POWER FUNCTIONS OF STATISTICAL CRITERIA DEFINED BY BANS

Alexander A. Grusho, Nick A. Grusho and Elena E. Timonina
Institute of Informatics Problems,
Federal Research Center "Computer Science and Control"
of the Russian Academy of Sciences
Vavilova 44-2,
119333, Moscow, Russia
Email: grusho@yandex.ru

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ABSTRACT

The basic problem for developers of monitoring systems for technological processes is to exclude the false alarms. False alarms generate the interruption of technological process and lead to the manual analysis of the reasons of the wrong system behavior.

In the paper it is offered to use the statistical techniques with probabilities of false alarms equal to zero. This class of statistical decisions is based on concept of bans of probability measures in a finite space. Conditions under which powers of statistical criteria accept value 1 on a finite step are found. These conditions are formulated in terms of supports of probability measures.

INTRODUCTION

The paper deals with the mathematical model of monitoring of a technological system behavior with finite set of states. Suppose that such monitoring systems solve the task with the help of statistical techniques. In the mathematical models the trajectories of functioning of such system are represented by infinite sequences in which each coordinate accepts value in the finite fixed alphabet.

Application of statistical techniques on a set of infinite sequences demands a probability measure P which describes the correct behavior of analyzable system. The wrong system behavior is described by a probability distribution Q. Different wrong behaviors of the technological system can be described by different distributions of probabilities on space of the infinite sequences. However in the elementary case it is possible to assume that distribution of the wrong behavior of technological system is unique and known.

In practice the monitoring system of technological process observes initial sections of trajectories and for each step n it tests the hypothesis $H_{0,n}$ that the distribution of the observed section of trajectory is defined by probablity distribution measure P_n which is the projection of measure P on the first n coordinates. The alternative hypothesis $H_{1,n}$ in the elementary case is defined by measure Q_n which is projection of measure Q on the first n coordinates. Criteria of testing of hypotheses $H_{0,n}$ against alternatives $H_{1,n}$ allow to make the decision about the wrong behavior of technological system.

The basic problem for developers of such monitoring systems is the false alarms appearance when the correct behavior of technological process is perceived as wrong (Axelson, 1999). False alarms generate interruption of technological process, and that the worst, they lead to necessity of the manual analysis of the reasons of the wrong system behavior.

For this purpose in the paper it is offered to use the statistical techniques for monitoring with probabilities of false alarms equal to zero. This class of statistical decisions is based on concept of the ban (Grusho and Timonina, 2011; Grusho et al., 2013). The ban of a probability measure in the considered scheme is a vector for which probability of its appearance is equal to 0 in a finite projection of measure.

Any statistical criterion for testing $H_{0,n}$ against $H_{1,n}$ is defined by a critical set S_n of vectors of length n. When the observed vector is in S_n then it leads to the acceptance of alternative $H_{1,n}$. If all vectors in S_n are bans of a measure P_n, say that the criterion is defined by bans of a measure P.

Existence and properties of the criteria determined by bans were researched in papers (Grusho and Timonina, 2011; Grusho et al., 2013, 2014). In particular, the behavior of power function of criteria was researched in case of $n \to \infty$. Conditions of consistency of sequence of the statistical criteria determined by bans, i.e. conditions when powers of criteria tend to 1 in case of $n \to \infty$ are found.

Specialists believed that all properties of power functions for finite n were defined by numerical values of probability distributions P and Q. However in this paper conditions under which power functions of criteria accept value 1 on a finite step are found. These conditions are formulated in terms of supports of probability measures for the main measure P on space of the infinite sequences and for alternatives.

Information about supports of measures is known not always. Therefore in the paper we built the constructive check of conditions for existence of criteria with the power function equals to 1 on a finite step N.

The article is structured as follows. Section 2 introduces definitions and previous results. In Section 3 the

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main results are proved. In Conclusion we shortly analyze applications of constructed sequences of tests.

