

INDEXED BONDS WITH MEAN-REVERTING RISK FACTORS

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ABSTRACT

In this paper, we focus on the value of inflation-indexed bonds in an extended short rate model, which is a specific case of the general framework provided by Jarrow and Yildirim (2003). In the model, we assume mean-reverting stochastic dynamics under the risk neutral measure for both the short interest rate and the instantaneous inflation rate. We define the zero-coupon inflation-indexed bond, and first estimate its value by Monte Carlo simulation, then deduce an analytical formula as well. We briefly touch on the yield and inflation curves the model is able to produce.

INTRODUCTION

Standard nominal bonds are considered one of the simplest financial products, serving as building blocks for complex ones as well. While government bonds are usually considered riskless – apart from default risk which we ignore in this paper –, it is important to note that this property means no risk in the future nominal cash-flow, not in the present value of the security itself. The main subject of our examination is a bond with a similar idea: the inflation-indexed bond. This security can also be considered riskless, but in a different sense: instead of the nominal values, the real values of future cash flows are fixed. To achieve this, the face value of an inflation-indexed bond is adjusted according to a price index, and the coupon and principal payments are based on this adjusted amount.

An example for the cash flows of nominal and inflation-indexed bonds is given in Table 1. While the future cash flows of the nominal bond are deterministic, those of the indexed bond are random variables, since we do not know the inflation rates of in advance.

However, if we care more about the purchasing power rather than the exact dollar figures, the inflation-indexed bond suddenly becomes deterministic, while the future

cash flows of the nominal bond can be considered random variables. The indexed bond described in Table 1 will provide the holder with (5,5,105) units of the basket of goods that is used for calculating the price index.

	0	1	2	3	
inflation		2%	4%	3%	
nominal cash flow	100.0	100.0	100.0	100.0	nominal bond
indexed cash flow	100.0	102.0	106.1	109.3	indexed bond

Table 1: Cash flows of standard nominal and inflation-indexed bonds. Coupon rate: 5%. Frequency: annual.

An example – and perhaps the most important one in terms of gross market value – of such an indexed bond is the Treasury Inflation-Protected Security [TIPS] issued by the Treasury of the United States. Issued at maturities of 5, 10, and 30 years, adjustment of the principal amount is based on the Consumer Price Index [CPI]. Real coupon rates are fixed, and are paid semi-annually. TIPS take up more than 6% of total public debt of the United States (United States Department of the Treasury, 2017).

The main holders of TIPS are pension funds, who are especially motivated to hedge against inflation risk. Pension funds' future obligations are tied to inflation, so by investing in TIPS, inflation risk will be present on both sides of their balance sheet, thus cancelling each other. For more information about inflation-indexed securities in general see Deacon et al. (2004). Affine models for inflation-indexed derivatives and related securities are discussed thoroughly by Ho, Huang and Yildirim (2014). Our research follows the path of Mercurio (2005), who considers a specific case of such affine models.

The rest of the paper is organized as follows. First we introduce two basic financial instruments and describe

the stochastic model within which we can price them. Afterwards, we detail a simple but computationally expensive estimation of this price by Monte Carlo simulation. Then, we deduce an analytical formula for this price by applying stochastic calculus and pricing theory. Finally, we investigate the yield and inflation curves the model is able to produce, with some emphasis on the model's most prominent parameter. For detailed analysis of the mathematical background of our paper see Cairns (2004), Medvegyev (2007), Medvegyev and Száz (2010).

THE MODEL

We assume both the short interest rate and the instantaneous rate of inflation to follow an Ornstein-Uhlenbeck process under the risk neutral measure (Q), described by the following stochastic differential equations:

$$\begin{aligned} dr(t) &= \alpha_r[\bar{r} - r(t)]dt + \sigma_r dW_r^Q(t) \\ di(t) &= \alpha_i[\bar{i} - i(t)]dt + \sigma_i dW_i^Q(t) \end{aligned} \quad (1)$$

The process for the short interest rate is of course the classic mean-reverting process of the Vasicek model (Vasicek, 1977). Mean reversion is present in most short interest rate models, and it seems reasonable to assume mean reversion for the inflation rate as well.

