Modeling and simulation of reliability function of a homogeneous hot double redundant repairable system

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KEYWORDS

System reliability function, mathematical modeling and simulation, redundant systems, markovization method

ABSTRACT

Calculation of a system reliability function is one of the principal problems in reliability theory. It is well known that even for the simplest double redundant repairable system this function has not been analytically calculated yet in case when elements reliability and recovery functions have general distributions.

In current paper we study the reliability function and the mean time to failure of a homogeneous hot standby repairable system with exponential distribution of its elements life time and general distribution of their repair time. With the help of the developed general discrete-event simulation model we extend the previous studies to a general case of a homogeneous hot double redundant repairable system with general distributions of both life and repair time lengths of elements.

It is shown that the results of exact analytical calculation and simulation results have close agreement.

INTRODUCTION AND MOTIVATION

Calculation of a system reliability function is one of the principal problems in reliability theory. It is well known that even for the simplest double redundant repairable system this function has not been analytically calculated yet in case when elements' reliability and recovery functions have general distributions.

First this problem was discussed in the book by Gnedenko, Belyaev, Solov'ev [1] for the case of homogeneous warm double redundant repairable system with the help of renewal theory methods. The same problem for the system with heterogeneous elements by the same methods was discussed also in [2]. In paper [3] multi-dimensional alternative processes have been used for studying the complex reliability systems. A system has been considered in case of no limitations on the number of repair facilities. When there is enough repair units, the components of the process describing the system behavior become independent allowing to calculate the system characteristics. However, in the case of limited number of repair facilities the problem of calculation of reliability function has not been solved yet, and it is considered in the current paper for the simplest case of a homogeneous hot standby system with only one repair unit.

On the other hand in series of works by B.V. Gnedenko, A.D. Solov'ev [4], [5], [6] and others it was shown that in case of "quick" restoration the system life time distribution becomes asymptotically insensitive to the shapes of its elements' life and repair time distributions and in scale of the system mean life time it tends to the exponential one.

In papers [7], [8], [9] a homogeneous cold standby double redundant system has been considered in the case when one of the input distributions (either of life or repair time lengths) is exponential. For these models explicit expressions for stationary probabilities have been obtained which show their evident dependence on the non-exponential distributions in the form of their Laplace-Stiltjes transforms. At that, the numerical investigations, performed by V.Rykov and D.Kozyrev and presented at the Eighth International Workshop on Simulation (Vienna, September 21st – 25th, 2015), show that this dependence becomes vanishingly small under "quick" restoration. However the problem of the reliability function calculation was not considered.

In current paper we study the reliability function and the mean time to failure of a homogeneous hot standby repairable system with exponential distribution of its elements' life time and general distribution of their repair time. With the help of the developed general discrete-event simulation model we extend the previous studies to a general case of a homogeneous hot double redundant repairable system with general distributions of both life and repair time lengths of elements.

PROBLEM SETTING AND NOTATIONS

Consider a hot standby repairable system with one repair unit. The elements of the system (units) have exponentially distributed times to failure with parameter α and general repair time distribution $B(x)$. Through-

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out the paper we will use a generalization of Kendall's notation [11] for queuing systems. In this notation the symbols $\langle GI_n|GI|m\rangle$ stand for a closed system, i.e. a system where the flow of customers is generated by a finite number n of sources that is shown by index in the first position. Symbol GI means "General Independent" and in the first position of this notation it denotes the general distribution of independent life times of the elements of the system and in the second one — the general distribution of their independent repair times. These symbols can be substituted by M for exponential ($exp(\cdot)$), Erlang $E(\cdot,\cdot)$, Gnedenko-Weibull ($GW(\cdot,\cdot)$) with appropriate parameters or any other symbol describing the distribution of life and/or repair time. Finally, the last factor m denotes the number of repair units in the system. In the current paper we consider a simple hot double redundant model, namely $\langle M_2|GI|1\rangle$ and study its reliability function (RF) under different distributions.