MATHEMATICAL MODEL. BASIC DEFINITIONS AND PREVIOUS RESULTS

Let's consider mathematical model of some technological process. Let $X = \{x_1, ..., x_m\}$ be a finite set, X^n be a Cartesian product of X, X^∞ be a set of all sequences when i-th element belongs to X. Define $\mathcal{A}$ be a σ-algebra on X^∞, generated by cylindrical sets. $\mathcal{A}$ is also Borel σ-algebra in Tychonoff product X^∞, where X has a discrete topology (Bourbaki, 1968; Prokhorov and Rozanov, 1993).

On $(X^\infty, \mathcal{A})$ a probability measure P is defined. For any $n = 1, 2, ...$, assume that probability distribution P_n is a projection of measure P on the first n coordinates of random sequences from X^∞. It is clear that for every $B_n \subseteq X^n$

$$P_n(B_n) = P(B_n \times X^\infty). \quad (1)$$

Let $D_n(P)$ be the support of a measure P_n (Prokhorov and Rozanov, 1993):

$$D_n(P) = \{\overline{x}_n \in X^n,\ P_n(\overline{x}_n) > 0\}.$$

Define cylindrical set $\Delta_n(P)$ as follows:

$$\Delta_n(P) = D_n(P) \times X^\infty.$$

The sequence of cylindrical sets $\Delta_n(P)$, n=1,2,..., is not increasing and

$$\Delta(P) = \lim_{n\to\infty} \Delta_n(P) = \bigcap_{n=1}^{\infty} \Delta_n(P). \quad (2)$$

Assertion 1. The set $\Delta(P)$ is closed and it is a support of probability measure P.

Proof. For every $n = 1, 2, ...$, the set $\Delta_n(P)$ is simultaneously an open and a closed set. Then $\Delta(P)$ is a closed set. Due to continuity of probability measure P it follows from (2) that $P(\Delta(P)) = 1$. Denote Δ be the support of measure P. By the definition of a support of a probability measure $\Delta \subseteq \Delta(P)$. Let's suppose that there is sequence $\omega \in \Delta(P)$ which doesn't belong to the set Δ. From the separation property of Tychonoff product it follows that there exists an open set O such that $\omega \in O$ and $O \cap \Delta = \emptyset$. From the fact that Tychonoff product is a space with countable basis (Bourbaki, 1968) it follows that $O = \bigcup_{n=1}^\infty I(n)$, where $I(n)$ is an elementary cylindrical set. There is n such that the sequence $\omega \in I(n)$. It means that $I(n) = I(\overline{\omega}_N)$ for some N, where the set $I(\overline{\omega}_N)$ is the elementary cylindrical set which is defined by vector $\overline{\omega}_N$ and this vector coincides with the first N coordinates of the sequence ω. Due to definition of $\Delta_n(P)$:

$$\overline{\omega}_N \in D_N(P),\ I(\overline{\omega}_N) \subseteq \Delta_N(P),$$

and

$$P_N(\omega_N) > 0.$$

Then for initial n for which $I(n) = I(\overline{\omega}_N)$ we get the inequality $P(I(n)) > 0$ and

$$1 = P(\Delta_N(P)) \geq P(\Delta \bigcup I(n)) = P(\Delta) + P(I(n)) > 1.$$

That means that supposition is wrong and $\Delta = \Delta(P)$, and assertion 1 is proved.

Let $\overline{\omega}_k \in X^k$ and $\widetilde{\omega}_{k-1}$ is obtained from $\overline{\omega}_k$ by dropping the last coordinate.

