences, more precisely, of random sums, to normal scale-location mixtures in Sect. 5.

In order to present the result that is a partial inversion of theorem 1, for fixed random variables Z and Y with the characteristic functions $f(t)$ and $h(t)$ introduce the set $\mathcal{W}(Z|Y)$ containing all pairs of random variables (U, V) such that the characteristic function $f(t)$ can be represented as

$$f(t) = \mathsf{E}h(tU)e^{itV}, \quad t \in \mathbb{R}, \qquad (7)$$

and $\mathsf{P}(U \ge 0) = 1$. Whatever random variables Z and Y are, the set $\mathcal{W}(Z|Y)$ is always nonempty since it trivially contains the pair $(0, Z)$. It is easy to see that representation (7) is equivalent to relation (6) between the distribution functions $F(x)$ and $H(x)$ of the random variables Z and Y.

The set $\mathcal{W}(Z|Y)$ may contain more that one element. For example, if Y is the standard normal random variable and $Z \stackrel{d}{=} W_1 - W_2$ where W_1 and W_2 are independent random variables with the same standard exponential distribution, then along with the pair $(0, W_1 - W_2)$ the set $\mathcal{W}(Z|Y)$ contains the pair $\big(\sqrt{W_1}, 0\big)$. In this case $F(x)$ is the symmetric Laplace distribution.

Let $L_1(X_1,\, X_2)$ be the Lévy distance between the distributions of random variables X_1 and X_2: if $F_1(x)$ and $F_2(x)$ are the distribution functions of X_1 and X_2, respectively, then

$$L_1(X_1,\, X_2) = \inf\{y \geq 0 :\\ F_2(x-y) - y \leq F_1(x) \leq F_2(x+y) + y,\ \forall x \in \mathbb{R}\}.$$

As is well known, the Lévy distance metrizes weak convergence. Let $L_2\big((X_1, X_2),\, (Y_1, Y_2)\big)$ be any probability metric which metrizes weak convergence in the space of two-dimensional random vectors. An example of such a metric is the Lévy–Prokhorov metric (see,e. g., (Zolotarev, 1997)).

THEOREM 2. *Let the family of random variables* $\{U_n\}_{n\in\mathbb{N}}$ *be weakly relatively compact. Assume that coherency condition* (2) *holds. Then a random variable* Z *such that*

$$Z_n \Longrightarrow Z \quad (n \to \infty) \tag{8}$$

with some $c_n \in \mathbb{R}$ *exists if and only if there exists a weakly relatively compact sequence of pairs* $(U_n',\, V_n') \in \mathcal{W}(Z|Y)$, $n \in \mathbb{N}$, *such that*

$$\lim_{n\to\infty} L_2\big((U_n,\, V_n),\, (U_n',\, V_n')\big) = 0. \tag{9}$$

REMARK 2. It should be noted that in (Korolev, 1993) and some subsequent papers a stronger and less convenient version of the coherency condition was used. Furthermore, in (Korolev, 1993) and the subsequent papers the statements analogous to lemma 1 and theorems 1 and 2 were proved under the additional assumption of the weak relative compactness of the family $\{Y_{n,k}\}_{n,k\in\mathbb{N}}$.

LIMIT THEOREMS FOR RANDOM SUMS OF INDEPENDENT RANDOM VARIABLES

Let $\{X_{n,j}\}_{j\geq 1}$, $n \in \mathbb{N}$, be a double array of row-wise independent not necessarily identically distributed random variables. For $n, k \in \mathbb{N}$ denote

$$S_{n,k} = X_{n,1} + \ldots + X_{n,k}. \tag{10}$$

If $S_{n,k}$ is a sum of independent random variables, then the condition of weak relative compactness of the sequence $\{U_n\}_{n\in\mathbb{N}}$ used in the preceding section can be replaced by the condition of weak relative compactness of the family $\{Y_{n,k}\}_{n,k\in\mathbb{N}}$ which is in fact considerably less restrictive. Indeed, let, for example, the random variables $S_{n,k}$ possess moments of some order $\delta > 0$. Then, if we choose $b_{n,k} = (\mathsf{E}|S_{n,k} - a_{n,k}|^{\delta})^{1/\delta}$, then by the Markov inequality