Intuition behind the parameters is the following: \bar{r} and \bar{i} are the long-term means of the processes. Whenever the process is below (above) its long-term mean, the drift becomes positive (negative), thus tending towards the long-term mean. The factors $\alpha_{\{r,i\}}$ represent the strength of this mean-reverting property, and $\sigma_{\{r,i\}}$ are the volatility parameters.

There are two Wiener processes present on this market, so we must define a correlation parameter through their cross quadratic variation:

$$d\langle W_r^Q(t), W_i^Q(t) \rangle = \rho dt$$

It is important to note that neither of these processes have independent increments: their increments are dependent on the current value of the processes themselves, which stems from their essential mean-reverting property. Thus, the parameter ρ does not perfectly describe the correlation of the increments of the two processes, but only the correlation of the stochastic shocks represented by the Wiener-processes.

We assume two financial products to be driven by these processes: the bank deposit $B(t)$ and the price index $I(t)$:

$$\begin{aligned} dB(t) &= r(t)B(t)dt \\ dI(t) &= i(t)I(t)dt \end{aligned}$$

The solution of these differential equations takes the well-known form for $t < T$:

$$\begin{aligned} B(T) &= B(t)e^{\int_t^T r(s)ds} \\ I(T) &= I(t)e^{\int_t^T i(s)ds} \end{aligned} \quad (2)$$

We are investigating the zero-coupon inflation indexed bond [ZCIIB]. This theoretical financial security – together with the standard zero-coupon bond – is going to define the term structure of inflation rates (and interest rates).

The ZCIIB has the following parameters: it is issued at time T_0 , it matures at time T , and its single payoff at maturity is $I(T)/I(T_0)$, thus the payoff is indexed by the increase of the price index during the bond's term. We are looking for the value of the ZCIIB at time t , where naturally $T_0 \leq t \leq T$. Notation: $P_i(t, T_0, T)$.

It should be noted here that while ZCIIBs are mostly theoretical in nature, a real world inflation-indexed coupon bond like the TIPS described in the introduction can be thought of as a portfolio of ZCIIBs with maturities that coincide with the coupon and notional payments of the coupon bond.

Since the model is a simple extension of the Vasicek model to include inflation-indexed bonds, it does not lose standard nominal bonds either. Thus, we also define the zero-coupon bond [ZCB]: a security with a single payoff of 1 (notional) at maturity T . The notation for the value of this security at time $t \leq T$: $P_r(t, T)$.

NUMERICAL APPROACH

First, we will calculate the value of a ZCIIB (issued at $T_0 = 0$) at time $t = 0$ by Monte-Carlo simulation.

Without violating generality, we can assume $B(0) = I(0) = 1$. The value of a ZCIIB maturing at time T is thus given by the pricing formula:

$$P_i(0, 0, T) = \mathbb{E}_Q \left(\frac{I(T)}{B(T)} \middle| \mathcal{F}(0) \right) \quad (3)$$

Since the stochastic processes detailed above are given under the martingale measure, we can estimate this expectation by Monte Carlo simulation directly.

The two processes are correlated in their stochastic shocks, but most software can only generate independent pseudo-random variables, so we will use Cholesky-decomposition to exchange $W_i^Q(t)$ of equation 1 for two independent Wiener processes:

$$\begin{aligned} di(t) &= \alpha_i[\bar{i} - i(t)]dt \\ &\quad + \sigma_i \left(\rho dW_r^Q(t) + \sqrt{1 - \rho^2} d\tilde{W}_r^Q(t) \right) \end{aligned}$$

where $W_r^Q(t)$ and $\tilde{W}_r^Q(t)$ are independent.

A given pair of trajectories for the short interest rate and the instantaneous inflation rate will be simulated recursively as a first order approximation of the dynamics given in equation 1:

$$\begin{aligned} r(t_{n+1}) &= r(t_n) + \alpha_r[\bar{r} - r(t_n)]\Delta t + \sigma_r\sqrt{\Delta t}Z_{n+1} \\ i(t_{n+1}) &= i(t_n) + \alpha_i[\bar{i} - i(t_n)]\Delta t \\ &\quad + \sigma_i\sqrt{\Delta t} \left(\rho Z_{n+1} + \sqrt{1 - \rho^2}\tilde{Z}_{n+1} \right) \end{aligned}$$

where $0 = t_0 < \dots < t_N = T$ is an equidistant partition of the interval $[0, T]$, $\Delta t = t_{n+1} - t_n$ is the length of a subinterval, and $\{\{Z_n\}_{n=1}^N, \{\tilde{Z}_n\}_{n=1}^N\}$ are independent standard normal random variables.