Definition 1. Ban of measure P_n (Grusho et al., 2013) is a vector $\overline{\omega}_k \in X^k$, $k \leq n$, such that

$$P_n(\overline{\omega}_k \times X^{n-k}) = 0.$$

Definition 2. Ban $\overline{\omega}_k$ of measure P_n is the smallest ban of measure P (Grusho et al., 2013) if

$$P_{k-1}(\widetilde{\omega}_{k-1}) > 0.$$

If $\overline{\omega}_k$ is a ban of measure P_n then for every $k \leq s \leq n$ and for every sequence $\overline{\omega}_s$ starting with $\overline{\omega}_k$ we have

$$P_s(\overline{\omega}_s) = 0.$$

In fact, if

$$P_k(\overline{\omega}_k) = 0,$$

then it follows from (1) that

$$P(\overline{\omega}_k \times X^\infty) = 0,$$

and

$$P(\overline{\omega}_k \times X^{s-k} \times X^\infty) = 0.$$

It follows that

$$P_s(\overline{\omega}_s) = P(\overline{\omega}_s \times X^\infty) \leq$$

$$\leq P(\overline{\omega}_k \times X^{s-k} \times X^\infty) = 0.$$

If there exists a vector $\overline{\omega}_n \in X^n$ such that $P_n(\overline{\omega}_n) = 0$, then there exists a smallest ban which is defined by the values of the first coordinates of vector $\overline{\omega}_n$.

Further under Λ_n we will understand a set of the smallest bans of measure P_n, which have lengths equal to n.

We also consider a probability measure Q on $(X^\infty, \mathcal{A})$ for which Q_n, $D_n(Q)$, $\Delta_n(Q)$, $\Delta(Q)$ are defined.

Consider the sequence of criteria for testing of hypotheses $H_{0,n} : P_n$ against alternatives $H_{1,n} : Q_n$, $n = 1, 2, ...$

The statistical criterion for testing $H_{0,n}$ against $H_{1,n}$ is defined by a critical set S_n of vectors of length n. When the observed vector is in S_n then it leads to the acceptance of alternative $H_{1,n}$. If all vectors from S_n are bans of measure P_n, say that the criterion is defined by bans of measure P. Note that for every n we have $P_n(S_n) = 0$, if S_n is defined by bans.

Let W_n be the power function of criterion for testing $H_{0,n}$ against $H_{1,n}$. It is known that $W_n = Q_n(S_n)$.

The basic problem considered in the paper is to find conditions when there exists such N that for all $n > N$ the power function $W_n = 1$.

MATHEMATICAL RESULTS

Let P and Q are probability measures defined in Section 2. The solution for the basic problem is described in the next theorem.

Theorem 1. *There exists a sequence of criteria for testing $H_{0,n}$ against $H_{1,n}$ with critical sets S_n, $n = 1, 2, ...,$ defined by bans, for which exists such N, that for every $n \geq N$ power function $W_n = 1$ if and only if*

$$\Delta(P) \cap \Delta(Q) = \emptyset.$$

The proof of the theorem 1 is based on several lemmas.

Let $\overline{x}_k$ be the smallest ban of measure P. Then define $I(\overline{x}_k)$ be the elementary cylindrical set in X^∞, which is generated by the vector $\overline{x}_k$.

Lemma 1. For every sequence $\omega \in I(\overline{x}_k)$ it follows that $\omega \notin \Delta(P)$.

Proof. Suppose that there exists $\omega \in I(\overline{x}_k)$ that belongs to $\Delta(P)$. From formula (2) it follows that $\omega \in \Delta_n(P)$ for every $n = 1, 2, ...$ By the definition of $\Delta_k(P)$ the vector $\overline{\omega}_k$ defined by the first k coordinates of ω belongs to the set $D_k(P)$. Then

$$P_k(\overline{\omega}_k) > 0.$$

Besides

$$\overline{\omega}_k = \overline{x}_k,$$

that contradicts to supposition. The lemma 1 is proved.

