ASYMPTOTIC ANALYSIS OF MARKOVIAN RETRIAL QUEUE WITH TWO-WAY COMMUNICATION UNDER LOW RATE OF RETRIALS CONDITION [1]

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ABSTRACT

In this paper we are reviewing the retrial queue with two-way communication and Poisson arrival process. If the server free, incoming call occupies it. The call that finds the server being busy joins an orbit and retries to enter the server after some exponentially distributed time. If the server is idle, it causes the outgoing call from the outside. The outgoing call can find server free, then it starts making an outgoing call in an exponentially distributed time. If the outgoing call finds the server occupied, then it is lost. To research the system in question we have derived first and second order asymptotics of a number of calls in the orbit in an asymptotic condition of a low rate of retrials. Based on found asymptotics we have built the Gaussian approximation of a number of calls in the orbit.

INTRODUCTION

Recently a lot of attention is being paid to the research of the retrial queues such as mathematical models of real call center systems, telecommunication networks, computer networks, economical systems (Artalejo and Gomez-Corral 2008). These systems are characterized by the fact that if the clients (calls, phone calls, messages etc.) couldn't be served immediately they have to enter the virtual orbit where they wait out some delay before they could access the server for service again (Flajolet and Sedgewick 2009).

As a rule, the ones that are considered are the retrial queues in which arriving calls are either served immediately or join the orbit where they are wait out a random delay before accessing the server again. Recently, however, server is more likely to have the ability to make an outgoing call. The example of that could be the common cellphone that has function of both incoming and outgoing calls. In different call centers operators could receive arriving calls but as soon as they have free time and are in standby mode they could make outgoing calls to advertise, promote and sell packages and services of the centre.

Falin (Falin 1979) derives integral formulas for partial generating functions and some explicit expressions for characteristics of the M|G|1|1 retrial queues with outgoing calls. Choi et al. (Choi et al. 1995) extends Falin's model for the M/G/1/K retrial queues. Artalejo and Resing (Artalejo and Resing 2010) have derived first moments for characteristics of the M/G/1/1 retrial queues, in which the times of serving arriving and outgoing calls are different.

Martin and Artalejo (Martin and Artalejo 1995) are considering M|G|1|1 retrial queues with outgoing calls in which calls from an orbit access the server after an exponentially distributed delay in the order of arrival.

Artalejo and Phung-Duc (Artalejo and Tuan 2012) are considering M|M|1|1 retrial queues with outgoing calls and a different service time for incoming and outgoing calls. In their paper the authors have found an explicit solution for two-dimensional probability distribution of a server state and a number of calls in an orbit. Likewise, the factorial moments are found, based on which the proposed numerical and recurrent algorithms may be applied.

In this paper the main method of research is the asymptotic analysis method which allows to find in M|M|1|1 retrial queue with two-way communication type of limit distribution of a number of calls in the orbit in an asymptotic condition of a low rate of retrials and to show that limit distribution is Gaussian.

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This result is achieved by using the original asymptotic analysis method without needing to find the nonlimiting distribution. Furthermore, the discrete distribution is constructed which approximates discrete distribution of a number of calls in an orbit. This distribution will be addressed as Gaussian approximation. Research of retrial queueing system under the asymptotic condition that the retrial rate is extremely low is stated in the following papers (Nazarov and Chernikova 2014) (Nazarov and Izmailova 2016).

Furthermore, we have defined conditions of applicability of obtained approximation according to system defining parameters.

The remainder of the paper is presented as follows. In Section "Mathematical Model", we describe the model in detail and preliminaries for later asymptotic analysis. In Sections "Ferst order asymptotic" and "Second order acymptotic", we present our main contribution for the model with Poisson input. In Section "Approximation accuracy $P^{(2)}(i)$ and its application area" we have defined the conditions of applicability of the obtained approximation depending on values of system-defining parameters. Section "Conclusions" is devoted to concluding remark and future work.

MATHEMATICAL MODEL

Let's consider retrial queue (Figure 1) with Poisson arrival process of incoming calls with rate λ.

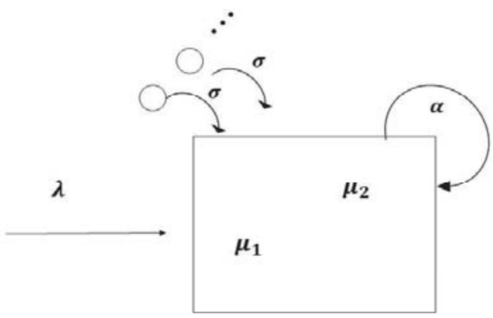

Figure 1: Retrial queue with two-way communication

The incoming call finds the server and goes into service for an exponentially distributed time with rate μ_1. If upon entering the system the call finds the server being busy the call immediately joins the orbit, where it stays during a random time distributed exponentially with rate σ.

