ON SHARP BOUNDS OF THE RATE OF CONVERGENCE FOR SOME QUEUEING MODELS

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ABSTRACT

We consider a class of homogeneous SZK models with finite state space and consider sharp bounds on the rate of convergence to stationary distribution.

INTRODUCTION

The problem of relating the rate of convergence to stationarity of an ergodic continuous-time Markov processes is very important in queueing theory. If we have a sufficiently accurate bound, then we can correctly estimate the time at which the queueing system enters the stationary mode. The second application of sharp bounds on the rate of convergence is the possibility of obtaining the correspondent perturbation bounds, see (Zeifman and Korolev 2014).

In this note we deal with the class of queueing systems which was introduced and studied in our previous papers (Satin et al 2013, Zeifman et al. 2014a). Here we consider only finite homogeneous situation and discuss sharp bounds on the rate of convergence. The correspondent problems for birth-death processes were studied in (Van Doorn et al 2010, Granovsky and Zeifman 1997, 2000, 2005). A general approach to obtaining two-sided bounds of the so-called decay parameter α^* (see below) and its asymptotic behavior was considered in details in (Van Doorn et al 2010). This method is based on two main ingredients. The first is the concept of logarithmic norm of a square matrix, developed independently by (Dahlquist 1958) and (Lozinskii 1958) as a tool to derive error bounds in the numerical integration of initial-value problems for a system of ordinary differential equations. The second ingredient is a suitable transformation of the Kolmogorov forward equations. Here we apply this approach to SZK models.

Let $X(t)$, $t \geq 0$ be a queue-length process for a finite SZK model. This is a time-homogeneous continuous-time Markovian queueing model on the state space $E = \{0, 1, \ldots, r\}$ with possible batch arrivals and group services under some special assumptions.

Denote by $p_{ij}(s,t) = Pr\{X(t) = j\,|X(s) = i\}$, $i, j \geq 0$, $0 \leq s \leq t$ the transition probabilities for $X(t)$, and let $p_i(t) = Pr\{X(t) = i\}$ be its state probabilities. Throughout the paper we assume that

$$\Pr\left(X\left(t+h\right)=j/X\left(t\right)=i\right)= \qquad (1)$$
$$\begin{cases} q_{ij}h+\alpha_{ij}\left(t,h\right), \text{ if } j\neq i, \\ 1-\sum\limits_{k\neq i} q_{ik}h+\alpha_{i}\left(t,h\right), \text{ if } j=i, \end{cases}$$

where all $\alpha_i(t,h)$ are $o(h)$. We also assume $q_{i,i+k} = \lambda_k$, $q_{i,i-k} = \mu_k$ for any $k > 0$. In other words, we suppose that the arrival rates λ_k and the service rates μ_k do not depend on the the length of the queue. In addition, we assume that $\lambda_{k+1} \leq \lambda_k$ and $\mu_{k+1} \leq \mu_k$ for any k

Then the probabilistic dynamics of the process is represented by the forward Kolmogorov system:

$$\frac{d\mathbf{p}}{dt} = A\mathbf{p}(t), \qquad (2)$$

where

$$A = \begin{pmatrix} a_{00} & \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots & \mu_r \\ \lambda_1 & a_{11} & \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{r-1} \\ \lambda_2 & \lambda_1 & a_{22} & \mu_1 & \mu 2 & \cdots & \mu_{r-2} \\ \cdots \\ \lambda_r & \lambda_{r-1} & \lambda_{r-2} & \cdots & \lambda_2 & \lambda_1 & a_{rr} \end{pmatrix}, \qquad (3)$$

and $a_{ii} = -\sum_{k=1}^{i} \mu_k - \sum_{k=1}^{r-i} \lambda_{r-k}$.

Throughout the paper by $\|\cdot\|$ we denote the l_1-norm, i. e. $\|\mathbf{x}\| = \sum |x_i|$, and $\|B\| = \sup_j \sum_i |b_{ij}|$ for $B = (b_{ij})_{i,j=0}^{r}$.