Let's define the open set S:

$$S = \bigcup_{k=1}^{\infty} \bigcup_{\overline{x}_k \in \Lambda_k} I(\overline{x}_k). \quad (3)$$

From lemma 1 it follows that

$$S \cap \Delta(P) = \emptyset.$$

Lemma 2. The set S can be represented in the next form

$$S = X^\infty \backslash \Delta(P).$$

Proof. From $S \cap \Delta(P) = \emptyset$ it follows that

$$S \subseteq X^\infty \backslash \Delta(P).$$

Let's assume that

$$\omega \in X^\infty \backslash \Delta(P).$$

If $\omega \in X^\infty \backslash \Delta(P)$ then

$$\omega \notin \Delta(P) = \bigcap_{n=1}^{\infty} \Delta_n(P).$$

The sequence of sets $\{\Delta_n(P)\}$ is not increasing. Then there exists n such that for every $t \geq n$ we have $\omega \notin \Delta_t(P)$. That means that $P_t(\overline{x}_t) = 0$. Thus there exists the smallest ban $\overline{x}_k$ such that $\omega \in I(\overline{x}_k)$, so $\omega \in S$. Lemma is proved.

Lemma 3. $\Delta(Q) \cap \Delta(P) = \emptyset$ if and only if

$$\Delta(Q) \subseteq S.$$

Proof. From the condition of lemma 3 it follows that

$$\Delta(Q) \subseteq X^\infty \backslash \Delta(P).$$

Then from lemma 2 $\Delta(Q) \subseteq S$.

On the other hand if $\Delta(Q) \subseteq S$, then

$$\Delta(Q) \subseteq X^\infty \backslash \Delta(P),$$

and it follows that

$$\Delta(Q) \cap \Delta(P) = \emptyset.$$

Lemma is proved.

Lemma 4. If $\Delta(Q) \cap \Delta(P) = \emptyset$ then $\exists N$ such that

$$\Delta(Q) \subseteq \bigcup_{k=1}^{N} \bigcup_{\overline{x}_k \in \Lambda_k} I(\overline{x}_k).$$

Proof. Tychonoff product X^∞ is a compact space (Bourbaki, 1968) and therefore from an every infinite cover of a compact by open sets it is possible to select a finite cover. The closed set $\Delta(Q)$ is a compact and $\Delta(Q) \subseteq S$. That's why due to definition (3) there exists N such that

$$\Delta(Q) \subseteq \bigcup_{k=1}^{N} \bigcup_{\overline{x}_k \in \Lambda_k} I(\overline{x}_k) = \sigma_N. \quad (4)$$

Lemma 4 is proved.

The set σ_N is a cylindrical set. Therefore it can be represented in the next form

$$\sigma_N = C_N \times X^\infty,$$

where

$$C_N \subseteq X^N.$$

Lemma 5. The support of measure Q_N satisfies to the following condition

$$D_N(Q) \subseteq C_N.$$

Proof. Let's denote by $\overline{x}_N x$ the concatenation of vector $\overline{x}_N \in X^N$ and an element x of alphabet X. If $\overline{x}_N \in D_N(Q)$, then there exists $x \in X$ for which

$$\overline{x}_N x \in D_{N+1}(Q).$$

Otherwise

$$Q_{N+1}(\overline{x}_N \times X) = 0.$$

Due to the consistency of sequence of probability measures $\{Q_n\}$ it follows that $Q_N(\overline{x}_N) = 0$. Then for every natural number t there exists $\overline{x}_t \in X^t$, such that

$$\overline{x}_N \overline{x}_t \in D_{N+t}(Q),$$

where the vector $\overline{x}_N \overline{x}_t$ is the concatenation of vectors $\overline{x}_N$ and $\overline{x}_t$.

Then there exists $\omega \in \Delta(Q)$, such that $\overline{\omega}_N = \overline{x}_N$. It was proved in lemma 4 that $\Delta(Q) \subseteq \sigma_N$. That's why $\overline{x}_N \in C_N$. The lemma 5 is proved.

