ON TRUNCATIONS FOR A CLASS OF FINITE MARKOVIAN QUEUING MODELS

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Markovian queueing models; SZK model; approximation bounds

ABSTRACT

We consider a class of finite Markovian queueing models and obtain uniform approximation bounds of truncations.

INTRODUCTION

It is well known that explicit expressions for the probability characteristics of stochastic models can be found only in a few special cases, moreover, if we deal with an inhomogeneous Markovian model, then we must approximately calculate the limiting probability characteristics of the process. The problem of calculation of the limiting characteristics for inhomogeneous birth-death process via truncations was firstly mentioned in (Zeifman 1991) and was considered in details in (Zeifman et al. 2006). In (Zeifman et al. 2014b) we have proved uniform (in time) error bounds of truncation this class of Markov chains. First uniform bounds of truncations for the class of Markovian time-inhomogeneous queueing models with batch arrivals and group services (SZK models) introduced and studied in our recent papers (Satin et al. 2013, Zeifman et al. 2014a), were obtained in (Zeifman et al. 2014c). In this note we deal with approximations of finite SZK model via the same models with smaller state space and obtain the correspondent bounds of error of truncation bounds.

Consider a time-inhomogeneous continuous-time Markovian queueing model on the state space $E = \{0, 1, \dots, r\}$ with possible batch arrivals and group services.

Let $X(t)$, $t \geq 0$ be the queue-length process for the queue, $p_{ij}(s,t) = \mathsf{P}\{X(t) = j\,|X(s) = i\}$, $i, j \geq 0$, $0 \leq s \leq t$, be transition probabilities for $X = X(t)$, and $p_i(t) = \mathsf{P}\{X(t) = i\}$ be its state probabilities. Throughout the paper we assume that

$$\mathsf{P}\left(X\left(t+h\right)=j|X\left(t\right)=i\right)= \\ =\begin{cases} q_{ij}\left(t\right)h+\alpha_{ij}\left(t,h\right), \text{ if } j\neq i,\\ 1-\sum\limits_{k\neq i} q_{ik}\left(t\right)h+\alpha_i\left(t,h\right), \text{ if } j=i, \end{cases} \quad (1)$$

where all $\alpha_i(t,h)$ are $o(h)$ uniformly in i, i. e., $\sup_i |\alpha_i(t,h)| = o(h)$. We also assume $q_{i,i+k}(t) = \lambda_k(t)$, $q_{i,i-k}(t) = \mu_k(t)$ for any $k > 0$. In other words, we suppose that the arrival rates $\lambda_k(t)$ and the service rates $\mu_k(t)$ do not depend on the queue length. In addition, we assume that $\lambda_{k+1}(t) \leq \lambda_k(t)$ and $\mu_{k+1}(t) \leq \mu_k(t)$ for any k and almost all $t \geq 0$. Hence, $X(t)$ is a so-called SZK model, which was studied in (Satin et al. 2013, Zeifman et al. 2014a, 2014c). We suppose that all intensity functions are locally integrable on $[0, \infty)$, and

$$\lambda_k(t) \leq \lambda_k, \quad \mu_k(t) \leq \mu_k, \quad (2)$$

for any k and almost all $t \geq 0$, and put

$$L_\lambda = \sum_{i=1}^{r} \lambda_i, \quad L_\mu = \sum_{i=1}^{r} \mu_i. \quad (3)$$

Then the probabilistic dynamics of the process is represented by the forward Kolmogorov system

$$\frac{d\mathbf{p}}{dt} = A(t)\mathbf{p}(t), \quad (4)$$

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where $A(t) =$

$$= \begin{pmatrix} a_{00}(t) & \mu_1(t) & \mu_2(t) & \cdots & \mu_r(t) \\ \lambda_1(t) & a_{11}(t) & \mu_1(t) & \cdots & \mu_{r-1}(t) \\ \lambda_2(t) & \lambda_1(t) & a_{22}(t) & \cdots & \mu_{r-2}(t) \\ \cdots \\ \lambda_r(t) & \lambda_{r-1}(t) & \lambda_{r-2}(t) & \cdots & a_{rr}(t) \end{pmatrix}, \quad (5)$$

and $a_{ii}(t) = -\sum_{k=1}^{i} \mu_k(t) - \sum_{k=1}^{r-i} \lambda_k(t)$.