Let Ω be a set all stochastic vectors, i. e. l_1 vectors with nonnegative coordinates and unit norm.

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Recall the definition of decay parameter α^* (or spectral gap) for a homogeneous Markov chain:

$$\alpha^* := \sup\{a > 0 : \|\mathbf{p}(t) - \pi\| = \mathcal{O}(e^{-at})\}$$

as $t \to \infty$ for any $\mathbf{p}(0)$.

BOUNDS ON THE RATE OF CONVERGENCE

Let $\{d_i\}$, $i = 1, \dots, r$ be a sequence of positive numbers. Consider

$$\alpha_i = -a_{ii} + \lambda_{r-i+1} - \sum_{k=1}^{i-1} (\mu_{i-k} - \mu_i) \frac{d_k}{d_i} - \sum_{k=1}^{r-i} (\lambda_k - \lambda_{i+r-1}) \frac{d_{k+i}}{d_i}, \tag{4}$$

and

$$\alpha = \min_{1 \le i \le r} \alpha_i, \quad \beta = \max_{1 \le i \le r} \alpha_i. \tag{5}$$

Let D be upper triangular matrix

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots & d_1 \\ 0 & d_2 & d_2 & \cdots & d_2 \\ \cdots & & & & \\ 0 & 0 & \cdots & 0 & d_r \end{pmatrix}, \tag{6}$$

and $\|\mathbf{z}\|_D = \|D\mathbf{z}\|_1$ be the correspondent norm, where $\mathbf{z}(t) = (p_1(t), \dots, p_r(t))^T$. Below we will identify $\|\mathbf{z}\|_D$ and the respective norm of the whole vector $\|\mathbf{p}\|_D$.

Theorem 1. Let $X(t)$ be a given SZK process. Then the following bound for the correspondent decay parameter α^* holds:

$$\alpha \le \alpha^* \le \beta, \tag{7}$$

for any positive $\{d_j\}$. Moreover,

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \le e^{-\alpha t} \|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1D} \tag{8}$$

for any $\mathbf{z}^*(0), \mathbf{z}^{**}(0)$, and

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \ge e^{-\beta t} \|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1D}, \tag{9}$$

for any $D\left(\mathbf{z}^*(0) - \mathbf{z}^{**}(0)\right) \ge \mathbf{0}$, and any $t \ge 0$.

Proof.

Put $p_0(t) = 1 - \sum_{i \ge 1} p_i(t)$, then from (2) we obtain the equation

$$\frac{d\mathbf{z}}{dt} = B\mathbf{z}(t) + \mathbf{f}, \tag{10}$$

where $\mathbf{f} = (\lambda_1, \lambda_2, \cdots, \lambda_r)^T$, $\mathbf{z}(t) = (p_1, p_2, \cdots, p_r)^\top$,

$$B = \begin{pmatrix} a_{11} - \lambda_1 & \mu_1 - \lambda_1 & \mu_2 - \lambda_1 \cdots & \mu_{r-1} - \lambda_1 & \\ \lambda_1 - \lambda_2 & a_{22} - \lambda_2 & \mu_1 - \lambda_2 & \cdots & \mu_{r-2} - \lambda_2 \\ \lambda_2 - \lambda_3 & \lambda_1 - \lambda_3 & a_{33} - \lambda_3 \cdots & \mu_{r-3} - \lambda_3 & \\ \cdots & & & & \\ \lambda_{r-1} - \lambda_r & \lambda_{r-2} - \lambda_r & \cdots & \lambda_1 - \lambda_r & a_{rr} - \lambda_r \end{pmatrix}, \tag{11}$$

see a detailed discussion in (Granovsky and Zeifman 2004, Zeifman 1995a, Zeifman et al. 2006, 2008).