The cumulative distribution functions (CDF) of the random life time A and random repair time B are denoted respectively by $A(x)$ and $B(x)$. We suppose the existence of the corresponding probability density functions (PDF), which are denoted by $a(x) = A'(x)$ and $b(x) = B'(x)$. The mean time between failures, the mean service (repair) time, the failure and repair hazard functions are denoted as follows:

$$a = \int_0^\infty (1 - A(x))dx, \text{ and } b = \int_0^\infty (1 - B(x))dx.$$

and

$$\alpha(x) = \frac{a(x)}{1 - A(x)}, \text{ and } \beta(x) = \frac{b(x)}{1 - B(x)}.$$

Define also the moment-generating functions (m.g.f.) of life A and repair B times, the Laplace-Stiltjes transforms (LST) of their distributions by the following expressions:

$$\tilde{a}(s) = \int_0^\infty e^{-sx}a(x)dx$$

and

$$\tilde{b}(s) = \int_0^\infty e^{-sx}b(x)dx, \; Re[s] \geq 0.$$

Life and repair times are assumed independent. The "up" (working) states of each unit and the system will be marked by 0 and the "down" (failed) states – by 1. Under considered assumptions the system behavior can be described by a random process which takes values from the system state space $E = \{0, 1, 2\}$. Denote by $J(t)$ the random process, describing the system behavior: $J(t) = i$ if the system is in state i. At that the system state subset $E_0 = \{0, 1\}$ represents its working (up) states, and the subset $E_1 = \{2\}$ represents the system failure (down) state. The process $J = \{J(t), t \geq 0\}$ represents the system states, and takes the values from E_0 if at least one of the system components is in up state and takes the value 1 otherwise. The corresponding system state probabilities are denoted by $\pi_0(t)$, $\pi_1(t;x)$, $\pi_2(t)$.

We are interested in studying the system lifetime T, which is the duration of time when at least one unit is working. Thus the system lifetime T can be represented as follows: $T = \inf\{t : J(t) = 2\}$ and the system reliability function as

$$R(t) = \mathbf{P}\{T \leq t\} = \mathbf{P}\{J(\tau) \in E_0, \; \tau \in (0,t)\} \quad (1)$$

ANALYTICAL RESULTS

Using a standart method of comparing the state probabilities at time instants t and $t + \Delta$, we develop the following system of Kolmogorov forward partial differential equations for these probabilities:

$$\frac{d}{dt}\pi_0(t) = -2\alpha\pi_0(t) + \int_0^t \pi_1(t,u)\beta(u)du,$$
$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\pi_1(t;x) = -(\alpha + \beta_1(x))\pi_1(t;x),$$
$$\frac{d}{dt}\pi_2(t) = \alpha \int_0^t \pi_1(t;u)du.$$

with boundary and initial conditions

$$\pi_1(t,0) = 2\alpha\pi_0(t), \quad \pi_0(0) = 1.$$

Theorem. Laplace transform of the reliability function for the considered system has the following form:

$$\tilde{R}(s) = \frac{1}{s} - \tilde{\pi}_2(s) = \frac{s + \alpha + 2\alpha(1 - \tilde{b}(s + \alpha))}{(s + \alpha)[s + 2\alpha(1 - \tilde{b}(s + \alpha))]} \quad (2)$$

which coincides with the results obtained before in [7], [9].

Corollary. The mean time to failure of the system under consideration equals

$$m = \mathbf{E}[T] = \tilde{R}(0) = \frac{1 + 2(1 - \tilde{b}(\alpha))^2}{2\alpha(1 - \tilde{b}(\alpha))} \quad (3)$$

SIMULATION RESULTS

In this section we present the results of simulation of a homogeneous two-unit cold standby repairable system $\langle GI_2|GI|1\rangle$ with one repair unit and general distributions of both life and repair times of its elements.

A. *General simulation model*

We perform the simulation using the discrete event modeling method. We consider the functioning of the system being modeled as a sequence of operations being performed across entities (events). The simulation model is specified graphically as a process flowchart (see Fig. 1).

In order to ensure the precise understanding and reproducibility of the simulation model, we present an algorithm for a simulation process which is represented in the form of pseudocode (see Algorithm 1).

B. *Comparison of analytical solution and simulation results for $\langle GI_2|GI|1\rangle$*

In this section we perform the comparison of analytical (where possible) and simulation results for the reliability of the considered system. As a model parameter