If the server is idle (empty) it starts making outgoing calls from the outside with rate α. If the outgoing call finds the server free the call goes into service for an exponentially distributed time with rate μ_2. If upon entering the system the outgoing call finds the server being busy the call is lost and is not considered in the future. Let's denote:

$i(t)$ – number of calls in the orbit at the time t,

$n(t)$ – server state: 0 – server is free, 1 – server is busy serving an incoming call, 2 – server is busy serving an outgoing call.

Let's consider two-dimensional Markovian process $\{i(t), n(t)\}$ for probability distribution

$$P\{i(t) = i, n(t) = n\} = P_n(i, t)$$

setting up system of Kolmogorov equations

$$-(\lambda + i\sigma + \alpha)P_0(i) + \mu_1 P_1(i) + \mu_2 P_2(i) = 0,$$
$$-(\lambda + \mu_1)P_1(i) + \lambda[P_1(i-1) + P_0(i)] +$$
$$+(i+1)\sigma P_0(i+1) = 0,$$
$$-(\lambda + \mu_2)P_2(i) + P_0(i)\alpha + P_2(i-1)\lambda = 0. \quad (1)$$

Introducing partial characteristic functions (Nazarov and Paul 2016), denoting $j = \sqrt{-1}$,

$$H_n(u) = \sum_{i=0}^{\infty} e^{jui} P_n(i).$$

Rewriting system (1) in the following form

$$-(\lambda + \alpha)H_0(u) + j\sigma \frac{dH_0(u)}{du} +$$
$$+\mu_1 H_1(u) + \mu_2 H_2(u) = 0,$$
$$\left[\lambda\left(e^{ju} - 1\right) - \mu_1\right]H_1(u) + \lambda H_0(u) - j\sigma e^{-ju} \frac{dH_0(u)}{du} = 0,$$
$$\left[\lambda\left(e^{ju} - 1\right) - \mu_2\right]H_2(u) + \alpha H_0(u) = 0. \quad (2)$$

Characteristic function $H(u)$ of a number of incoming calls in an orbit and server states probability distribution r_n are relatively easy expressed through partial characteristic functions $H_n(u)$ by the following equations

$$H(u) = Me^{jui(t)} = H_0(u) + H_1(u) + H_2(u),$$
$$r_n = H_n(0), \; n = 0, 1, 2.$$

The task is put to find these characteristics of retrial queue with two-way communication. The main content of this paper is the solution of system (2) by using asymptotic analysis method in limit condition of a low rate of retrials, when $\sigma \to 0$.

This is due to the fact that for the more complicated queues with an incoming MMPP, the equation system similar to (2) is analytically unsolvable, but a solution by using asymptotic analysis method is allowed.

Application of asymptotic results in prelimit situation is causing the necessity of specifying the area of its applicability, which is obtainable only through comparison of asymptotic and prelimit characteristics and that is relatively easy implemented for the retrial queue in question. For more complex systems prelimit characteristics are usually defined by results of imitational modeling or by using pretty complicated numerical algorithms. The asymptotic analysis method suggested below is implemented by sequential determination of first and second order asymptotics.

FIRST ORDER ASYMPTOTIC

We introduce the following notations

$$\sigma = \varepsilon, \quad u = \varepsilon w, \quad H_n(u) = F_n(w, \varepsilon),$$

then we will get this system

$$-(\lambda+\alpha)F_0(w,\varepsilon)+j\varepsilon\frac{\partial F_0(w,\varepsilon)}{\partial w}+$$
$$+\mu_1 F_1(w,\varepsilon)+\mu_2 F_2(w,\varepsilon)=0,$$
$$\{\lambda(e^{jw\varepsilon}-1)-\mu_1\}F_1(w,\varepsilon)+$$
$$+\lambda F_0(w,\varepsilon)-j\varepsilon e^{-jw\varepsilon}\frac{\partial F_0(w,\varepsilon)}{\partial w}=0,$$
$$\{\lambda(e^{jw\varepsilon}-1)-\mu_1\}F_2(w,\varepsilon)+\alpha F_0(w,\varepsilon)=0. \qquad (3)$$

Theorem 1. (First order asymptotic) Suppose $i(t)$ is a number of calls in an orbit of stationary M|M|1 retrial queue with two-way communication, then the following equation is true

$$\lim_{\sigma\to 0} Me^{jw\sigma i(t)}=e^{jw\kappa_1},$$

where the parameter κ_1 is defined by the following

$$\kappa_1=\frac{\lambda}{\mu_2}\cdot\frac{\lambda\mu_2+\alpha\mu_1}{(\mu_1-\lambda)}.$$

