REACHABILITY OF FRACTIONAL CONTINUOUS-TIME LINEAR SYSTEMS USING THE CAPUTO-FABRIZIO DERIVATIVE

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ABSTRACT

The Caputo-Fabrizio definition of the fractional derivative is applied to analysis of the positivity and reachability of continuous-time linear systems. Necessary and sufficient conditions for the reachability of standard and positive fractional continuous-time linear systems are established.

INTRODUCTION

A dynamical system is called positive if its trajectory starting from any nonnegative initial condition state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive system theory is given in the monographs (Farina and Rinaldi 2000; Kaczorek 2001) and in the papers (Kaczorek 1997, 1998, 2011b, 2014a, 2014b, 2015b). Models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The positive standard and descriptor systems and their stability have been analyzed in (Kaczorek 1997, 1998, 2001, 2011b, 2014b, 2015b). The positive linear systems with different fractional orders have been addressed in (Kaczorek 2011b, 2012) and the descriptor discrete-time linear systems in (Kaczorek 1998). Descriptor positive discrete-time and continuous-time nonlinear systems have been analyzed in (Kaczorek 2014a) and the positivity and linearization of nonlinear discrete-time systems by state-feedbacks in (Kaczorek 2014b). New stability tests of positive standard and fractional linear systems have been investigated in (Kaczorek 2011a). The stability and robust stabilization of discrete-time switched systems have been analyzed in (Zhang, Xie, Zhang and Wang 2014; Zhang, Han, Wu and Hung 2014). Minimum energy control of 2D systems in Hilbert spaces has been analyzed in (Klamka 1983). Controllability of dynamical systems has been investigated in (Kalman 1960; Klamka 1991, 1997, 1998).

Recently a new definition of the fractional derivative without singular kernel has been proposed in (Caputo and Fabrizio 2015; Losada and Nieto 2015).

In this paper the Caputo-Fabrizio definition of the fractional derivative will be applied to analysis of the reachability of the standard and positive linear systems.

The paper is organized as follows. In section 2 necessary and sufficient conditions for the reachability of fractional standard continuous-time linear systems are established. Necessary and sufficient conditions for the positivity of the fractional systems and sufficient conditions for the reachability of the positive systems are proposed in section 3. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices, I_n - the $n\times n$ identity matrix.

REACHABILITY OF STANDARD FRACTIONAL SYSTEMS

The Caputo-Fabrizio definition of fractional derivative of order α of the function $f(t)$ for $0<\alpha<1$ has the form (Caputo and Fabrizio 2015; Losada and Nieto 2015)

$$ {}^{CF}D^{\alpha} f(t) = \frac{1}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)\dot{f}(\tau)d\tau, \\ \dot{f}(\tau) = \frac{df(\tau)}{d\tau},\ t\ge 0. \tag{1}$$

Consider the fractional differential state equations

$$ {}^{CF}D^{\alpha} x(t) = \frac{d^{\alpha}x(t)}{dt^{\alpha}} = Ax(t) + Bu(t),\ 0<\alpha<1, \tag{2a}$$

$$ y(t) = Cx(t) + Du(t), \tag{2b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Theorem 1. The solution $x(t)$ of the equation (2a) for a given initial condition $x(0)=x_0$ and input $u(t)$ has the form

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$$x(t)=e^{\hat{A}t}(\hat{x}_0+\hat{B}u_0)+\int_0^t e^{\hat{A}(t-\tau)}\hat{B}[\beta u(\tau)+\dot{u}(\tau)]d\tau, \quad (3a)$$

where

$$\begin{aligned}&\hat{A}=\alpha[I_n-(1-\alpha)A]^{-1}A,\\ &\hat{B}=[I_n-(1-\alpha)A]^{-1}(1-\alpha)B, \quad \beta=\frac{\alpha}{1-\alpha},\\ &\hat{x}_0=[I_n-(1-\alpha)A]^{-1}x_0, \ e^{\hat{A}t}=\mathcal{L}^{-1}\{[I_n s-\hat{A}]^{-1}\},\\ &\dot{u}(\tau)=\frac{du(\tau)}{d\tau}, \ u(0)=u_0.\end{aligned} \quad (3b)$$

Proof. The proof is given in (Kaczorek 2015a).