Consider now the logarithmic norm of matrix B in 1_D-norm, see the respective motivation in (Van Doorn et al. 2010, Granovsky and Zeifman 2004, Zeifman et al. 2008) and detailed proofs in (Zeifman 1995b). Recall that the logarithmic norm of matrix B is defined as

$$\gamma(B) = \lim_{h \to +0} h^{-1} \left(\|I + hB\| - 1\right).$$

The important inequality

$$\|V(t,s)\| \le \exp{(s-t)\gamma(B)}$$

holds, where $V(t,s) = \exp{(t-s)B}$ is the Cauchy matrix of equation (10). Further, in l_1-norm we have the simple formula

$$\gamma(B) = \max_j \left(b_{jj} + \sum_{i \ne j} |b_{ij}| \right).$$

Hence we obtain the following bound for the logarithmic norm of the matrix B:

$$\gamma\left(B\right)_{1D} = \gamma\left(DBD^{-1}\right)_1 = \max\left(a_{ii} - \lambda_{r-i+1} + \sum_{k=1}^{i-1} (\mu_{i-k} - \mu_i) \frac{d_k}{d_i} + \sum_{k=1}^{r-i} (\lambda_k - \lambda_{i+r-1}) \frac{d_{k+i}}{d_i} \right) = -\min \alpha_i = -\alpha, \tag{12}$$

where

$$DBD^{-1} = \begin{pmatrix} a_{11} - \lambda_r & (\mu_1 - \mu_2)\frac{d_1}{d_2} & (\mu_2 - \mu_3)\frac{d_1}{d_3} & \cdots & (\mu_{r-1} - \mu_r)\frac{d_1}{d_r} \\ (\lambda_1 - \lambda_r)\frac{d_2}{d_1} & a_{22} - \lambda_{r-1} & (\mu_1 - \mu_3)\frac{d_2}{d_3} & \cdots & (\mu_{r-2} - \mu_r)\frac{d_2}{d_r} \\ (\lambda_2 - \lambda_r)\frac{d_3}{d_1} & (\lambda_1 - \lambda_{r-1})\frac{d_3}{d_2} & a_{33} - \lambda_{r-2} & \cdots & (\mu_{r-3} - \mu_r)\frac{d_3}{d_r} \\ \cdots & & & & \\ (\lambda_{r-1} - \lambda_r)\frac{d_r}{d_1} & (\lambda_{r-2} - \lambda_{r-1})\frac{d_r}{d_2} & (\lambda_{r-3} - \lambda_{r-2})\frac{d_r}{d_3} & \cdots & a_{rr} - \lambda_1 \end{pmatrix}. \tag{13}$$

Therefore

$$\|V(t,s)\|_{1D} \le e^{-(t-s)\alpha}, \tag{14}$$

and

$$\|\mathbf{z}^*(t) - \mathbf{z}^{**}(t)\|_{1D} \le e^{-\alpha t} \|\mathbf{z}^*(s) - \mathbf{z}^{**}(s)\|_{1D} \tag{15}$$

for any $0 \le s \le t$ and any initial conditions $\mathbf{z}^*(s), \mathbf{z}^{**}(s)$.

On the other hand, the matrix DBD^{-1} is essentially nonnegative (i. e. all its off-diagonal elements are nonnegative). Putting $\mathbf{v}(t) = D\left(\mathbf{z}^*(t) - \mathbf{z}^{**}(t)\right)$, we obtain the nonnegativity $\mathbf{v}(t)$ if $\mathbf{v}(s) \ge \mathbf{0}$ and $t \ge s$. Moreover, $\|\mathbf{v}(t)\| = \sum_i v_i(t)$, hence (13) implies $\frac{d\|\mathbf{v}\|}{dt} \ge -\beta\|\mathbf{v}\|$, and

$$\|\mathbf{z}^*(t) - \mathbf{z}^{**}(t)\|_{1D} \ge e^{-\beta t} \|\mathbf{z}^*(s) - \mathbf{z}^{**}(s)\|_{1D} \tag{16}$$

for any $s, t, \quad 0 \le s \le t$ and initial conditions $\mathbf{z}^*(s), \mathbf{z}^{**}(s)$ such that $D\left(\mathbf{z}^*(s) - \mathbf{z}^{**}(s)\right) \ge 0$.