Proof. Consider $\varepsilon\to 0$, then we will get

$$-(\lambda+\alpha)F_0(w)+\mu_1 F_1(w)+\mu_2 F_2(w)=0,$$
$$-\mu_1 F_1(w)+\lambda F_0(w)=0,$$
$$-\mu_2 F_2(w)+\alpha F_0(w)=0, \qquad (4)$$

by denoting

$$\lim_{\varepsilon\to 0} F_n(w,\varepsilon)=F_n(w).$$

We will look for solution of the last system in form of

$$F_n(w)=\Phi(w)r_n, \qquad (5)$$

where r_n is the scalar server state probability distribution, and the function $\Phi(w)$ is defined in the following form

$$\Phi(w)=\exp\{jw\kappa_1\}.$$

Then, if we look at the system, we could see that the first equation is a sum of the second and the third in the system (4), and whilst keeping that in mind, let's review the normalization condition for stationary server state probability distribution

$$r_0+r_1+r_2=1,$$

we have the system

$$\begin{cases}\mu_1 r_1=(\lambda+\kappa_1)r_0,\\ \mu_2 r_2=\alpha r_0,\\ r_0+r_1+r_2=1.\end{cases} \qquad (6)$$

Value of the parameter κ_1 will be defined below. By summing equations of the system (3) we will get the following equation

$$F_1(w,\varepsilon)(e^{j\varepsilon w}-1)\lambda+F_2(w,\varepsilon)(e^{j\varepsilon w}-1)\lambda+$$
$$+je^{-j\varepsilon w}(e^{j\varepsilon w}-1)F_0'(w,\varepsilon)=o(\varepsilon), \qquad (7)$$

in which we will execute the limit transition while $\varepsilon\to 0$. Then we could write down this equation

$$F_1(w)\lambda+F_2(w)\lambda-jF_0'(w,\varepsilon)=0.$$

Let's substitute the product (5) in the obtained equation

$$\Phi_0'(w)=j\Phi(w)\frac{[r_1+r_2]\lambda}{r_0},$$

Then

$$\Phi(w)=\exp j\frac{[r_1+r_2]\lambda}{r_0}w.$$

Let's denote

$$\kappa_1=\frac{[r_1+r_2]\lambda}{r_0}. \qquad (8)$$

By taking into consideration the normalization condition for server state probability distribution and solving the system (6) alongside with the system (8) we will get the values of probabilities r_n and the value of parameter κ_1.

$$\begin{cases}r_1=\frac{\lambda}{\mu_1},\\ r_2=\frac{\alpha}{\mu_2}r_0,\\ r_0+\frac{\lambda}{\mu_1}+\frac{\alpha}{\mu_2}r_0=1.\end{cases}$$

will have this system in the end

$$\begin{cases}r_1=\frac{\lambda}{\mu_1},\\ r_2=\frac{\alpha}{\mu_1}\cdot\frac{\mu_1-\lambda}{\mu_2+\alpha},\\ r_0=\frac{\mu_1-\lambda}{\mu_1}\cdot\frac{\mu_2}{\mu_2+\alpha}.\end{cases} \qquad (9)$$

Parameter κ_1 is defined by equality

$$\kappa_1=\frac{[r_1+r_2]\lambda}{r_0}=\frac{\lambda}{\mu_2}\cdot\frac{\lambda\mu_2+\alpha\mu_1}{(\mu_1-\lambda)}.$$

First order asymptotic i.e. the proven theorem, only defines the mean asymptotic value κ_1/σ of a number of calls in an orbit in prelimit situation of nonzero values of σ. For more detailed research of a number $i(t)$ of calls in an orbit let's consider the second order asymptotic.

SECOND ORDER ASYMPTOTIC

Let's substitute the following in the system (2)

$$H_n(u)=\exp\left(j\frac{u}{\sigma}\kappa_1\right)H_n^{(2)}(u),$$

we will get the system

$$-H_0^{(2)}(u)(\lambda+\alpha+\kappa_1)+j\sigma\frac{dH_0^{(2)}(u)}{du}+$$
$$+\mu_1 H_1^{(2)}(u)+\mu_2 H_2^{(2)}(u)=0,$$
$$H_1^{(2)}(u)((e^{ju}-1)\lambda-\mu_1)+H_0^{(2)}(u)\lambda+$$
$$+\kappa_1 e^{-ju}H_0^{(2)}(u)-j\sigma e^{-ju}\frac{dH_0^{(2)}(u)}{du}=0,$$
$$H_2^{(2)}(u)((e^{ju}-1)\lambda-\mu_2)+\alpha H_0^{(2)}(u)=0. \qquad (10)$$