Representing a single event from the sample space, Figure 1 shows generated trajectories for the short interest rate and the instantaneous inflation rate. We chose $T = 10$ and $N = 1000$ (thus $\Delta t = 0.01$) for the simulation. The strong negative correlation parameter can be traced visually: while both processes fluctuate around their long-term mean, they tend to move in the opposite direction.

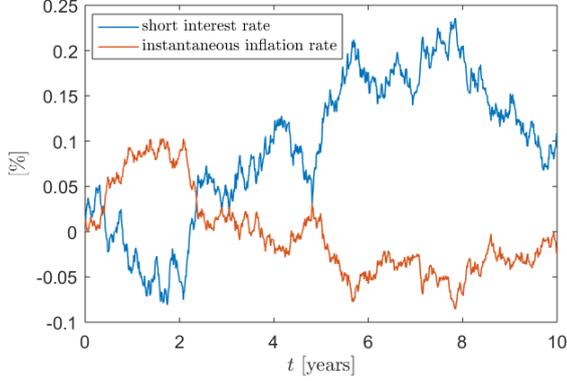


Figure 1: Generated trajectories for the interest rate and inflation rate. Parameters: $\alpha_r = 0.4$, $\alpha_i = 0.4$, $\bar{r} = 0.06$, $\bar{i} = 0.04$, $\sigma_r = 0.06$, $\sigma_i = 0.04$, $r(0) = 0.02$, $i(0) = 0.01$, $\rho = -0.9$.

Given the trajectories for the interest rate and the inflation rate, the value processes for the bank deposit and the price index can be calculated by linear approximation of the dynamics given in equation 2:

$$\begin{aligned} B(t_{n+1}) &= B(t_n)(1 + r(t_n)\Delta t) \\ I(t_{n+1}) &= I(t_n)(1 + i(t_n)\Delta t) \end{aligned}$$

where $B(0) = I(0) = 1$.

While the processes $r(t)$ and $i(t)$ are of unbounded variation, the value processes $B(t)$ and $I(t)$ are defined by the integral of these, thus becoming of bounded variation. This ‘‘smoothness’’ property can be seen in Figure 2.

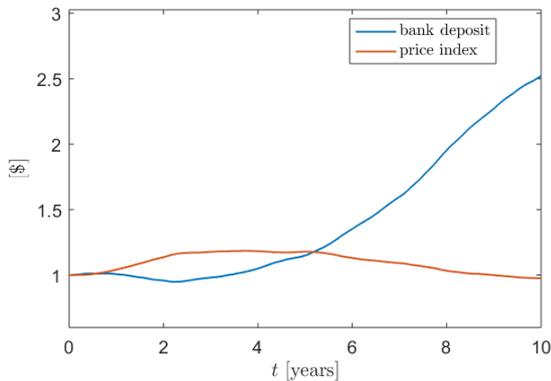


Figure 2: Value processes for the bank deposit and the price index.

The processes of Figure 2 correspond to the instantaneous rate processes in Figure 1: compounding the interest (inflation) rates of Figure 1 will result a bank deposit (price index) seen in Figure 2.

Finally, we calculate the ratio of these value processes, thus receiving a single observation for $I(T)/B(T)$. It is important to note that a single observation is a trajectory itself, since we calculate the ratio for $\forall \{t_n\}_{n=1}^N \in [0, T]$. We produce a multitude of such observations in order to get an estimate for the expectation of equation 3.

$$\hat{P}_i(0,0,T) = \frac{\sum_{m=1}^M \frac{I_m(T)}{B_m(T)}}{M}$$

We chose a sample size of $M = 10,000$ for this simulation. Figure 3 shows the 95% confidence interval of the empirical mean, and it also shows the exact theoretical value, which was calculated by using the closed formula detailed in the next section.