Corollary. The following bound for the decay parameter of SZK model holds:

$$\min_{1 \le i \le S} \left(i\mu_i + (r+1-i)\lambda_{r+1-i}\right) \le \alpha^* \le \max_{1 \le i \le S} \left(i\mu_i + (r+1-i)\lambda_{r+1-i}\right). \tag{17}$$

Proof.

Put all $d_i = 1$. Then (5) implies

$$\alpha_i = -a_{ii} + \lambda_{r-i+1} - \sum_{k=1}^{i-1}(\mu_{i-k} - \mu_i) -$$

$$\sum_{k=1}^{r-i}(\lambda_k - \lambda_{i+r-1}) = i\mu_i + (r+1-i)\lambda_{r+1-i}, \qquad (18)$$

and our bound.

Let now

$$A^* =$$

$$\begin{pmatrix} a_{11} - \lambda_r & (\mu_1 - \mu_2) & (\mu_2 - \mu_3) & \cdots & (\mu_{r-1} - \mu_r) \\ (\lambda_1 - \lambda_r) & a_{22} - \lambda_{r-1} & (\mu_1 - \mu_3) & \cdots & (\mu_{r-2} - \mu_r) \\ (\lambda_2 - \lambda_r) & (\lambda_1 - \lambda_{r-1}) & a_{33} - \lambda_{r-2} & \cdots & (\mu_{r-3} - \mu_r) \\ \cdots & & & & \\ (\lambda_{r-1} - \lambda_r) & (\lambda_{r-2} - \lambda_{r-1}) & (\lambda_{r-3} - \lambda_{r-2}) & \cdots & a_{rr} - \lambda_1 \end{pmatrix}. \qquad (19)$$

This matrix A^* is *essentially nonnegative*, i. e. all off-diagonal elements of this matrix are nonnegative. We recall that A^* is essentially positive, if in addition all elements of matrix ${A^*}^k$ are strictly positive for some natural k.

Theorem 2. Let A^* be essentially positive. Then there exists a sequence $\{d_j\}$ such that

$$\alpha = \beta = \alpha^*. \qquad (20)$$

Moreover, bound (8) with $\alpha = \alpha^*$ holds for any initial conditions, and, if in addition, the initial conditions are such that $D\left(\mathbf{z}^*(0) - \mathbf{z}^{**}(0)\right) \geq \mathbf{0}$, then

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} = e^{-\alpha^* t}\|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1D}. \qquad (21)$$

Proof. Let λ_0 be the greatest eigenvalue of the matrix A^* (it is real and simple). Let H be a diagonal matrix with diagonal entries d_j. Then the transformed matrix $A^*_H = HA^*H^{-1} = DBD^{-1}$ is also essentially positive. Denote by $m = \max|d_{jj}|$ and consider matrix $C' = {A^*}^T + mI$. This matrix is also essentially positive and *all* elements of C' are nonnegative. Hence C' has strictly positive simple eigenvalue $\lambda^* = \lambda_0 + m$, and the correspondent eigenvector $\mathbf{x} = (x_1, \ldots, x_r)^T$ is also strictly positive. Let $d_k = x_k^{-1}$. Then $\mathbf{e} = (1, \ldots, 1)^T$ is eigenvector for $C'_H = HC'H^{-1}$, therefore all row sums for this matrix are equal to λ^*. Hence all row sums of ${A^*}^T_H = C'_H - mI$ are $\lambda^* - m = \lambda_0$, and all column sums of matrix A^*_H are the same. It implies our claim.

We note that another of this statement one can find in (Van Doorn et al 2010).

Remark. Essential positivity of matrix (19) is equivalent to the existence of a path of nonsingularity, i. e. a chain of positive elements $a^*_{i_1,i_2}, a^*_{i_2,i_3}, \ldots, a^*_{i_k,i_1}$, such that it contains all indexes $1, \ldots, r$, and $i_j \neq i_{j+1}$ for any j.