Let's make substitutions as shown below

$$\sigma=\varepsilon^2,\quad u=\varepsilon w,\quad H_n^{(2)}(u)=F_n^{(2)}(w,\varepsilon),$$

then we will get the system

$$-F_0^{(2)}(w,\varepsilon)(\lambda+\alpha+\kappa_1)+j\sigma\frac{\partial F_0^{(2)}(w,\varepsilon)}{\partial w}+$$

$$+\mu_1 F_1^{(2)}(w,\varepsilon)+\mu_2 F_2^{(2)}(w,\varepsilon)=0,$$
$$F_1^{(2)}(w,\varepsilon)\left(\left(e^{jw\varepsilon}-1\right)\lambda-\mu_1\right)+F_0^{(2)}(w,\varepsilon)\lambda+$$
$$+\kappa_1 e^{-jw\varepsilon}F_0^{(2)}(w,\varepsilon)-j\sigma e^{-jw\varepsilon}\frac{\partial F_0^{(2)}(w,\varepsilon)}{\partial w}=0,$$
$$F_2^{(2)}(w,\varepsilon)\left(\left(e^{jw\varepsilon}-1\right)\lambda-\mu_2\right)+\alpha F_0^{(2)}(w,\varepsilon)=0. \quad (11)$$

Theorem 2. (Second order asymptotic) In the context of Theorem 1 the following equation is true
$$\lim_{\sigma\to 0} Me^{jw\sqrt{\sigma}\left(i(t)-\frac{\kappa_1}{\sigma}\right)}=e^{\frac{(jw)^2}{2}\kappa_2},$$
where parameter κ_2 is defined by the following expression
$$\kappa_2=\frac{\lambda^3\mu_2^2+\lambda^3\alpha\mu_2+\alpha\mu_1^2\lambda^2-\lambda^3\alpha\mu_1}{\mu_2^2(\mu_1-\lambda)^2}+\kappa_1.$$