It is important to note the computational costliness of the Monte Carlo approach: a single pair of trajectories required $2 \times N = 2,000$ standard normal random variables, and we generated $M = 10,000$ such pairs, thus requiring 20 million in total.

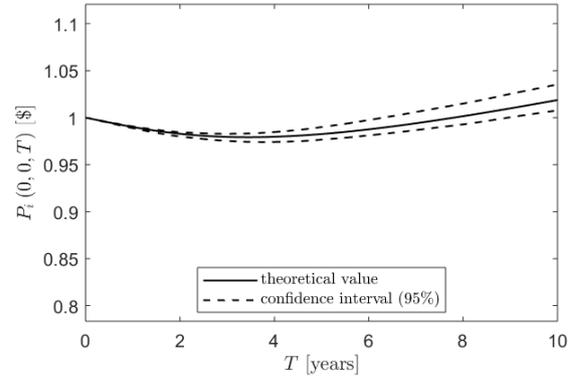


Figure 3: Price of a ZCIIB issued today for maturities $T \in [0,10]$. Parameters used are the same as in Figure 1.

An important property of the ZCIIB can be traced in Figure 3: the value of such a bond does not necessarily have to be decreasing in maturity. While ZCBs’ simple payoff of 1 at maturity means that their price does have to be decreasing in maturity almost by definition because of the time value of money, ZCIIBs’ payoff is indexed by inflation, thus a longer bond can be worth more than a shorter one if market participants expect high inflation in the future.

CLOSED FORMULA

In this section, we will deduce a closed formula for the price of the ZCIIB. The value of a security at time t with the payoff $I(T)/I(T_0)$ is given by the pricing formula:

$$P_i(t, T_0, T) = B(t) \mathbb{E}_Q \left(\frac{I(T)/I(T_0)}{B(T)} \middle| \mathcal{F}(t) \right)$$

After substituting the solutions for $B(T)$ and $I(T)$ given in equation 2, and taking the $\mathcal{F}(t)$ -measurable factors out of the expectation, we arrive at:

$$P_i(t, T_0, T) = \frac{I(t)}{I(T_0)} \mathbb{E}_Q \left(e^{\int_t^T r(s) - i(s) ds} \middle| \mathcal{F}(t) \right)$$

The value of this conditional expectation could be calculated directly, but we follow another route. Since both $r(t)$ and $i(t)$ are Markov processes, the condition $\mathcal{F}(t)$ is equivalent to the condition $\{r(t), i(t)\}$. The conditional expectation thus becomes a function with the following arguments:

$$P_i(t, T_0, T) = \frac{I(t)}{I(T_0)} V(r(t), i(t), t, T) \quad (4)$$

Since the Q martingale measure is the specific measure under which any value process divided by the bank deposit (our numeraire) becomes a martingale, $P_i(t, T_0, T)/B(t)$ must also be a martingale, thus its drift must equal 0. We calculate this drift by applying Ito's lemma, and arrive at a partial differential equation for $V(r(t), i(t), t, T)$. We omit the arguments for $V(r(t), i(t), t, T)$, $r(t)$, $i(t)$, and indicate partial derivatives in the subscript:

$$\begin{aligned} V_t + V_r \alpha_r (\bar{r} - r) + V_i \alpha_i (\bar{i} - i) + V_{r_i} \sigma_r \sigma_i \rho \\ + \frac{1}{2} V_{rr} \sigma_r^2 + \frac{1}{2} V_{ii} \sigma_i^2 + V_i - Vr = 0 \end{aligned} \quad (5)$$

$$V(r(T), i(T), T, T) = 0$$

The boundary condition arises from the ZCIIB's payoff at maturity: it pays $I(T)/I(T_0)$ at maturity, thus the bond's value at time T must also equal this amount, which means from equation 4 we get:

$$\frac{I(T)}{I(T_0)} = P_i(T, T_0, T) = \frac{I(T)}{I(T_0)} V(r(T), i(T), T, T)$$

which implies the boundary condition of equation 5: $V(r(T), i(T), T, T) = 1$.