Definition 1. A state $x_f \in \Re^n$ of the standard system (2) is called reachable in time $t\in[0,t_f]$ if there exists an input $u(t)\in\Re^m$ for $t\in[0,t_f]$ which steers the state of the system from zero initial condition $x_0=0$ to the final state $x_f=x(t_f)$. If every state $x_f\in\Re^n$ is reachable in time $t\in[0,t_f]$ then the system is called reachable in time $t\in[0,t_f]$. The system (2) is called reachable if for every $x_f\in\Re^n$ there exists t_f and an input $u(t)\in\Re^m$ for $t\in[0,t_f]$ which steers the state of the system from $x_0=0$ to x_f.

Theorem 2. The standard fractional system (2) is reachable in time $t\in[0,t_f]$ if and only if the matrix

$$R_f=R(t_f)=\int_0^{t_f} e^{\hat{A}t}\hat{B}\hat{B}^T e^{\hat{A}^T t}dt \quad (4)$$

is invertible.

The input which steers the state of the system from $x_0=0$ to x_f is given by

$$u(t)=\int_0^t e^{-\beta\tau}\hat{B}^T e^{\hat{A}^T(t_f-\tau)}d\tau R_f^{-1}x_f, \ t\in[0,t_f] \quad (5)$$

and $u_0=u(0)=0$.

Proof. Substituting

$$\overline{u}(t)=\beta u(t)+\dot{u}(t) \quad (6)$$

into (3a) for $x_0=0$, $u_0=0$ we obtain

$$x(t)=\int_0^t e^{\hat{A}(t-\tau)}\hat{B}\overline{u}(\tau)d\tau. \quad (7)$$

The solution of the differential equation (6) for $u_0=u(0)=0$ has the form

$$u(t)=\int_0^t e^{-\beta\tau}\overline{u}(t-\tau)d\tau. \quad (8)$$

To show that the input

$$\overline{u}(t)=\hat{B}^T e^{\hat{A}(t_f-t)}R_f^{-1}x_f, \ t\in[0,t_f] \quad (9)$$

steers the state from $x_0=0$ to x_f in time $t\in[0,t_f]$ we substitute (9) into (7) and we obtain

$$\begin{aligned}x(t_f)&=\int_0^{t_f} e^{\hat{A}(t_f-\tau)}\hat{B}\hat{B}^T e^{\hat{A}^T(t_f-\tau)}d\tau R_f^{-1}x_f\\ &=R_f R_f^{-1}x_f=x_f.\end{aligned} \quad (10)$$

Substituting (9) into (8) we obtain (5). □

From Theorem 1 and its proof follows the corollary.

Corollary 1. The fractional system (2) is reachable in time $t\in[0,t_f]$ if and only if the fractional system

$$\frac{d^\alpha x(t)}{dt^\alpha}=\hat{A}x(t)+\hat{B}u(t) \quad (11)$$

is reachable in time $t\in[0,t_f]$.

The input $\overline{u}(t)$ steers the state $x(t)$ from $x_0=0$ to x_f in time $t\in[0,t_f]$ of the system (11) if and only if the input (8) steers the state from $x_0=0$ to x_f in time $t\in[0,t_f]$ of the system (2a).