To prove the sufficient condition of the theorem 1 it is enough to denote $S_N = C_N$. According to lemma 5

$$Q_N(S_N) = 1.$$

Let's prove the necessary condition of the theorem 1. Let S_N be such a critical set for which $Q_N(S_N) = 1$ and S_N is defined by bans of measure P. Using the definition of σ in (4) we conclude that every set S_N defined by bans satisfies to

$$S_N \times X^\infty \subseteq \sigma.$$

Then $D_N(Q) \subseteq S_N$. Therefor for cylindrical sets we have

$$\Delta_N(Q) \subseteq \sigma.$$

Thus

$$\Delta(Q) \subseteq \sigma.$$

As

$$\sigma = X^\infty \backslash \Delta(P),$$

then it follows that

$$\Delta(P) \cap \Delta(Q) = \emptyset.$$

The theorem 1 is proved.

Let's consider a set of probability measures $\{Q_\theta, \theta \in \Theta\}$ on $(X^\infty, \mathcal{A})$, for which $Q_{\theta, n}$, $D_n(Q_\theta)$, $\Delta_n(Q_\theta)$, $\Delta(Q_\theta)$ are defined as in Section 2.

Consider a problem of testing a sequence of hypotheses $H_{0, n} : P_n$ against complex alternatives $H_{1, n} : \{Q_{\theta, n}, \theta \in \Theta\}$. Let $W_n(\theta)$ be a power function.

Theorem 2. *There exists a sequence of criteria for testing $H_{0, n}$ against $H_{1, n}$ with critical sets S_n, $n = 1, 2...$, defined by bans, for which exists such N that for every $n \geq N$ the power function $W_n(\theta) = 1$ if and only if there exists a closed set Δ such that for every θ, $\theta \in \Theta$*

$$\Delta(Q_\theta) \subseteq \Delta,$$

and

$$\Delta(P) \cap \Delta = \emptyset.$$

Proof. By the same way as in the theorem 1 for the set Δ there exists a finite cover

$$\sigma_N = \bigcup_{k=1}^{N} \bigcup_{\overline{x}_k \in \Lambda_k} I(\overline{x}_k),$$

where $I(\overline{x}_k)$ be the elementary cylindrical set in X^∞, which is generated by the smallest ban $\overline{x}_k$.

The set σ_N is a cylindrical set and $\sigma_N = C_N \times X^\infty$. From the conditions of the theorem 2 it follows that $Q_{\theta, N}(C_N) = 1$, $\theta \in \Theta$. If critical set S_N of criterion for testing $H_{0, N} : P_N$ against the complex alternatives $H_{1, N} : \{Q_{\theta, N}, \theta \in \Theta\}$ is chosen as $S_N = C_N$, then $W_n(\theta) = 1$ for all θ. The sufficiency is proved.

Let' prove the necessity. Let S_N be such a critical set for which $Q_{\theta, N}(S_N) = 1$ for all θ, $\theta \in \Theta$, and S_N is defined by bans of measure P. Then for all θ, $\theta \in \Theta$, the set $D_N(Q_\theta) \subseteq S_N$, and using the definition of σ_N in (4) we conclude that every set S_N defined by bans satisfies to

$$S_N \times X^\infty \subseteq \sigma_N.$$

For cylindrical sets we have

$$\Delta_N(Q_\theta) \subseteq \sigma_N, \ \theta \in \Theta.$$

Thus

$$\Delta(Q_\theta) \subseteq \sigma_N, \ \theta \in \Theta.$$

By the definition

$$S_N \cap D_N(P) = \emptyset.$$

As

$$\Delta(P) \subseteq \Delta_N(P).$$

then it follows that

$$\sigma_N \cap \Delta(P) = \emptyset.$$

σ_N is a cylindrical set and in discrete topology on X it is a closed set. So we can define $\Delta = \sigma_N$. The theorem 2 is proved.