We are going to solve the PDE of equation 5 by assuming the solution to be of a specific – so-called affine – form:

$$V(r(t), i(t), t, T) = e^{A(t, T) - C(t, T)r(t) + D(t, T)i(t)} \quad (6)$$

After substituting this solution form into equation 5, the boundary condition becomes:

$$A(T, T) - C(T, T)r(T) + D(T, T)i(T) = 0$$

We are looking for a solution for $\forall \{r(T), i(T)\} \in \mathbb{R} \times \mathbb{R}$. When $r(T) = i(T) = 0$, the boundary condition becomes $A(T, T) = 0$. When $r(T) \neq i(T) = 0$, we get $C(T, T) = 0$; and similarly, when $i(T) \neq r(T) = 0$, we get $D(T, T) = 0$. Since the boundary condition has to hold true for $\forall \{r(T), i(T)\}$, it has fallen apart into three separate boundary conditions:

$$\begin{aligned} A(T, T) &= 0 \\ C(T, T) &= 0 \\ D(T, T) &= 0 \end{aligned}$$

After substituting the solution form of equation 6 into the PDE of equation 5 itself, we can apply the same reasoning, which results in it falling apart into three separate PDEs as well:

$$\begin{aligned} A_t - \alpha_r \bar{r} C + \alpha_i \bar{i} D - \sigma_r \sigma_i \rho CD + \frac{1}{2} \sigma_r^2 C^2 + \frac{1}{2} \sigma_i^2 D^2 &= 0 \\ -C_t + \alpha_r C - 1 &= 0 \\ D_t - \alpha_i D + 1 &= 0 \end{aligned}$$

The solutions for $C(t, T)$ and $D(t, T)$ are the following:

$$\begin{aligned} C(t, T) &= \frac{1 - e^{-\alpha_r(T-t)}}{\alpha_r} \\ D(t, T) &= \frac{1 - e^{-\alpha_i(T-t)}}{\alpha_i} \end{aligned} \quad (7)$$

From here, the solution for $A(t, T)$ can be derived by simple integration. We will omit the arguments of $A(t, T)$, $C(t, T)$ and $D(t, T)$ in the result:

$$\begin{aligned} A = C \left[\bar{r} + \frac{\sigma_r \sigma_i \rho}{\alpha_r \alpha_i} \left(1 - \frac{\alpha_r}{\alpha_r + \alpha_i} \right) - \left(\frac{\sigma_r}{2\alpha_r} \right)^2 (\alpha_r C + 2) \right] \\ + D \left[-\bar{i} + \frac{\sigma_r \sigma_i \rho}{\alpha_r \alpha_i} \left(1 - \frac{\alpha_i}{\alpha_r + \alpha_i} \right) - \left(\frac{\sigma_i}{2\alpha_i} \right)^2 (\alpha_i D + 2) \right] \\ + (T - t) \left[\bar{i} - \bar{r} - \frac{\sigma_r \sigma_i \rho}{\alpha_r \alpha_i} + \frac{\left(\frac{\sigma_r}{\alpha_r} \right)^2 + \left(\frac{\sigma_i}{\alpha_i} \right)^2}{2} \right] \\ + CD \frac{\sigma_r \sigma_i \rho}{\alpha_r + \alpha_i} \end{aligned}$$

Thus, the value of a ZCIIB (issued at T_0 with maturity date T) at time t is:

$$P_i(t, T_0, T) = \frac{I(t)}{I(T_0)} e^{A(t, T) - C(t, T)r(t) + D(t, T)i(t)}$$

where the functions $A(t, T)$, $C(t, T)$ and $D(t, T)$ are defined above.

Figure 3 shows results of this closed formula for the current price of ZCIIBs for different maturities issued today: $P_i(0, 0, T)$, $T \in [0, 10]$. The theoretical values fall well inside the 95% confidence interval of the Monte Carlo simulation.

A logical control point for our result arises here: if we choose the parameters in a way that eliminates the effect of inflation, we should get the value of the standard nominal zero-coupon bond $P_r(t, T)$ of the Vasicek model.