Example 1. Consider the fractional system described by the equation (2a) with $\alpha=0.5$, zero initial condition $x_0=0$, $u_0=0$ and

$$A=\begin{bmatrix}-1 & a\\ 0 & -2\end{bmatrix}, \ B=\begin{bmatrix}2\\ 2\end{bmatrix}, \ a\text{ - parameter.} \quad (12)$$

Compute the input $u(t)$ which steers the system from $x_0=0$ to $x_f=[1 \quad 1]^T$ (T denotes transpose) in time $t\in[0,1]$. Using (3b) and (12) we obtain

$$\begin{aligned}\hat{A}&=\alpha[I_2-(1-\alpha)A]^{-1}A\\ &=0.5\begin{bmatrix}1.5 & -0.5a\\ 0 & 2\end{bmatrix}^{-1}\begin{bmatrix}-1 & a\\ 0 & -2\end{bmatrix}\\ &=\frac{1}{6}\begin{bmatrix}-2 & a\\ 0 & -3\end{bmatrix},\end{aligned} \quad (13a)$$

$$\hat{B} = [I_2 - (1-\alpha)A]^{-1}(1-\alpha)B$$
$$= \begin{bmatrix} 1.5 & -0.5a \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} a+4 \\ 3 \end{bmatrix} \quad (13b)$$

Taking into account that the eigenvalues of the matrix (13a) are $\lambda_1 = -\frac{1}{2}$, $\lambda_2 = -\frac{1}{3}$ and using the Sylvester formula we obtain

$$e^{\hat{A}t} = \frac{\hat{A} - I_2\lambda_2}{\lambda_1 - \lambda_2}e^{\lambda_1 t} + \frac{\hat{A} - I_2\lambda_1}{\lambda_2 - \lambda_1}e^{\lambda_2 t}$$
$$= \begin{bmatrix} 0 & -a \\ 0 & 1 \end{bmatrix} e^{-\frac{1}{2}t} + \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} e^{-\frac{1}{3}t} \quad (14)$$
$$= \begin{bmatrix} e^{-\frac{1}{3}t} & a\left(e^{-\frac{1}{3}t} - e^{-\frac{1}{2}t}\right) \\ 0 & e^{-\frac{1}{2}t} \end{bmatrix}.$$

Using (4) for $t_f = 1$ and (14), (13b) we obtain

$$R_f = \int_0^{t_f} e^{\hat{A}t}\hat{B}\hat{B}^T e^{\hat{A}^T t}dt = \int_0^1 (e^{\hat{A}t}\hat{B})(e^{\hat{A}t}\hat{B})^T dt$$
$$= \int_0^1 \begin{bmatrix} a\left(\frac{2}{3}e^{-\frac{1}{3}t} - \frac{1}{2}e^{-\frac{1}{2}t}\right) + \frac{2}{3}e^{-\frac{1}{3}t} \\ \frac{1}{2}e^{-\frac{1}{2}t} \end{bmatrix}$$
$$\times \begin{bmatrix} a\left(\frac{2}{3}e^{-\frac{1}{3}t} - \frac{1}{2}e^{-\frac{1}{2}t}\right) + \frac{2}{3}e^{-\frac{1}{3}t} & \frac{1}{2}e^{-\frac{1}{2}t} \end{bmatrix} dt$$
$$= \int_0^1 \begin{bmatrix} \left[a\left(\frac{2}{3}e^{-\frac{1}{3}t} - \frac{1}{2}e^{-\frac{1}{2}t}\right) + \frac{2}{3}e^{-\frac{1}{3}t}\right]^2 & \left[a\left(\frac{2}{3}e^{-\frac{1}{3}t} - \frac{1}{2}e^{-\frac{1}{2}t}\right) + \frac{2}{3}e^{-\frac{1}{3}t}\right]\frac{1}{2}e^{-\frac{1}{2}t} \\ \left[a\left(\frac{2}{3}e^{-\frac{1}{3}t} - \frac{1}{2}e^{-\frac{1}{2}t}\right) + \frac{2}{3}e^{-\frac{1}{3}t}\right]\frac{1}{2}e^{-\frac{1}{2}t} & \left[\frac{1}{2}e^{-\frac{1}{2}t}\right]^2 \end{bmatrix} dt$$
$$= \begin{bmatrix} 0.0301a^2 + 0.1965a + 0.3244 & 0.0681a + 0.2262 \\ 0.0681a + 0.2262 & 0.158 \end{bmatrix}. \quad (15)$$

The matrix (15) is nonsingular since $\det R_f = 0.0001a^2 + 0.0002a + 0.0001 \neq 0$ for $a \neq -1$ and by Theorem 1 the fractional system with (12) is reachable in time $t \in [0,1]$ for $a \neq -1$.