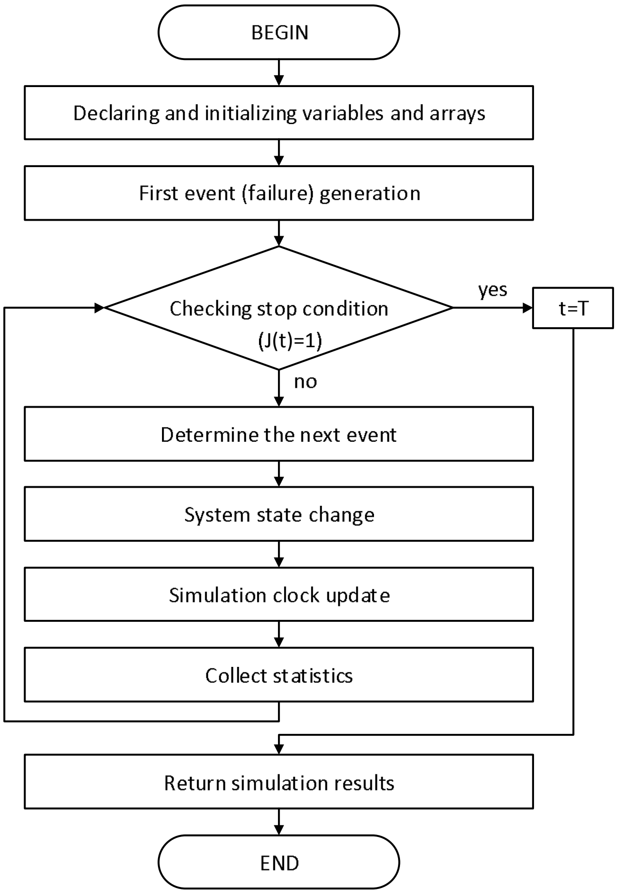

Fig. 1. Flowchart of the discrete-event simulation model

we consider the value $\rho = \frac{\mathbf{E}[A]}{\mathbf{E}[B]}$, which can be interpreted as a relative rate of system recovery [12]. It will be shown that as $\rho \to \infty$ the sensitivity of the model to shapes of input distributions becomes negligible. Distributions that we've used in our experiments include, but are not limited to the following ones: Exponential (Exp(α)), Erlang (E(k, α)), Gnedenko-Weibull (GW) and Pareto (P). The simulation time has been chosen equal to $T = 10000$ and number of replications equal to 100 and 1000 (for more precise simulation results).

For illustrative purposes we conduct this comparison graphically at Figure 2 and Figure 3 where results of simulation are represented for different distributions $GI^{(1)}$, $GI^{(2)}$ and $\rho = \frac{5}{5} = 1$. In all cases the parameters of distributions have been chosen so that the value of $\mathbf{E}[B]$ remained fixed ($\mathbf{E}[B] = 5$) and the mean time to failure of an element would ascend $\mathbf{E}[A] = \rho\mathbf{E}[B]$ according to the values of ρ. Instead of parameters of distributions the coefficient of variation c (the ratio of the standard deviation to the mean) is indicated in parentheses in the legends of all figures.

Figure 4 depicts the curves of the system reliability function $R(t)$ under rare failures of system elements. As it can be seen from the figure, the differences between both simulation and analytical curves become indistinguishable very quickly. At a relatively small value of $\rho = 10$ the reliability function curves for are already very close to each other for all considered special

Input: a, b, T, '$GI^{(1)}$', '$GI^{(2)}$'
a – mean life time of an element, b – mean repair time, T – maximum simulation time, '$GI^{(1,2)}$' denote CDFs of life and repair time lengths, respectively.
Output: empirical reliability function R_{empir}, system mean time to failure ET
begin
double $t := 0.0$;
int $i := 0$; $j := 0$;
double t_nextfail$:= 0.0$;
double t_nextrepair$:= 0.0$;
int $k := 1$;
array $r[\,] := [0, 0, 0]$;
$s := df\_Exp(\frac{1}{a})$;
$t\_nextfail := t + s$;
while $t < T$ **do**
if $i = 0$ **then**
$t\_nextrepair := \infty$;
$j := j + 1$; $t := t\_nextfail$;
else if $i = 1$ **then**
$s_1 := df\_GI1(..)$;
$s_2 := df\_GI2(..)$ $t\_nextfail := t + s_1$;
$t\_nextrepair := t + s_2$;
if $t\_nextfail < t\_nextrepair$ **then**
$j := j + 1$; $t := t\_nextfail$;
else
$j := j - 1$; $t := t\_nextrepair$
end
end
else
$i = 2$; $t\_nextfail := \infty$;
$j := j - 1$; $t := t\_nextrepair$;
end
if $t > T$ **then**
$t := T$
end
$r[k] := [t, i, j]$;
$i := j$; $k := k + 1$;
end
Evaluate duration of overall time spent in working states;
Calculate estimates of the reliability function and the system mean life time.
end

Algorithm 1: Pseudocode for the simulation process of $\langle GI_2|GI|1\rangle$ model

cases of the $\langle GI_2|GI|1\rangle$ model. The observed behavior is fairly expected, as it was proved by B.V. Gnedenko and A.D. Solov'ev [5],[6]. What is more important and interesting — is that we can assess the rate of this convergence and it will be done in the full-text version of the current paper with the means of quantiles for the given reliability level.