Throughout the paper by $\|\cdot\|$ we denote the l_1-norm, i. e., $\|\mathbf{x}\| = \sum |x_i|$, and $\|B\| = \sup_j \sum_i |b_{ij}|$ for $B = (b_{ij})$.

Let Ω be the set all stochastic vectors, i. e., l_1 vectors with nonnegative coordinates and unit norm. Hence $\|A(t)\| \le 2\sum_{k=1}^{r}(\lambda_k(t) + \mu_k(t)) \le 2(L_\lambda + L_\mu)$ for almost all $t \ge 0$.

By $E(t,k) = \mathsf{E}\{X(t)\,|X(0) = k\}$ denote the mathematical expectation of the process at a moment t under the initial condition $X(0) = k$.

Recall that a Markov chain $X(t)$ is called *weakly ergodic*, if $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $\mathbf{p}^*(0), \mathbf{p}^{**}(0)$, where $\mathbf{p}^*(t)$ and $\mathbf{p}^{**}(t)$ are the corresponding solutions of (4), and a Markov chain $X(t)$ has the *limiting mean* $\varphi(t)$, if $\lim_{t\to\infty}(\varphi(t) - E(t,k)) = 0$ for any k.

TRUNCATIONS

Consider the family of "truncated" processes $X_{N-1}(t)$, and let $E_{N-1} = \{0, 1, \ldots, N-1\}$ be the corresponding state space and $A_{N-1} =$

$$= \begin{pmatrix} b_{00} & \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{N-1} \\ \lambda_1 & b_{11} & \mu_1 & \mu_2 & \cdots & \mu_{N-2} \\ \lambda_2 & \lambda_1 & b_{22} & \mu_1 & \cdots & \mu_{N-3} \\ \cdots \\ \lambda_{N-1} & \lambda_{N-2} & \lambda_{N-3} & \cdots & \lambda_1 & b_{N-1,N-1} \end{pmatrix} \quad (6)$$

be the corresponding intensity matrix, where $b_{ii}(t) = -\sum_{k=1}^{i} \mu_k(t) - \sum_{k=1}^{N-1-i} \lambda_k(t)$.

Instead of (4), for $X_{N-1}(t)$ we have the following forward Kolmogorov system:

$$\frac{d\mathbf{p}_{N-1}}{dt} = A_{N-1}(t)\mathbf{p}_{N-1}. \quad (7)$$

Setting $p_0(t) = 1 - \sum_{i\ge1} p_i(t)$, from (4) we obtain the equation

$$\frac{d\mathbf{z}}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t), \quad (8)$$

where $\mathbf{f}(t) = (\lambda_1, \lambda_2, \cdots, \lambda_r)^\top$, $\mathbf{z}(t) = (p_1, p_2, \cdots, p_r)^\top$, $B = \left(b_{ij}(t)_{i,j=1}^{r}\right) =$

$$\begin{pmatrix} a_{11}-\lambda_1 & \mu_1-\lambda_1 & \cdots & \mu_{r-1}-\lambda_1 \\ \lambda_1-\lambda_2 & a_{22}-\lambda_2 & \cdots & \mu_{r-2}-\lambda_2 \\ \lambda_2-\lambda_3 & \lambda_1-\lambda_3 & \cdots & \mu_{r-3}-\lambda_3 \\ \cdots \\ \lambda_{r-1}-\lambda_r & \lambda_{r-2}-\lambda_r & \cdots & a_{rr}-\lambda_r \end{pmatrix}. \quad (9)$$