The input steering the system from $x_0 = 0$ to $x_f = [1 \quad 1]^T$ in time $t \in [0,1]$ is given by

$$u(t) = \int_0^t e^{-\beta(t-\tau)}\hat{B}^T e^{\hat{A}^T(t_f - \tau)}d\tau R_f^{-1}x_f$$
$$= e^{-t}\int_0^t e^{-\tau}\hat{B}^T e^{\hat{A}^T}e^{-\hat{A}^T\tau}d\tau R_f^{-1}x_f$$
$$= e^{-t}\int_0^t e^{\tau}\frac{1}{6}[a+4 \quad 3]\begin{bmatrix} 0.7165 & 0 \\ 0.11a & 0.6065 \end{bmatrix}$$
$$\times \begin{bmatrix} e^{\frac{1}{3}\tau} & 0 \\ a\left(e^{\frac{1}{3}\tau} - e^{\frac{1}{2}\tau}\right) & e^{\frac{1}{2}\tau} \end{bmatrix} d\tau$$
$$\times \begin{bmatrix} 0.0301a^2 + 0.1965a + 0.3244 & 0.0681a + 0.2262 \\ 0.0681a + 0.2262 & 0.158 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= e^{-t}\int_0^t e^{\tau}\begin{bmatrix} e^{\frac{1}{3}\tau}(0.4777a + 0.4777) - 0.333e^{\frac{1}{2}\tau} & 0.3033e^{\frac{1}{2}\tau} \end{bmatrix} d\tau$$
$$\times \begin{bmatrix} \dfrac{-0.0681a - 0.0682}{0.0001a^2 + 0.0002a + 0.0001} \\ \dfrac{0.0301a^2 + 0.1284a + 0.0982}{0.0001a^2 + 0.0002a + 0.0001} \end{bmatrix}. \quad (16)$$

For example for $a = 1$ we obtain

$$u(t) = e^{-t}\int_0^t e^{\tau}\begin{bmatrix} e^{\frac{1}{3}\tau}(0.4777a + 0.4777) - 0.303e^{\frac{1}{2}\tau} & 0.3033e^{\frac{1}{2}\tau} \end{bmatrix} d\tau \begin{bmatrix} -340.75 \\ 641.75 \end{bmatrix} \quad (17)$$
$$= -244.1644e^{\frac{1}{3}t} + 198.6615e^{\frac{1}{2}t} + 45.2029e^{-t}.$$

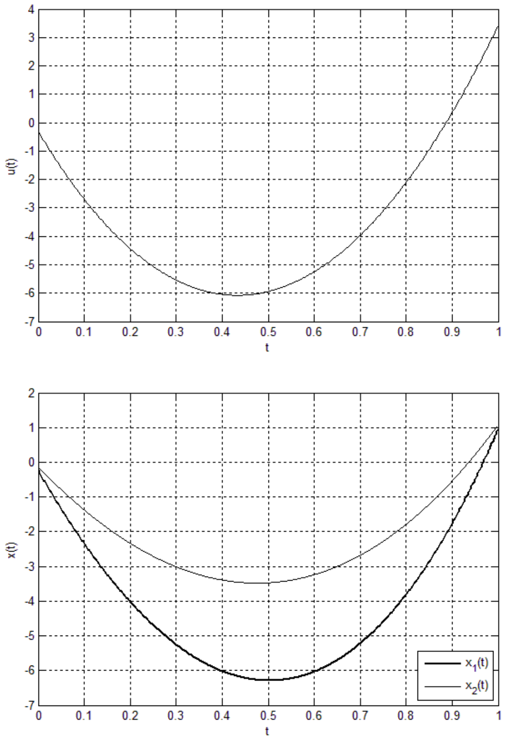

Figure 1: Input signal and state vector for $t_f = 1$[s].