$$\lim_{R\to\infty} \sup_{n,k\in\mathbb{N}} \mathsf{P}\big(|Y_{n,k}| > R\big) \leq \lim_{R\to\infty} \frac{1}{R^{\delta}} = 0,$$

that is, the family $\{Y_{n,k}\}_{n,k\in\mathbb{N}}$ is weakly relatively compact.

THEOREM 3. *Assume that the random variables* $S_{n,k}$ *have the form* (10). *Let the family of random variables* $\{Y_{n,k}\}_{n,k\in\mathbb{N}}$ *be weakly relatively compact. Assume that coherency condition* (2) *holds. Then convergence* (8) *of normalized random sums* Z_n *to some random variable* Z *takes place with some* $c_n \in \mathbb{R}$ *if and only if there exists a weakly relatively compact sequence of pairs* $(U_n',\, V_n') \in \mathcal{W}(Z|Y)$, $n \in \mathbb{N}$, *such that condition* (9) *holds.*

A VERSION OF THE CENTRAL LIMIT THEOREM FOR RANDOM SUMS WITH A NORMAL VARIANCE-MEAN MIXTURE AS THE LIMITING LAW

Let $\{X_{n,j}\}_{j\geq 1}$, $n \in \mathbb{N}$, be a double array of row-wise independent not necessarily identically distributed random variables. As in the preceding section, let $S_{n,k} = X_{n,1} + \ldots + X_{n,k}$, $n, k \in \mathbb{N}$. The distribution function of the random variable $X_{n,j}$ will be denoted $F_{n,j}(x)$. Denote $\mu_{n,j} = \mathsf{E}X_{n,j}$, $\sigma^2_{n,j} = \mathsf{D}X_{n,j}$ and assume that $0 < \sigma^2_{n,j} < \infty$, $n, j \in \mathbb{N}$. Denote

$$A_{n,k} = \mu_{n,1} + \ldots + \mu_{n,k}\ \big(= \mathsf{E}S_{n,k}\big),\\ B^2_{n,k} = \sigma^2_{n,1} + \ldots + \sigma^2_{n,k}\ \big(= \mathsf{D}S_{n,k}\big)$$

It is easy to make sure that $\mathsf{E}S_{n,N_n} = \mathsf{E}A_{n,N_n}$, $\mathsf{D}S_{n,N_n} = \mathsf{E}B^2_{n,N_n} + \mathsf{D}A_{n,N_n}$, $n \in \mathbb{N}$. In order to formulate a version of the central limit theorem for random sums with the limiting distribution being a normal variance-mean mixture, assume that nonrandom sums, as usual, are centered by their expectations and normalized by by their mean square deviations and put $a_{n,k} = A_{n,k}$, $b_{n,k} = \sqrt{B^2_{n,k}}$, $n, k \in \mathbb{N}$. Although it would have been quite natural to normalize random sums by their mean square deviations as well, for simplicity we will use slightly different normalizing constants and put $d_n = \sqrt{\mathsf{E}B^2_{n,N_n}}$.