Similarly, instead of (8), we obtain the corresponding reduced system for the truncated process in the form:

$$\frac{d\mathbf{z}_{N-1}}{dt} = B_{N-1}(t)\mathbf{z}_{N-1}(t) + \mathbf{f}_{N-1}(t), \quad (10)$$

where $\mathbf{f}_{N-1}(t) = (\lambda_1, \cdots, \lambda_{N-1})^\top$, $\mathbf{z}_{N-1}(t) = (p_1, p_2, \cdots, p_{N-1})^\top$, $B_{N-1} = (b_{ij}{}^*(t))_{i,j=1}^{N-1} =$

$$\begin{pmatrix} b_{11}-\lambda_1 & \mu_1-\lambda_1 & \cdots & \mu_{N-1}-\lambda_1 \\ \lambda_1-\lambda_2 & b_{22}-\lambda_2 & \cdots & \mu_{N-2}-\lambda_2 \\ \lambda_2-\lambda_3 & \lambda_1-\lambda_3 & \cdots & \mu_{N-3}-\lambda_3 \\ \cdots \\ \lambda_{N-2}-\lambda_{N-1} & \lambda_{N-3}-\lambda_{N-1} & \cdots & b_{N-1,N-1}-\lambda_{N-1} \end{pmatrix}. \quad (11)$$

Consider the system

$$\frac{d\mathbf{z}_{N-1}}{dt} = B_{N-1}(t)\mathbf{z}_{N-1}(t) + \mathbf{f}(t), \quad (12)$$

One can see that the solution of system (10) and the corresponding solution of system (12) with initial condition $p_0(0) = 1$ coincide.

Below we will identify the vector with entries, say, $(a_1, \ldots, a_{N-1})^\top$ and the r-dimensional vector with the same first $N-1$ coordinates and the rest equal to zero.

Let $\{d_i\}$, $i = 1, 2, \ldots$ be an increasing sequence of positive numbers, $d_1 = 1$, and

$$W = \min_{i\ge1} \frac{\sum_{k=1}^{i} d_k}{i}, \quad g_i = \sum_{n=1}^{i} d_n. \quad (13)$$

Put

$$\alpha_i(t) = -a_{ii}(t) + \lambda_{r-i+1}(t) - \quad (14)$$

$$\sum_{k\ge1}^{r-i}(\lambda_k(t) - \lambda_{r-i+1}(t))\frac{d_{k+1}}{d_i} - \sum_{k=1}^{i-1}(\mu_{i-k}(t) - \mu_i(t))\frac{d_k}{d_i},$$

and

$$\alpha(t) = \min_{i\ge1} \alpha_i(t). \quad (15)$$

Let D be upper triangular matrix,

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots & d_1 \\ 0 & d_2 & d_2 & \cdots & d_2 \\ 0 & 0 & d_3 & \cdots & d_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & d_r \end{pmatrix}, \quad (16)$$

and let $\|\bullet\|_{1D}$ be the corresponding norm $\|\mathbf{z}\|_{1D} = \|D\mathbf{z}\|_1$. Then the important inequality

$$\|V(t,s)\| \le \exp \int_s^t \gamma(B(u))\,du$$

holds, where $V(t,s) = V(t)V^{-1}(s)$ is the Cauchy matrix of equation (8), and $\gamma(B(t))$ is the logarithmic norm of the matrix function $B(t)$, see details in (Van Doorn et al. 2010, Granovsky and Zeifman 2004, Zeifman et al. 2008). Further, for an operator function from l_1 to itself we have the simple formula

$$\gamma(B(t)) = \sup_j \left(b_{jj}(t) + \sum_{i \neq j} |b_{ij}(t)| \right).$$

Hence we obtain the following bound for the logarithmic norm of $B(t)$:

$$\gamma(B(t))_{1D} = \gamma(DB(t)D^{-1}) = \sup_{i \geq 1}\{-\alpha_i(t)\} = -\alpha(t), \tag{17}$$

where

$$DBD^{-1} = \tag{18}$$

$$\begin{pmatrix} a_{11} - \lambda_r & (\mu_1 - \mu_2)\frac{d_1}{d_2} & \cdots & (\mu_{r-1} - \mu_r)\frac{d_1}{d_r} \\ (\lambda_1 - \lambda_r)\frac{d_2}{d_1} & a_{22} - \lambda_{r-1} & \cdots & (\mu_{r-2} - \mu_r)\frac{d_2}{d_r} \\ \cdots & \cdots & \cdots & \cdots \\ (\lambda_{r-1} - \lambda_r)\frac{d_r}{d_1} & (\lambda_{r-2} - \lambda_{r-1})\frac{d_r}{d_2} & \cdots & a_{rr} - \lambda_1 \end{pmatrix}.$$