REACHABILITY OF POSITIVE FRACTIONAL SYSTEMS

Consider the fractional system (2).

Definition 2. The fractional system (2) is called (internally) positive if the state vector $x(t) \in \Re_+^n$ and the output vector $y(t) \in \Re_+^p$, $t \geq 0$ for all initial conditions $x_0 \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $\dot{u}(t) \in \Re_+^m$, $t \geq 0$.

Definition 3. A real matrix $A = [a_{ij}] \in \Re^{n \times n}$ is called Metzler matrix if its off-diagonal entries are nonnegative, i.e. $a_{ij} \geq 0$ for $i \neq j$; $i, j = 1,...,n$.

Lemma 1. Let $\hat{A} \in M_n$ and $0 < \alpha < 1$. Then

$$e^{\hat{A}t} \in \Re_+^{n \times n} \text{ for } t \geq 0. \tag{18}$$

Proof. The proof is similar to the one given in (Kaczorek 2001).

Theorem 3. The fractional system (2) is positive if and only if

$$\hat{A} \in M_n,\ \hat{B} \in \Re_+^{n \times m},\ C \in \Re_+^{p \times n},\ D \in \Re_+^{p \times m}. \tag{19}$$

Proof. *Sufficiency.* If $\hat{A} \in M_n$ and $\hat{B} \in \Re_+^{n \times m}$ then from (3) we have $x(t) \in \Re_+^n$, $t \geq 0$ since by Lemma 1 $e^{\hat{A}t} \in \Re_+^{n \times n}$ and $x_0 \in \Re_+^n$, $u(t) \in \Re_+^m$, $\dot{u}(t) \in \Re_+^m$, $t \geq 0$.

Necessity. Let $u(t) = 0$, $t \geq 0$ and $x_0 = e_i$ (i-th column of the identity matrix I_n). The trajectory does not leave the orthant $\Re_+^n$ only if ${}^{CF}D^{\alpha} x(0) = \hat{A} e_i \geq 0$ what implies $\hat{a}_{ij} \geq 0$ for $i \neq j$, $i, j = 1,...,n$ and $\hat{A} \in M_n$. If $x_0 = 0$ then ${}^{CF}D^{\alpha} x(0) = Bu(0) \geq 0$ and this implies $B \in \Re_+^{n \times m}$ since $u(0) \in \Re_+^m$ is arbitrary. From (2b) for $u(t) = 0$, $t \geq 0$ we have $y(0) = Cx(0)$ and $C \in \Re_+^{p \times n}$ since $x(0) = x_0 \in \Re_+^n$ is arbitrary. Assuming $x_0 = 0$ from (2b) we have $y(0) = Du(0)$ and $D \in \Re_+^{p \times m}$ since $u(0) \in \Re_+^m$ is arbitrary. □

Lemma 2. If λ_k, $k = 1,...,n$ are the eigenvalues of the matrix A then the eigenvalues of the matrix $\hat{A} = \alpha [I_n - (1-\alpha)A]^{-1} A$ are given by

$$\hat{\lambda}_k = \alpha [1 - (1-\alpha)\lambda_k]^{-1} \lambda_k. \tag{20}$$

Proof. It is well-known (Gantmacher 1959) that if $f(\lambda_k)$ is well-defined on the spectrum λ_k, $k = 1,...,n$ of the matrix A then the eigenvalues of the matrix $f(A)$ are given by $f(\lambda_k)$, $k = 1,...,n$. In this case $f(A) = \alpha [I_n - (1-\alpha)A]^{-1} A$. □

Lemma 3. The matrix $\overline{A} = (1-\alpha)A \in \Re^{n \times n}$ for $0 < \alpha < 1$ is asymptotically stable if and only if the matrix A is asymptotically stable.