Let $\Phi(x)$ be the standard normal distribution function,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz, \quad x \in \mathbb{R}.$$

THEOREM 4. *Assume that the following conditions hold*:
(i) *for every* $n \in \mathbb{N}$ *there exist real numbers* α_n *such that*

$$\mu_{n,j} = \frac{\alpha_n \sigma_{n,j}^2}{\sqrt{\mathsf{E}B_{n,N_n}^2}}, \quad n, j \in \mathbb{N},$$

and

$$\lim_{n\to\infty} \alpha_n = \alpha, \quad 0 < |\alpha| < \infty;$$

(ii) (*the random Lindeberg condition*) *for any* $\epsilon > 0$

$$\mathsf{E}\left[\frac{1}{B_{n,N_n}^2}\sum_{j=1}^{N_n}\int\limits_{|x-\mu_{n,j}|>\epsilon B_{n,N_n}}(x-\mu_{n,j})^2 dF_{n,j}(x)\right] \to 0$$

as $n \to \infty$.
Then the convergence of the normalized random sums

$$\frac{S_{n,N_n}}{\sqrt{\mathsf{E}B_{n,N_n}^2}} \Longrightarrow Z \tag{11}$$

to some random variable Z *as* $n \to \infty$ *takes place if and only if there exists a random variable* U *such that*

$$\mathsf{P}(Z < x) = \mathsf{E}\Phi\Big(\frac{x-\alpha U}{\sqrt{U}}\Big), \quad x \in \mathbb{R}, \tag{12}$$

and

$$\frac{B_{n,N_n}^2}{\mathsf{E}B_{n,N_n}^2} \Longrightarrow U \quad (n \to \infty). \tag{13}$$

The proof uses a result of V. V. Petrov (Petrov, 1979) improved in (Korolev and Popov, 2012), the reasoning used to prove theorem 7 in Sect. 3, Chapt. V of (Petrov, 1987) and relation (3.8) and the fact recently proved in (Korolev, 2013) that normal variance-mean mixtures are identifiable.

REMARK 3. In accordance with what has been said in remark 1, the random Lindeberg condition (ii) can be used in the following form: for any $\epsilon > 0$

$$\frac{1}{B_{n,N_n}^2}\sum_{j=1}^{N_n}\int\limits_{|x-\mu_{n,j}|>\epsilon B_{n,N_n}}(x-\mu_{n,j})^2 dF_{n,j}(x) \longrightarrow 0$$

in probability as $n \to \infty$.

LIMIT THEOREMS FOR STATISTICS CONSTRUCTED FROM SAMPLES WITH RANDOM SIZES

Let $\{X_{n,j}\}_{j\geq 1}$, $n \in \mathbb{N}$, be a double array of row-wise independent not necessarily identically distributed random variables. For $n, k \in \mathbb{N}$ let $T_{n,k} = T_{n,k}(X_{n,1}, ..., X_{n,k})$ be a statistic, i.e., a real-valued measurable function of $X_{n,1}, ..., X_{n,k}$. For each $n \geq 1$ we define a r.v. T_{n,N_n} by setting $T_{n,N_n}(\omega) \equiv T_{n,N_n(\omega)}(X_{n,1}(\omega), ..., X_{n,N_n(\omega)}(\omega))$, $\omega \in \Omega$.

Let θ_n be real numbers, $n \in \mathbb{N}$. In this section we will assume that the random variables $S_{n,k}$ have the form $S_{n,k} = T_{n,k} - \theta_n$, $n, k \in \mathbb{N}$. Concerning the normalizing constants we will assume that there exist finite real numbers α, α_n, β, β_n $\sigma_n > 0$ such that

$$\alpha_n \to \alpha, \quad \beta_n \to \beta \quad (n \to \infty) \tag{14}$$

and for all $n, k \in \mathbb{N}$

$$b_{n,k} = \frac{1}{\sigma_n\sqrt{k}}, \quad d_n = \frac{1}{\sigma_n\sqrt{n}},$$

$$a_{n,k} = \frac{\alpha_n\sqrt{n}}{\sigma_n k}, \quad c_n = \frac{\beta_n}{\sigma_n\sqrt{n}} \tag{15}$$

so that

$$Y_{n,k} = \sigma_n\sqrt{k}(T_{n,k} - \theta_n) - \alpha_n\sqrt{n/k}$$

and

$$Z_n = \sigma_n\sqrt{n}(T_{n,N_n} - \theta_n) - \beta_n.$$

As this is so, σ_n^2 can be regarded as the asymptotic variance of $T_{n,k}$ as $k \to \infty$ whereas the bias of $T_{n,k}$ is $\alpha_n\sqrt{n}/(k\sigma_n)$.