Corollary. Let's consider a set of probability measures $\{Q_{\theta, n}, \theta \in \Theta, |\Theta| < \infty\}$. There exists a sequence of criteria for testing $H_{0, n}$ against $H_{1, n}$ with critical sets S_n, $n = 1, 2, ...$, defined by bans, for which exists such N, that for every $n \geq N$ power function $W_n(\theta) = 1$ if and only if that for every θ, $\theta \in \Theta$,

$$\Delta(P) \cap \Delta(Q_\theta) = \emptyset.$$

Proof. In any topological space a finite union of closed sets is a closed set (Bourbaki, 1968; Prokhorov and Rozanov, 1993). The union of all supports of probability measures Q_θ, $\theta \in \Theta$, satisfies to conditions of the theorem 2. That proves the corollary.

Not always the exact description of supports of measures can be received in an explicit form. At the same time, application of the theory explained above can be made more constructive.

In the further theorem 3 there are more constructive conditions for usage of theorems 1 and 2. These conditions are not completely constructive and only declare that there is a finite step N for which it is possible to check feasibility of sufficient conditions for the supports formulated in theorems 1 and 2. The number of this steps isn't defined, but membership functions for certain necessary sets are can be carried out with the help of constructive calculation.

The next theorem is formulated for the elementary case when the given measures are P and Q.

Theorem 3. *For the given probability measures P and Q on $(X^\infty, \mathcal{A})$*

$$\Delta(P) \cap \Delta(Q) = \emptyset$$

if and only if there exists N such that

$$\Delta_N(P) \cap \Delta_N(Q) = \emptyset.$$

Proof. If there exists N such that $\Delta_N(P) \cap \Delta_N(Q) = \emptyset$, then due to definitions of sets $\Delta(P)$ and $\Delta(Q)$ it follows that $\Delta(P) \subseteq \Delta_N(P)$ and $\Delta(Q) \subseteq \Delta_N(Q)$. It proves the sufficiency. Let

$$\Delta(P) \cap \Delta(Q) = \emptyset.$$

It means that

$$\bigcap_{n=1}^{\infty} (\Delta_n(P) \cap \Delta_n(Q)) = \emptyset.$$

In the considered case the topological Tychonoff product X^∞ is a compact space (Bourbaki, 1968). From the fact that if an infinite intersection of closed sets is the empty set then there exists N such that

$$\bigcap_{n=1}^{N} (\Delta_n(P) \cap \Delta_n(Q)) = \emptyset.$$

That is

$$(\bigcap_{n=1}^{N} (\Delta_n(P)) \cap (\bigcap_{n=1}^{N} (\Delta_n(Q)) = \emptyset.$$

Due to not increasing of the sequences of sets $\Delta_N(P)$ and $\Delta_N(Q)$ it follows that

$$\Delta_N(P) \cap \Delta_N(Q) = \emptyset.$$

The theorem 3 is proved.

CONCLUSION

Let's formulate the requirements for a monitoring system of some technological process which separates normal and abnormal behavior of the process. It's necessary to prevent false alarms and for sure to find the process deviations from the normal behavior. Usage of bans helps to exclude false alarms by definition.

The paper introduces the conditions to estimate the trust for statistical methods defined by bans.

In the next works we'll try to define algorithms for the constructive procedures defined by bans which can be implemented in a monitoring system.

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AUTHOR BIOGRAPHIES

ALEXANDER A. GRUSHO was born 28.08.1946, Doctor of sciences (Math), Professor of Moscow State University, leading scientist of Institute of Informatics Problems, Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences. His email is grusho@yandex.ru.

NICK A. GRUSHO was born 15.06.1982, PhD (Math), senior scientist of Institute of Informatics Problems, Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences. His email is info@itake.ru.

ELENA E. TIMONINA was born 27.02.1952, Doctor of sciences (Tech), Professor, leading scientist of Institute of Informatics Problems, Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences. Her email is eltimon@yandex.ru.