CONCLUSION

The problem of analytical calculation and simulation assessment of reliability function for a homogeneous hot

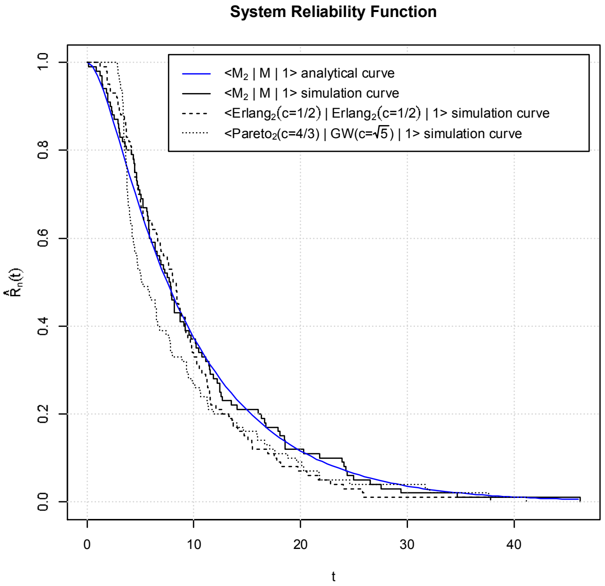

Fig. 2. Empirical and analytical values of $R(t)$ versus time t (values based on 100 replications), $\rho = 1$

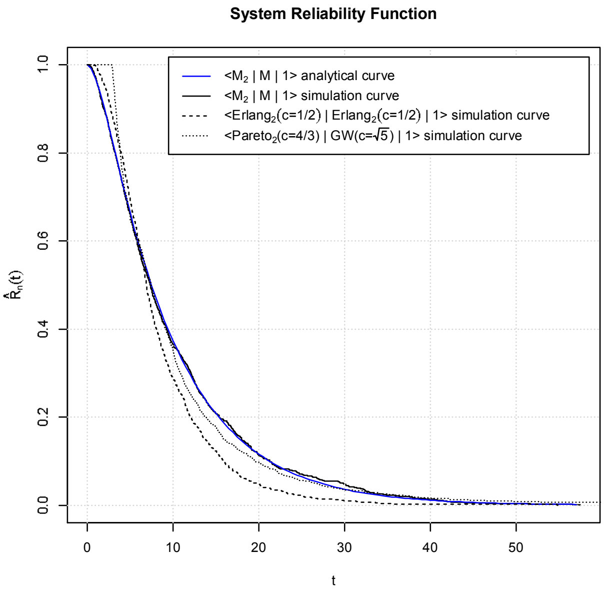

Fig. 3. Empirical and analytical values of $R(t)$ versus time t (values based on 1000 replications), $\rho = 1$

double redundant repairable system has been considered. Explicit representation of this function in terms of Laplace transform has been found. It was shown that under rare failures of system elements the reliability function is approximated with exponential distribution with appropriate mean life time. Also analysis of the obtained results shows that the results of exact analytical calculation (where possible) and simulation results have close agreement.

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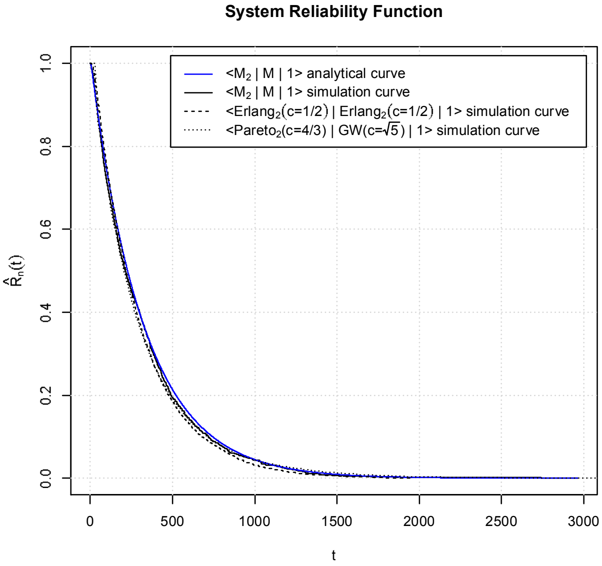

Fig. 4. Empirical and analytical values of $R(t)$ versus time t (values based on 1000 replications) under rare failures of system elements, $\rho = 10$

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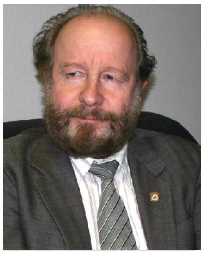

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