Proof. Let's substitute the following expansion into the system (5)
$$F_n^{(2)}(w,\varepsilon)=\Phi_2(w)\{r_n+j\varepsilon w f_n\}+o\left(\varepsilon^2\right), \quad (12)$$
then we will get
$$-\Phi_2(w)\{r_0+j\varepsilon w f_0\}(\lambda+\alpha+\kappa_1)+$$
$$+j\varepsilon\frac{d\Phi_2(w)}{dw}r_0+\mu_1\Phi_2(w)\{r_1+j\varepsilon w f_1\}+$$
$$+\mu_2\Phi_2(w)\{r_2+j\varepsilon w f_2\}=o\left(\varepsilon^2\right),$$
$$\Phi_2(w)\{r_1+j\varepsilon w f_1\}(j\varepsilon w\lambda-\mu_1)+$$
$$+\Phi_2(w)\{r_0+j\varepsilon w f_0\}(\lambda+\kappa_1(1-j\varepsilon w))-$$
$$-j\varepsilon\frac{d\Phi_2(w)}{dw}r_0=o\left(\varepsilon^2\right),$$
$$\Phi_2(w)\{r_2+j\varepsilon w f_2\}(j\varepsilon w\lambda-\mu_2)+$$
$$+\alpha\Phi_2(w)\{r_0+j\varepsilon w f_0\}=o\left(\varepsilon^2\right).$$
Transforming the last system
$$\Phi_2(w)\{r_0(-\lambda-\alpha-\kappa_1)+\mu_1 r_1+\mu_2 r_2+$$
$$+j\varepsilon w\left[f_0(-\lambda-\alpha-\kappa_1)+\mu_1 f_1+\mu_2 f_2\right]\}+$$
$$+j\varepsilon\frac{d\Phi_2(w)}{dw}r_0=o\left(\varepsilon^2\right),$$
$$\Phi_2(w)\{\mu r_1+r_0(\lambda+\kappa_1)+$$
$$+j\varepsilon w\left[-\mu f_1+r_1\lambda+f_0(\lambda+\kappa_1)-\kappa_1 r_0\right]-$$
$$-j\varepsilon\frac{d\Phi_2(w)}{dw}r_0=o\left(\varepsilon^2\right),$$
$$j\varepsilon w\left[-\mu_2 f_2+r_2\lambda+\alpha f_0\right]=o\left(\varepsilon^2\right).$$
Then
$$j\varepsilon w\Phi_2(w)\left[f_0(-\lambda-\alpha-\kappa_1)+\mu_1 f_1+\mu_2 f_2\right]+$$
$$+j\varepsilon\frac{d\Phi_2(w)}{dw}r_0=o\left(\varepsilon^2\right),$$
$$j\varepsilon w\Phi_2(w)\left[-\mu_1 f_1+r_1\lambda+f_0(\lambda+\kappa_1)-\kappa_1 r_0\right]-$$
$$-j\varepsilon\frac{d\Phi_2(w)}{dw}r_0=o\left(\varepsilon^2\right)$$
$$j\varepsilon w\left[-\mu_2 f_2+r_2\lambda+\alpha f_0\right]=o\left(\varepsilon^2\right).$$
Let's divide equation of the system by ε, we will get
$$\Phi_2(w)\left[f_0(-\lambda-\alpha-\kappa_1)+\mu_1 f_1+\mu_2 f_2\right]+\frac{d\Phi_2(w)}{wdw}r_0=0,$$
$$\Phi_2(w)\left[-\mu_1 f_1+r_1\lambda+f_0(\lambda+\kappa_1)-\kappa_1 r_0\right]-$$
$$-\frac{d\Phi_2(w)}{wdw}r_0=0$$
$$-\mu_2 f_2+r_2\lambda+\alpha f_0=0.$$
Take note that the scalar function $\Phi_2(w)$ is defined in the following form
$$\Phi_2(w)=\exp\left\{\frac{(jw)^2}{2}\kappa_2\right\},$$
then
$$\left[f_0(-\lambda-\alpha-\kappa_1)+\mu_1 f_1+\mu_2 f_2\right]-\kappa_2 r_0=0,$$
$$\left[-\mu_1 f_1+r_1\lambda+f_0(\lambda+\kappa_1)-\kappa_1 r_0\right]+\kappa_2 r_0=0,$$
$$-\mu_2 f_2+r_2\lambda+\alpha f_0=0.$$
We have
$$f_0(-\lambda-\alpha-\kappa_1)+\mu_1 f_1+\mu_2 f_2=\kappa_2 r_0,$$
$$-\mu_1 f_1+f_0(\lambda+\kappa_1)=(\kappa_1-\kappa_2)r_0-r_1\lambda,$$
$$-\mu_2 f_2+\alpha f_0=-r_2\lambda.$$
By summing equations of the system (11) we have
$$\kappa_1 e^{-j\varepsilon w}\left(1-e^{j\varepsilon w}\right)F_0^{(2)}(w,\varepsilon)-$$
$$-j\varepsilon e^{-j\varepsilon w}\left(1-e^{j\varepsilon w}\right)\frac{\partial F_0^{(2)}(w,\varepsilon)}{\partial w}-$$
$$-\left(1-e^{j\varepsilon w}\right)\left[F_1^{(2)}(w,\varepsilon)+F_2^{(2)}(w,\varepsilon)\right]\lambda=o\left(\varepsilon^2\right).$$
Transforming the last system
$$\kappa_1 e^{-j\varepsilon w}F_0^{(2)}(w,\varepsilon)-j\varepsilon e^{-j\varepsilon w}\frac{\partial F_0^{(2)}(w,\varepsilon)}{\partial w}-$$
$$-\left[F_1^{(2)}(w,\varepsilon)+F_2^{(2)}(w,\varepsilon)\right]\lambda=o\left(\varepsilon^2\right).$$
Let's substitute the expansion (12), we will get the following equation
$$\kappa_1(1-j\varepsilon w)\Phi_2(w)\{r_0+j\varepsilon w f_0\}-$$
$$-j\varepsilon(1-j\varepsilon w)\left[\frac{d\Phi_2(w)}{dw}\{r_0+j\varepsilon w f_0\}+\Phi_2(w)j\varepsilon f_0\right]-$$
$$-\Phi_2(w)\left[r_1+j\varepsilon w f_1+r_2+j\varepsilon w f_2\right]\lambda=o\left(\varepsilon^2\right).$$
By applying previously obtained equations we have
$$j\varepsilon w\Phi_2(w)\left[\kappa_1(f_0-r_0)-(f_1+f_2)\lambda\right]-$$
$$-j\varepsilon\frac{d\Phi_2(w)}{dw}r_0=o\left(\varepsilon^2\right).$$
Consider $\varepsilon\to 0$
$$\frac{d\Phi_2(w)}{dw}r_0=w\Phi_2(w)\left[\kappa_1(f_0-r_0)-(f_1+f_2)\lambda\right],$$
then
$$\frac{d\Phi_2(w)}{\Phi_2(w)}=\frac{\left[\kappa_1(f_0-r_0)-(f_1+f_2)\lambda\right]}{r_0}wdw.$$
Let's denote
$$\frac{\kappa_1 f_0-(f_1+f_2)\lambda}{r_0}-\kappa_1=$$
$$=-\left(\frac{(f_1+f_2)\lambda-\kappa_1 f_0}{r_0}+\kappa_1\right)=j^2\kappa_2,$$
where

$$\kappa_2 = \frac{(f_1 + f_2)\lambda - \kappa_1 f_0}{r_0} + \kappa_1 .$$

Then, considering that $\Phi_2(0) = 1$ we have

$$\Phi_2(w) = \exp\left\{ \frac{(jw)^2}{2} \kappa_2 \right\}.$$

Let's find κ_2, by expressing

$$f_1 = f_0 \frac{(\lambda + \kappa_1)}{\mu_1} - \frac{(\kappa_1 - \kappa_2) r_0}{\mu_1} + r_1 \frac{\lambda}{\mu_1} ,$$