To achieve this, first we set the price index at the issue date and current time to unity: $I(T_0) = I(t) = 1$. Next, we eliminate the volatility of the inflation rate: $\sigma_i = 0$. Finally, we make sure the inflation rate starts at zero and stays at zero by setting: $i(t) = \bar{i} = 0$. With these parameters, our result reduces to:

$$P_r(t, T) = P_i(t, T_0, T) = e^{A(t, T) - C(t, T)r(t)}$$

where $A(t, T)$ takes the reduced form:

$$\begin{aligned} A(t, T) = C(t, T) \left[\bar{r} - \left(\frac{\sigma_r}{2\alpha_r} \right)^2 (\alpha_r C(t, T) + 2) \right] \\ + (T - t) \left(\frac{\sigma_r^2}{2\alpha_r^2} - \bar{r} \right) \end{aligned}$$

and $C(t, T)$ is given in equation 7. This result does concur with the price of the zero-coupon bond of the Vasicek model.

TERM STRUCTURE OF INFLATION RATES

The yield curve of a given issuer – typically a government – shows the current market conditions for

annual yields at which the issuer could refinance its debt at different maturities. There is a one-to-one correspondence between the yield curve and the price of ZCBs, thus the formula of the Vasicek model for $P_r(0, T)$ produces a yield curve:

$$y_r(T) = \frac{1/P_r(0, T)}{T}$$

where $y_r(T)$ is the nominal yield for maturity T . The yield curve typically – although not necessarily – has an upward sloping shape, which the Vasicek model is able to reproduce, as seen in Figure 4.

We go one step further: since our model includes both nominal and inflation-indexed zero-coupon bonds, we will be able to deduce the market-projected annual inflation rate (Dodgson and Kainth, 2006), which we will simply refer to as inflation curve:

$$y_i(T) = \frac{\log \frac{P_i(0,0,T)}{P_r(0,0,T)}}{T} = y_r(T) - \frac{1/P_i(0,0,T)}{T}$$

where $y_i(T)$ shows the annual inflation rate which is projected by the market for maturity T .

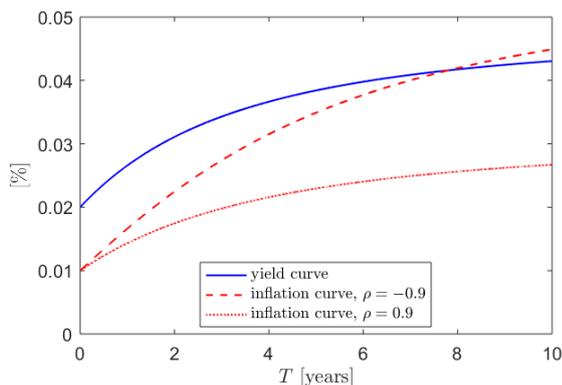


Figure 4: Possible yield and inflation curves described by the model. Parameters are the same as in Figure 1, apart from the correlation parameter as noted in the legend. Note that the mean reversion level of the interest rate process is well above that of the inflation rate process.

Figure 4 shows two inflation curves defined by the model, which differ only in their correlation parameter. Under typical market conditions we would expect the inflation curve to be below the yield curve, since investors usually demand a positive ex ante real yield from their investments, including their nominal bonds. While this does hold true for the case of strong positive correlation parameter, the strong negative correlation parameter makes the inflation curve steeper, even going above the yield curve for longer maturities.

This implies a negative connection between the value of the zero-coupon inflation-indexed bond and the correlation parameter of the stochastic processes for the short interest rate and the instantaneous inflation rate. The exact mechanics of this connection offers ground for further research.

CONCLUSION

In this paper, we presented the main features of inflation-indexed bonds. We defined the zero-coupon inflation-indexed bond, which can serve as building blocks for coupon bonds as well. We set up a mean-reverting stochastic model for the short interest rate and the instantaneous inflation rate. We simulated pairs of trajectories for these processes, and estimated the value of the zero-coupon inflation-indexed bond by Monte Carlo simulation. Then we presented an analytical solution for this problem as well, which corresponded with the results of the simulation. Finally, we used the analytical formula to graph typical inflation curves, and discussed the possible effect of the model's correlation parameter.

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