Therefore

$$\|V(t,s)\|_{1D} \leq e^{-\int_s^t \alpha(u)\,du}. \tag{19}$$

Now let $\{d_i\}$ and $\{d_i^*\}$ be two increasing sequences such that $d_1 = d_1^* = 1$, all $d_i < d_i^*$, $i \geq 2$, and the following assumptions hold:

$$\|V(t,s)\|_{1D} \leq Me^{-a(t-s)} \tag{20}$$

and

$$\|V(t,s)\|_{1D^*} \leq M^*e^{-a^*(t-s)} \tag{21}$$

for any $0 \leq s \leq t$, and some positive numbers M, M^*, a, a^*.

Let K_N^* be a positive number such that

$$d_1^*\lambda_1 + (d_1^* + d_2^*)\lambda_2 + \cdots + (d_1^* + \cdots + d_{N-1}^*)\lambda_{N-1} \leq K_N^*. \tag{22}$$

For bounding the truncation error we rewrite (12) as

$$\frac{d\mathbf{z}_{N-1}}{dt} = B(t)\mathbf{z}_{N-1}(t) + \mathbf{f}(t) - \hat{B}(t)\mathbf{z}_{N-1}(t), \tag{23}$$

where $\hat{B}(t) = B(t) - B_{N-1}(t)$. Then we have

$$\mathbf{z}_{N-1}(t) = V(t)\mathbf{z}_{N-1}(0) + \int_0^t V(t,\tau)\mathbf{f}(\tau)\,d\tau - \int_0^t V(t,\tau)\hat{B}(\tau)\mathbf{z}_{N-1}(\tau)\,d\tau. \tag{24}$$

Hence, if $\mathbf{z}(0) = \mathbf{z}_{N-1}(0) = \mathbf{0}$, then the sum of the first and the second summands gives us $\mathbf{z}(t)$, and we obtain in *any* norm the bound

$$\|\mathbf{z}(t) - \mathbf{z}_{N-1}(t)\| \leq \|\int_0^t V(t,\tau)\hat{B}(\tau)\mathbf{z}_{N-1}(\tau)\,d\tau\| \leq \int_0^t \|V(t,\tau)\|\|\hat{B}(\tau)\mathbf{z}_{N-1}(\tau)\|\,d\tau. \tag{25}$$

On the other hand,

$$\hat{B}(t)\mathbf{z}_{N-1}(t) = (B(t) - B_{N-1}(t))\,\mathbf{z}_{N-1}(t) = ((a_{11}(t) - b_{11}(t))p_{N-1,1}(t), \cdots, (a_{N-1,N-1}(t) - b_{N-1,N-1}(t))p_{N-1,N-1}(t))^\top, \tag{26}$$

and

$$\begin{aligned} \|\hat{B}(t)\mathbf{z}_{N-1}(t)\|_{1D} &= \|D\,(B(t) - B_{N-1}(t))\,\mathbf{z}_{N-1}(t)\|_1 = \\ d_1(b_{11}(t) - a_{11}(t))p_{N-1,1}(t) &+ (d_1 + d_2)(b_{22}(t) - a_{22}(t))p_{N-1,2}(t) + \cdots + \\ (d_1 + \cdots + d_{N-1})&(b_{N-1,N-1}(t) - a_{N-1,N-1}(t))p_{N-1,N-1}(t) = \\ d_1 \sum_{k \geq N-1} \lambda_k(t)p_{N-1,1}(t) &+ (d_1 + d_2)\sum_{k \geq N-2} \lambda_k(t)p_{N-1,2}(t) + \cdots + \\ &(d_1 + \cdots + d_{N-1})\sum_{k \geq 1} \lambda_k(t)p_{N-1,N-1}(t). \end{aligned} \tag{27}$$