$$f_2 = \frac{\alpha}{\mu_2} f_0 + r_2 \frac{\lambda}{\mu_2} .$$

Then

$$(\kappa_2 - \kappa_1) r_0 = (f_1 + f_2)\lambda - \kappa_1 f_0 =$$

$$= f_0 \left[\frac{(\lambda + \kappa_1)\lambda}{\mu_1} + \frac{\alpha\lambda}{\mu_2} - \kappa_1 \right] -$$

$$- \frac{(\kappa_1 - \kappa_2) r_0 \lambda}{\mu_1} + r_1 \frac{\lambda^2}{\mu_1} + r_2 \frac{\lambda^2}{\mu_2} .$$

Let's consider this expression separately

$$\frac{(\lambda + \kappa_1)\lambda}{\mu_1} + \frac{\alpha\lambda}{\mu_2} - \kappa_1 =$$

$$= \frac{\lambda(\lambda\mu_2 + \alpha\mu_1) - \kappa_1 \mu_2 (\mu_1 - \lambda)}{\mu_1 \mu_2} =$$

$$= \frac{\lambda(\lambda\mu_2 + \alpha\mu_1) - \lambda(\lambda\mu_2 + \alpha\mu_1)}{\mu_1 \mu_2} = 0 .$$

Then

$$(\kappa_2 - \kappa_1) r_0 \frac{\mu_1 - \lambda}{\mu_1} = r_1 \frac{\lambda^2}{\mu_1} + r_2 \frac{\lambda^2}{\mu_2} ,$$

and

$$\kappa_2 = \frac{\mu_1}{\mu_1 - \lambda} \left[\frac{r_1}{r_0} \frac{\lambda^2}{\mu_1} + \frac{r_2}{r_0} \frac{\lambda^2}{\mu_2} \right] + \kappa_1 =$$

$$= \frac{\lambda^3 \mu_2^2 + \lambda^3 \alpha \mu_2 + \alpha \mu_1^2 \lambda^2 - \lambda^3 \alpha \mu_1}{\mu_2^2 (\mu_1 - \lambda)^2} + \kappa_1 .$$

We have found that the parameter κ_2 equals

$$\kappa_2 = \frac{\lambda^3 \mu_2^2 + \lambda^3 \alpha \mu_2 + \alpha \mu_1^2 \lambda^2 - \lambda^3 \alpha \mu_1}{\mu_2^2 (\mu_1 - \lambda)^2} + \kappa_1 . \quad (13)$$

Second order asymptotic i.e. the proven theorem 2, shows that the asymptotic probability distribution of a number $i(t)$ of calls in an orbit is Gaussian with mean asymptotic κ_1/σ and dispersion κ_2/σ. Then, with the following prelimit distribution in mind

$$P(i) = P_0(i) + P_1(i) + P_2(i),\ i \geq 0 , \quad (14)$$

we could build an approximation for said distribution and in particular the $P^{(2)}(i)$ approximation

$$P^{(2)}(i) = (L(i + 0.5) - L(i - 0.5))(1 - L(-0.5))^{-1}, \quad (15)$$

where $L(x)$ is the normal distribution function with parameters κ_1/σ and κ_2/σ.

Gaussian approximation (15), as will be shown below, is fairly applicable at low values $\sigma < 0{,}05$ and gives relative error at $\sigma > 0{,}05$. Moreover, prelimit distribution (14) is asymmetrical whilst the Gaussian approximation (15) is built upon the basis of symmetrical normal distribution.

NUMERICAL ALGORITM FOR SOLVING SYSTEM (1)

Let's write down system (1) at $i = 0$, $i = 1$ and $i \geq 2$, then we will have three systems

$$-(\lambda + \alpha) P_0(0) + \mu_1 P_1(0) + \mu_2 P_2(0) = 0,$$
$$-(\lambda + \mu_1) P_1(0) + \lambda P_0(0) + \sigma P_0(1) = 0 ,$$
$$-(\lambda + \mu_2) P_2(0) + P_0(0)\alpha = 0 . \quad (16)$$

$$-(\lambda + \sigma + \alpha) P_0(1) + \mu_1 P_1(1) + \mu_2 P_2(1) = 0,$$
$$-(\lambda + \mu_1) P_1(1) + \lambda [P_1(0) + P_0(1)] +$$
$$+ 2\sigma P_0(2) = 0 ,$$
$$-(\lambda + \mu_2) P_2(1) + P_0(1)\alpha + P_2(0)\lambda = 0 . \quad (17)$$

$$-(\lambda + i\sigma + \alpha) P_0(i) + \mu_1 P_1(i) + \mu_2 P_2(i) = 0,$$
$$-(\lambda + \mu_1) P_1(i) + \lambda [P_1(i-1) + P_0(i)] +$$
$$+ (i+1)\sigma P_0(i+1) = 0 ,$$
$$-(\lambda + \mu_2) P_2(i) + P_0(i)\alpha + P_2(i-1)\lambda = 0, i \geq 2 . \quad (18)$$