Proof. The eigenvalues $\overline{\lambda}_k$, $k = 1,...,n$ of the matrix $\overline{A}$ are related with the eigenvalues λ_k, $k = 1,...,n$ of the matrix A by

$$\overline{\lambda}_k = (1-\alpha)\lambda_k,\ k = 1,...,n. \tag{21}$$

since the characteristic polynomials of the matrices are related by the equality

$$\begin{aligned} \det[I_n \overline{\lambda}_k - \overline{A}] &= \det[I_n \overline{\lambda}_k - (1-\alpha)A] \\ &= (1-\alpha)^n \det\left[I_n \frac{\overline{\lambda}_k}{1-\alpha} - A\right] \\ &= (1-\alpha)^n \det[I_n \lambda_k - A]. \end{aligned} \tag{22}$$

Therefore, from (21) it follows that $\operatorname{Re} \overline{\lambda}_k < 0$, $k = 1,...,n$ if and only if $\operatorname{Re} \lambda_k < 0$, $k = 1,...,n$. □

Lemma 4. The matrix

$$\hat{A} = \alpha [I_n - (1-\alpha)A]^{-1} A \in M_n \tag{23}$$

is asymptotically stable if and only if the eigenvalues $\lambda_k = -\alpha_k + j\beta_k$, $k = 1,...,n$ of the matrix A satisfy the condition

$$[1+(1-\alpha)\alpha_k]\alpha_k+(1-\alpha)\beta_k^2=n(k)>0.$$

Proof. From (20) for $\hat{\lambda}_k=-\hat{\alpha}_k+j\hat{\beta}_k$ and $\lambda_k=-\alpha_k+j\beta_k$, $k=1,...,n$ we have

$$\begin{aligned}\hat{\lambda}_k&=-\hat{\alpha}_k+j\hat{\beta}_k=\alpha[1-(1-\alpha)\lambda_k]^{-1}\lambda_k\\&=\alpha[1-(1-\alpha)(-\alpha_k+j\beta_k)]^{-1}(-\alpha_k+j\beta_k)\\&=\alpha\frac{1+(1-\alpha)\alpha_k+j(1-\alpha)\beta_k}{[1+(1-\alpha)\alpha_k]^2+[(1-\alpha)\beta_k]^2}(-\alpha_k+j\beta_k)\\&=\alpha\left(\frac{-[1+(1-\alpha)\alpha_k]\alpha_k-(1-\alpha)\beta_k^2}{[1+(1-\alpha)\alpha_k]^2+[(1-\alpha)\beta_k]^2}\right.\\&\quad\left.j\frac{[1+(1-\alpha)\alpha_k]\beta_k-(1-\alpha)\alpha_k\beta_k}{[1+(1-\alpha)\alpha_k]^2+[(1-\alpha)\beta_k]^2}\right)\end{aligned}\tag{24}$$

and

$$\begin{aligned}\hat{\alpha}_k&=\alpha\left(\frac{[1+(1-\alpha)\alpha_k]\alpha_k+(1-\alpha)\beta_k^2}{[1+(1-\alpha)\alpha_k]^2+[(1-\alpha)\beta_k]^2}\right)\\&=\alpha\frac{n(k)}{d(k)},\ k=1,...,n.\end{aligned}\tag{25}$$

From (25) it follows that $\hat{\alpha}_k>0$, $k=1,...,n$ if and only if $n(k)>0$, $k=1,...,n$. □

Lemma 5. The matrices

$$\begin{aligned}\hat{A}&=\alpha[I_n-(1-\alpha)A]^{-1}A\in M_n,\\\hat{B}&=[I_n-(1-\alpha)A]^{-1}(1-\alpha)B\in\Re_+^{n\times m}\end{aligned}\tag{26}$$

if $A\in M_n$ is asymptotically stable and $B\in\Re_+^{n\times m}$.