Now we will estimate $\|\hat{B}(t)\mathbf{z}_{N-1}(t)\|_{1D}$. Firstly,

$$\|\mathbf{z}_{N-1}(t)\|_{1D^*} \leq \|V(t)\|_{1D^*}\|\mathbf{z}_{N-1}(0)\|_{1D^*} + \int_0^t \|V(t,\tau)\|_{1D^*}\|\mathbf{f}_{N-1}(\tau)\|_{1D^*}\,d\tau \leq M^*e^{-a^*t}\|\mathbf{z}_{N-1}(0)\|_{1D^*} + \frac{K_N^*M^*}{a^*}, \tag{28}$$

because $\|\mathbf{f}_{N-1}(t)\|_{1D^*} \leq K_N^*$ for almost all $t \geq 0$.

Put $X(0) = X_{N-1}(0) = 0$, then $\mathbf{z}_{N-1}(0) = \mathbf{0}$, hence

$$\|\mathbf{z}_{N-1}(t)\|_{1D^*} \leq \frac{K_N^*M^*}{a^*}, \tag{29}$$

for any $t \geq 0$.

For definiteness suppose that N is odd. All $p_{N-1,i}(t) \geq 0$, then

$$\|\mathbf{z}_{N-1}(t)\|_{1D^*} = \sum_{i \geq 1} p_{N-1,i}(t)\sum_{k=1}^{i} d_k^* \geq \sum_{i \geq \frac{N-1}{2}} d_i^* p_{N-1,i}(t) \geq \sum_{i=\frac{N-1}{2}}^{N-1} d_i^* p_{N-1,i}(t). \tag{30}$$

On the other hand we have the bound:

$$\begin{aligned} &d_1 \sum_{k \geq N-1} \lambda_k(t)p_{N-1,1}(t) + \\ &(d_1 + d_2)\sum_{k \geq N-2} \lambda_k(t)p_{N-1,2}(t) + \cdots + \\ &(d_1 + \cdots + d_{N-1})\sum_{k \geq 1} \lambda_k(t)p_{N-1,N-1}(t) \leq \\ &(d_1 + \cdots + d_{\frac{N-1}{2}})\sum_{k \geq \frac{N-1}{2}} \lambda_k(t)\sum_{k=1}^{\frac{N-1}{2}} p_{N-1,k}(t) + \\ &\sum_{k \geq 1} \lambda_k(t)\Big((d_1 + \cdots + d_{\frac{N-1}{2}})p_{N-1,\frac{N-1}{2}}(t) + \cdots + \\ &(d_1 + \cdots + d_{N-1})p_{N-1,N-1}(t)\big). \end{aligned} \tag{31}$$

Denote $\Lambda_K = \sum_{k\ge K}\lambda_k$, where λ_k are defined by (2). Then from (3), (27), (30) and (31) we obtain

$$\|\hat{B}(t)\mathbf{z}_{N-1}(t)\|_{1D} \le g_{\frac{N-1}{2}}\Lambda_{\frac{N-1}{2}}\sum_{k=1}^{\frac{N-1}{2}} p_{N-1,k}(t) + L_\lambda\left(g_{\frac{N-1}{2}}p_{N-1,\frac{N-1}{2}}(t)+\cdots+g_{N-1}p_{N-1,N-1}(t)\right) \le g_{\frac{N-1}{2}}\Lambda_{\frac{N-1}{2}} + L_\lambda\frac{g_{N-1}}{d^*_{\frac{N-1}{2}}}\left(d^*_{\frac{N-1}{2}}p_{N-1,\frac{N-1}{2}}(t)+\cdots+d^*_{N-1}p_{N-1,N-1}(t)\right) \le g_{\frac{N-1}{2}}\Lambda_{\frac{N-1}{2}} + L_\lambda\frac{g_{N-1}}{d^*_{\frac{N-1}{2}}}\|\mathbf{z}_{N-1}(t)\|_{1D^*} \le g_{\frac{N-1}{2}}\Lambda_{\frac{N-1}{2}} + L_\lambda\frac{g_{N-1}}{d^*_{\frac{N-1}{2}}}\frac{K_N^*M^*}{a^*}, \quad (32)$$

for any $t \ge 0$.