Let's consider $P_0(0) = 1$. Using the third and the first equations of the system (16) we could write down

$$P_2(0) = \frac{\alpha}{\lambda + \mu_2}, \quad P_1(1) = \frac{1}{\mu_1} \{(\lambda + \alpha) P_0(0) - \mu_2 P_2(0)\} .$$

Using the second equation of the system (16) we could write down

$$P_0(1) = \frac{1}{\sigma} \{(\lambda + \mu_1) P_1(0) - \lambda P_0(0)\} .$$

Using the third and the first equations of the system (17) we could write down

$$P_2(1) = \frac{1}{\lambda + \mu_2} \{\alpha P_0(1) + \lambda P_2(0)\} ,$$

$$P_1(1) = \frac{1}{\mu_1} \{(\lambda + \alpha + \sigma) P_0(1) - \mu_2 P_2(1)\} .$$

Further at $2 \leq i \leq N$ the recurrent procedure is implemented by the following equations

$$P_0(i) = \frac{1}{i\sigma} \{(\lambda + \mu_1) P_1(i-1) - \lambda P_0(i-2) - \lambda P_0(i-1)\} ,$$

$$P_2(i) = \frac{1}{\lambda + \mu_2} \{\alpha P_0(i) + \lambda P_2(i-1)\} ,$$

$$P_1(i) = \frac{1}{\mu_1} \{(\lambda + \alpha + i\sigma) P_0(i) - \mu_2 P_2(i)\} .$$

By normalizing the obtained results we have found the solution $P_n(i)$ of system (1) for all $0 \leq i \leq N$ .Suggested numerical algorithm is fairy effective as it allows finding the solution $P_n(i)$ for large values (up to thousands) of N.

APPROXIMATION ACCURACY $P^{(2)}(i)$ AND ITS APPLICATION AREA

Approximation accuracy $P^{(2)}(i)$ will be defined by using Kolmogorov equation

$$\Delta = \max_{0 \le i \le N} \left| \sum_{v=0}^{i} \left(P(v) - P^{(2)}(v) \right) \right|$$

For range between distributions $P(i)$ and $P^{(2)}(i)$, where distribution $P(i)$ is defined by using numerical algorithm and the approximation $P^{(2)}(i)$ is built upon the basis of the second asymptotic and the obtained Gaussian distribution. Tables 1-5 contain values for this range Δ for various values of rate λ and σ.We consider $\mu_1 = 1$ and $\mu_2 = 2$ for all Tables. Let's consider α = 1.

Table 1: Kolmogorov range

	λ = 0,5	λ = 0,6	λ = 0,7
σ = 1	0,092	0,108	0,123
σ = 0,5	0,066	0,079	0,092
σ = 0,1	0,064	0,039	0,045
σ =0,05	0,026	0,028	0,032

Table 2: Kolmogorov range

	λ = 0,8	λ = 0,9	λ = 0,95
σ = 1	0,116	0,163	0,174
σ = 0,5	0,106	0,123	0,131
σ = 0,1	0,052	0,060	0,064
σ =0,05	0,037	0,042	0,045

Analysis of values of Gaussian approximation tabulated in tables 1-2 lets us make the following conclusions.The approximation accuracy naturally increases with the deterioration of parameter σ value. With increasing values of rate λ (intensity of the incoming flow) the Gaussian approximation accuracy decreases. Let's say that the approximation error is allowed if the Kolmogorov range Δ < 0,05 and the second order approximation is allowed for fairly small values of σ parameter, to be precise σ < 0,05. Consider α = 10.

Table 3: Kolmogorov range

	λ = 0,5	λ = 0,6	λ = 0,7
σ = 1	0,053	0,063	0,072
σ = 0,5	0,039	0,046	0,053
σ = 0,1	0,018	0,021	0,024
σ =0,05	0,013	0,014	0,017

Table 4: Kolmogorov range

	λ = 0,8	λ = 0,9	λ = 0,95
σ = 1	0,083	0,094	0,100
σ = 0,5	0,060	0,068	0,072
σ = 0,1	0,027	0,030	0,032
σ =0,05	0,019	0,021	0,023

Considering λ = 0,8, $\mu_1 = 1$, $\mu_2 = 2$, by changing values of parameters α and σ and by numerically solving the probability distribution system (1), we could find Kolmogorov range between Gaussian approximation of probability distribution of a number of calls in an orbit and the numerical distribution.