The l_1-distance between distribution functions P_1 and P_2 will be denoted $\|P_1 - P_2\|$:

$$\|P_1 - P_2\| = \int_{-\infty}^{\infty} |P_1(x) - P_2(x)| dx.$$

Recall that the distribution function of the random variable $Y_{n,k}$ is denoted $H_{n,k}(x)$. Let $\Phi(x)$ be the standard normal distribution function. In what follows we will assume that the statistic $T_{n,k}$ is asymptotically normal in the following sense: for any $\gamma > 0$

$$\lim_{n\to\infty} \mathsf{E}\|H_{n,N_n} - \Phi\| = 0. \tag{16}$$

THEOREM 5. *Let the family of random variables* $\{n/N_n\}_{n\in\mathbb{N}}$ *be weakly relatively compact, the normalizing constants have the form* (15) *and satisfy condition* (14). *Assume that the statistic* $T_{n,k}$ *is asymptotically normal so that condition* (16) *holds. Then a random variable* Z *such that*

$$\sigma_n\sqrt{n}(T_{n,N_n} - \theta_n) - \beta_n \Longrightarrow Z \quad (n \to \infty)$$

exists if and only if there exists a nonnegative random variable W *such that* ($x \in \mathbb{R}$)

$$\mathsf{P}(Z < x) = \int_0^{\infty} \Phi\Big(\frac{x - \beta - \alpha w}{\sqrt{w}}\Big) d\mathsf{P}(W < w),$$

and

$$\mathsf{P}(N_n < nx) \Longrightarrow \mathsf{P}(W^{-1} < x) \quad (n \to \infty).$$

CONCLUDING REMARKS

The class of normal variance-mean mixtures is very wide and, in particular, contains the class of generalized hyperbolic distributions which, in turn, contains (a) symmetric and skew Student distributions (including the Cauchy distribution) with inverse gamma mixing distributions; (b) variance gamma distributions (including symmetric and non-symmetric Laplace distributions) with gamma mixing distributions; (c) normal\\inverse Gaussian distributions with inverse Gaussian mixing distributions including symmetric stable laws. By variance-mean mixing many other initially symmetric types represented as pure scale mixtures of normal laws can be skewed, e. g., as it was done to obtain non-symmetric exponential power distributions in (Grigoryeva and Korolev, 2013).

According to theorem 4, all these laws can be limiting for random sums of independent non-identically distributed random variables. For example, to obtain the skew Student distribution for Z it is necessary and sufficient that in (12) and (13) the random variable U has the inverse gamma distribution (Korolev and Sokolov, 2012). To obtain the variance gamma distribution for Z it is necessary and sufficient that in (12) and (13) the random variable U has the gamma distribution (Korolev and Sokolov, 2012). In particular, for Z to have the asymmetric Laplace distribution it is necessary and sufficient that U has the exponential distribution.

Remark 4. Note that the non-random sums in the coherency condition are centered, whereas in (11) the random sums are not centered, and if $\alpha \neq 0$, then the limit distribution for random sums becomes skew unlike usual non-random summation, where the presence of the systematic bias of the summands results in that the limit distribution becomes just shifted. So, if non-centered random sums are used as models of some real phenomena and the limit variance-mean mixture is skew, then it can be suspected that the summands are actually biased.

Remark 5. In limit theorems of probability theory and mathematical statistics, centering and normalization of random variables are used to obtain non-trivial asymptotic distributions. It should be especially noted that to obtain reasonable approximation to the distribution of the basic random variables (in our case, S_{n,N_n}), both centering and normalizing values should be non-random. Otherwise the approximate distribution becomes random itself and, say, the problem of evaluation of quantiles becomes senseless.

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