Finally, from (32) we obtain the following bound of truncation error:

$$\|\mathbf{z}(t)-\mathbf{z}_{N-1}(t)\| \le \int_0^t Me^{-a(t-\tau)}\left(g_{\frac{N-1}{2}}\Lambda_{\frac{N-1}{2}} + L_\lambda\frac{g_{N-1}}{d^*_{\frac{N-1}{2}}}\frac{K_N^*M^*}{a^*}\right)d\tau \le \frac{M}{a}\left(g_{\frac{N-1}{2}}\Lambda_{\frac{N-1}{2}} + L_\lambda\frac{g_{N-1}}{d^*_{\frac{N-1}{2}}}\frac{K_N^*M^*}{a^*}\right). \quad (33)$$

Now consider $\|\bullet\|_{1E}$ norm: $\|z\|_{1E} = \sum n|p_n|$, then $\|\mathbf{z}\|_{1E} \le W^{-1}\|\mathbf{z}\|_{1D}$, see, for instance, (Zeifman et al 2006), and we obtain the following statement.

Theorem 1. Let (20) and (21) hold. Then $X(t)$ is exponentially weakly ergodic, has the limiting mean, say, $E(t,0)$, and the following bounds of truncation error hold:

$$\|\mathbf{p}(t)-\mathbf{p}_{N-1}(t)\| \le \frac{M}{a}\Big(g_{\frac{N-1}{2}}\Lambda_{\frac{N-1}{2}} + L_\lambda\frac{g_{N-1}}{d^*_{\frac{N-1}{2}}}\frac{K_N^*M^*}{a^*}\Big), \quad (34)$$

and

$$|E(t,0)-E_{N-1}(t,0)| \le \frac{M}{aW}\Big(g_{\frac{N-1}{2}}\Lambda_{\frac{N-1}{2}} + L_\lambda\frac{g_{N-1}}{d^*_{\frac{N-1}{2}}}\frac{K_N^*M^*}{a^*}\Big), \quad (35)$$

for any $t \ge 0$, where $X(0) = X_{N-1}(0) = 0$, and $E_{N-1}(t,k) = E\{X_{N-1}(t)\,|X_{N-1}(0)=k\}$ is the mathematical expectation of the truncated process at the moment t under initial condition $X_{N-1}(0) = k$.

EXAMPLES

Let $r = 10^{10}$, $\lambda^*(t) = 1+\sin(2\pi t)$, $\mu^*(t) = 4+\cos(2\pi t)$, $\mu_i(t) = \frac{\mu^*(t)}{i}$.

1. Let $\lambda_1(t) = \lambda^*(t)$, $\lambda_i(t) = 0$, for $i \ge 2$.

Put all $d_k = 1$. Then we have $\alpha(t) = \mu^*(t)$, $L_\lambda = 2$, $g_{N-1} \le N$, $W = 1$, and (20) holds for $M = 1$, $a = 3$.

Let now $d^*_{k+1} = 2^k$. Then the respective $\alpha^*(t) = \mu^*(t) - \lambda^*(t)$, and (21) holds for $M^* = 1$, $a^* = 1$. Moreover, we have $K_N^* = 2$, $d^*_{\frac{N-1}{2}} = 2^{\frac{N-1}{2}}$, and Theorem 1 yields the bounds

$$\|\mathbf{p}(t)-\mathbf{p}_{N-1}(t)\| \le \frac{4N}{9\cdot 2^{\frac{N-1}{2}}}, \quad (36)$$

and

$$|E(t,0)-E_{N-1}(t,0)| \le \frac{4N}{9\cdot 2^{\frac{N-1}{2}}}, \quad (37)$$

for any $t \ge 0$, where $X(0) = X_{N-1}(0) = 0$.

Hence, if $N = 41$, then the truncation error for vector of state probabilities and for the mathematical expectation of the process $X(t)$ smaller than $2\cdot 10^{-5}$.