Table 5: Kolmogorov range

	σ = 0,2	σ = 0,1	σ =0,03	σ =0,01
α = 1	0,072	0,052	0,028	0,016
α = 3	0,058	0,041	0,022	0,013
α = 5	0,049	0,035	0,019	0,011

Analysis of values tabulated in tables 3-5 shows that the accuracy of Gaussian approximation greatly increases while increasing α, and therefore the area of applicability increases too. The area of applicability doubles in size and is applicable at $\sigma \le 0,05$. Density diagrams of probability distributions and distribution function diagrams of a number of calls in an orbit are shown in figures 2-4. The dotted line represents designated density of asymptotical distribution probabilities.

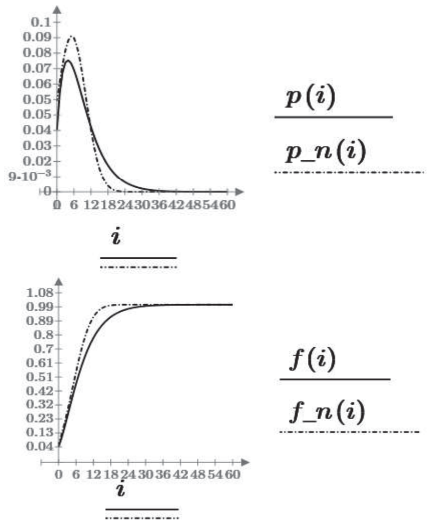

Figure 2: λ = 0,8 σ = 1 α = 1 $\mu_1 = 1$ $\mu_2 = 2$ Δ = 0,116

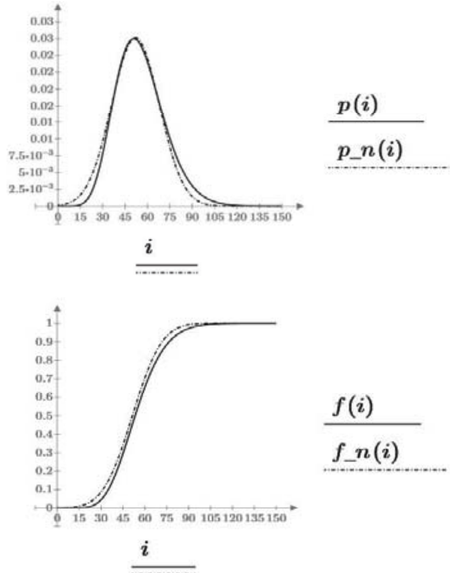

Figure 3: λ = 0,8 σ = 0,1 α = 1 $\mu_1 = 1$ $\mu_2 = 2$ Δ = 0,052

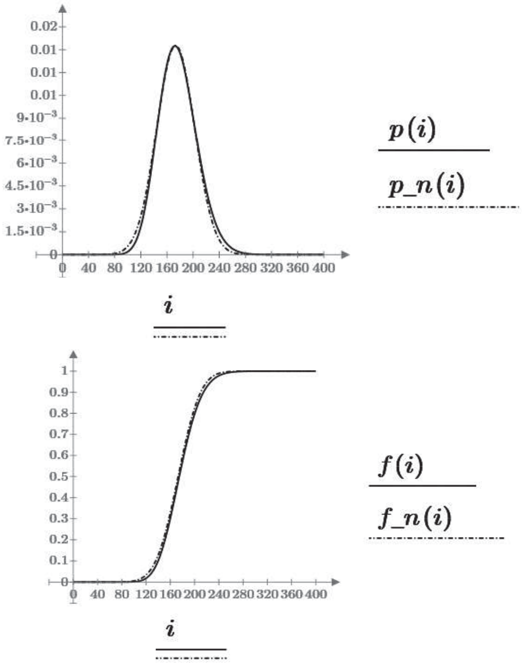

Figure 4: $\lambda = 0{,}8$ $\sigma = 0{,}03$ $\alpha = 1$ $\mu_1 = 1$ $\mu_2 = 2$ $\Delta = 0{,}016$

CONCLUSIONS

In this paper we have considered retrial queue with two-way communication. To research the system in question we have found first and second order asymptotics of a number of calls in an orbit in asymptotic condition of a low rate of retrials. Based on the obtained asymptotics we have built the Gaussian approximation of a probability distribution of a number of calls in an orbit. We have defined the conditions of applicability of the obtained approximation depending on values of system-defining parameters. As criteria we have chosen the Kolmogorov range assuming that the allowed approximation error is less than 0,05. By analyzing the obtained results we can make the conclusion that the accuracy of Gaussian approximation increases while decreasing values of σ parameter, increasing values of λ parameter and/or increasing values of α parameter

The results obtained in this paper are planned to be generalized for the case of correlated incoming flow and random time of serving in retrial queues with two-way communication.

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