In particular, this assumption certainly holds, if $\lambda_1 > \lambda_2$ and $\mu_1 > \mu_2$.

AN EXAMPLE OF SZK SYSTEM

Consider the following queueing model. Let the length of the queue is $X(t) \leq S$, and assume that a group of $k \leq M \leq S$ customers may arrive to the queue with intensity $\lambda_k = \frac{\lambda}{k}$, and a group of $k \leq N \leq S$ customers may leave the queue after their service with intensity $\mu_k = \frac{\mu}{k}$, where M and N are fixed natural numbers.

If $M = N = 1$, then $X(t)$ is simple birth-death process with birth (arrival) rate λ and death (service) rate μ.

If $M = S$ and $N = 1$ then $X(t)$ is the generalized Erlang model, this process was introduced and studied in (Zeifman et al. 2013, 2014a, 2014b, 2014c). Bounds for the spectral gap and its asymptotic as $S \to \infty$ for Erlang model was studied in various situations by a number of authors, see (Van Doorn and Zeifman 2009, Van Doorn et al. 2010, Fricker et al. 1999, Voit 2000, Zeifman 2009).

1. Consider firstly the simplest SZK model with $M = N = 1$ and the correspondent intensities $\lambda_1 = \lambda > 0$, $\mu_1 = \mu > 0$, and $\lambda_k = \mu_k = 0$,for $k \geq 1$. Then $X(t)$ is a birth-death process which was studied in (Granovsky and Zeifman 1997), Example 2.3. The authors found the correspondent transformation D and the correspondent spectral gap

$$\alpha^*(S) = \lambda + \mu - 2\sqrt{\lambda\mu}\cos\frac{\pi}{S+1},$$

and $\alpha^*(S) \to \left(\sqrt{\lambda} - \sqrt{\mu}\right)$ as $S \to \infty$.

2. Consider now the situation with maximal possible size of simultaneous arrivals and services, i. e. let $M = N = S$. Then all $\alpha_i = i\frac{\mu}{i} + (r+1-i)\frac{\lambda}{r+1-i} = \lambda + \mu$, and we also obtain sharp decay parameter (spectral gap) $\alpha^* = \lambda + \mu$.

3. Consider now a more general case.

Proposition. Let $M = S$ and N be fixed. Then $\alpha^*(S) \to \mu$ as $S \to \infty$.

Proof. Put $d_i = 1$ for $i \leq S - N$, and $d_i = d^* = 1 + o(1)$ if $i = S - N + 1, \ldots, S$, and consider α_i in the form: $\alpha_i = \alpha_{i,\lambda} + \alpha_{i,\mu}$, where $\alpha_{i,\lambda} = \sum_{k=1}^{i-1}\lambda_k + \lambda_{r-i+1} - \sum_{k=1}^{r-i}(\lambda_k - \lambda_{i+r-1})\frac{d_{k+i}}{d_i}$, and $\alpha_{i,\mu} = \sum_{k=1}^{i}\mu_k - \sum_{k=1}^{i-1}(\mu_{i-k} - \mu_i)\frac{d_k}{d_i}$.

Then we obtain $\alpha_{i,\lambda} = 0$ for $i < S - N - 1$, and $\alpha_{i,\lambda} = \lambda + o(1)$ for $i \geq S - N$.

On the other hand, $\alpha_{i,\mu} = \mu$, if $i < S - N - 1$, and

$$\alpha_{i,\mu} = \mu H_S\left(1 - \frac{1}{d^*}\right) + \frac{\mu}{d^*} + o(1),$$

where $H_S = \sum_{k=1}^{S}\frac{1}{k}$, for $i \geq S - N$.

Let $d^* = \frac{(H_S-1)\mu}{(H_S-1)\mu+\lambda}$. Then we have $\alpha_{i,\mu} = \mu - \lambda + o(1)$ for $i \geq S - N$.

Therefore, $\alpha^* = \alpha^*(S) = \mu + o(1)$.

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