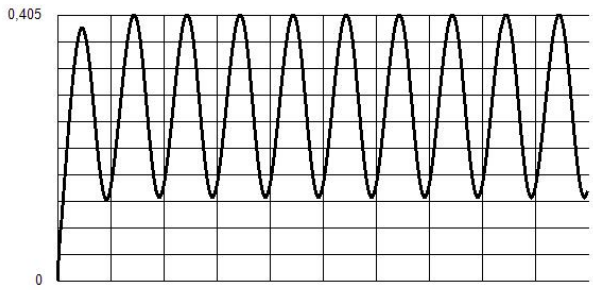

Fig. 1. First example, approximation of the mean $E(t,0)$ on $[0,10]$.

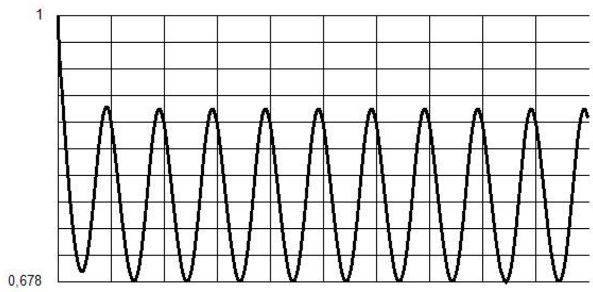

Fig. 2. First example, approximation of the probability of empty queue $\mathsf{P}\{X(t)=0|X(0)=0\}$ on $[0,10]$.

2. Let $\lambda_i(t) = \frac{\lambda^*(t)}{3^i}$.

Put all $d_k = 1$. Then we have $\alpha(t) = \mu^*(t)$, $L_\lambda = 1$, $g_{N-1} \le N$, $W = 1$, and (20) holds for $M = 1$, $a = 3$.

Let now $d^*_{k+1} = 2^k$. Then the respective $\alpha^*(t) = \mu^*(t) - \lambda^*(t)$, and (21) holds for $M^* = 1$, $a^* = 1$. Moreover, we have $K_N^* = 2$, $d^*_{\frac{N-1}{2}} = 2^{\frac{N-1}{2}}$, and Theorem 1 yields the bounds

$$\|\mathbf{p}(t)-\mathbf{p}_{N-1}(t)\| \le \frac{4N}{9\cdot 2^{\frac{N-1}{2}}}, \quad (38)$$

and

$$|E(t,0)-E_{N-1}(t,0)| \le \frac{4N}{9\cdot 2^{\frac{N-1}{2}}}, \quad (39)$$

for any $t \ge 0$, where $X(0) = X_{N-1}(0) = 0$.

Hence, if $N = 41$, then the truncation error for vector of state probabilities and for the mathematical expectation of the process $X(t)$ smaller than $2 \cdot 10^{-5}$.

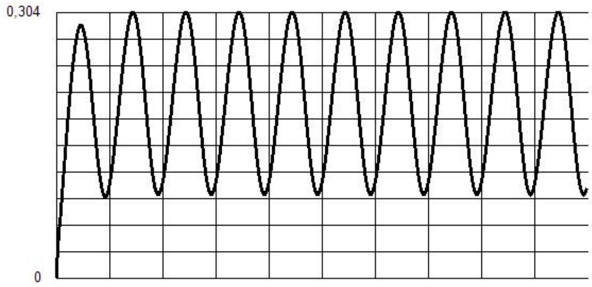

Fig. 3. Second example, approximation of the mean $E(t, 0)$ on $[0, 10]$.

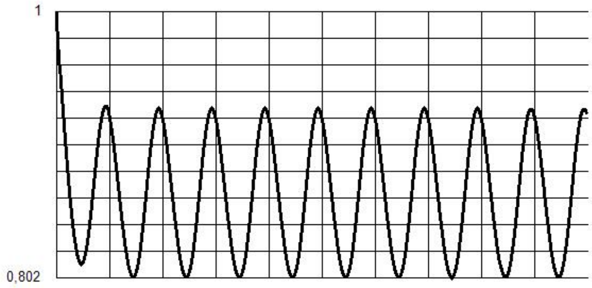

Fig. 4. Second example, approximation of the probability of empty queue $\mathsf{P}\{X(t) = 0|X(0) = 0\}$ on $[0, 10]$.

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