Proof. The matrix $[I_n-(1-\alpha)A]^{-1}\in\Re_+^{n\times n}$ if the matrix $A\in M_n$ is asymptotically stable (Kaczorek 2001). Therefore, by Lemma 3 and $(1-\alpha)B\in\Re_+^{n\times m}$ for $0<\alpha<1$ (25) holds if $A\in M_n$ is asymptotically stable. □

From Lemma 4 and Theorem 3 we have the following.

Theorem 4. The fractional system (2) is positive if $A\in M_n$ is asymptotically stable and $B\in\Re_+^{n\times m}$, $C\in\Re_+^{p\times n}$, $D\in\Re_+^{p\times m}$.

Definition 4. A state $x_f\in\Re_+^n$ of the positive system (2) is called reachable in time $t\in[0,t_f]$ if there exists an input $u(t)\in\Re_+^m$ for $t\in[0,t_f]$ which steers the state of the system from zero initial condition $x_0=0$ to the final state $x_f\in\Re_+^n$. If every state $x_f\in\Re_+^n$ is reachable in time $t\in[0,t_f]$ then the system is called reachable in time $t\in[0,t_f]$. The positive system (2) is called reachable if for every $x_f\in\Re_+^n$ there exists t_f and an input $u(t)\in\Re_+^m$ for $t\in[0,t_f]$ which steers the state of the system from $x_0=0$ to x_f.

Definition 5. A matrix $A\in\Re^{n\times n}$ is called monomial if in each row and in each column only one entry is positive and the remaining entries are zero.

Theorem 5. The positive fractional system (2) is reachable in time $t\in[0,t_f]$ if the matrix

$$R_f=R(t_f)=\int_0^{t_f}e^{\hat{A}t}\hat{B}\hat{B}^Te^{\hat{A}^Tt}dt\tag{27}$$

is monomial.

The input which steers the state of the system from $x_0=0$ to x_f is given by

$$u(t)=\int_0^t e^{-\beta t}\hat{B}^Te^{\hat{A}^T(t_f-\tau)}d\tau R_f^{-1}x_f\tag{28}$$

Proof. It is well-known (Kaczorek 2001) that $R_f^{-1}\in\Re_+^{n\times n}$ if and only if the matrix $R_f\in\Re_+^{n\times n}$ is monomial. In a similar way as in proof of Theorem 1 it can be shown that the input (28) steers the state of positive system from $x_0=0$ to $x_f\in\Re_+^n$ in time $t\in[0,t_f]$. From (28) it follows that $u(t)\in\Re_+^m$ since $e^{-\beta t}>0$ for $\beta=\frac{\alpha}{1-\alpha}>0$, $0<\alpha<1$, $\hat{B}^Te^{\hat{A}(t_f-\tau)}\in\Re_+^{m\times n}$ and $R_f^{-1}x_f\in\Re_+^n$. □

Example 2. (Continuation of Example 1) Note that the matrix R_f given by (15) is monomial only for $a=-3.3216$. Therefore, we cannot say anything about the reachability of the positive system with (12) in time $t\in[0,1]$ for $a\geq 0$.

CONCLUDING REMARKS

The Caputo-Fabrizio definition of the fractional derivative has been applied to analysis of the positivity and reachability of continuous-time linear systems. Necessary and sufficient conditions for the reachability of standard continuous-time linear systems have been established (Theorem 1). Necessary and sufficient conditions for the positivity of the fractional linear systems have been given (Theorems 3 and 4). Sufficient conditions for the reachability of the fractional positive linear systems have been also established (Theorem 5). The considerations are illustrated by numerical examples of standard and positive fractional linear systems.

The considerations can be extended to continuous-discrete linear systems.

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