NONSTATIONARY STOCHASTIC MOTION MODELING BY DYNAMICAL SYSTEMS

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ABSTRACT

In this paper we discuss two aspects of kinetic approach for time series modeling in terms of dynamical system. One method is based on the interpretation of kinetic equation for empirical distribution function density as a reduced description of statistical mechanics for appropriate dynamical system. For example, if distribution function density is satisfied to Liouville equation with some velocity, then this velocity can be treated as an average velocity of particle in phase space. The second method is based on the so-called Chernoff theorem from the group theory. According to the consequence from this theorem some iteration procedure exists for construction of group or semigroup, which is equivalent in some sense to average shift generator over the trajectory of appropriate dynamical system. Connection between these two methods enables us to construct a strict approach to nonstationary time series modeling with non-parametric estimation of statistical properties of corresponding sample distribution function. Also the notion of Chernoff-equivalent semigroup can be used for the calculation optimization procedure.

INTRODUCTION

In the traditional approach to nonstationary time series analysis (see e.g. Harris 1995) the cointegration method is applied for construction some linear combination of finite differences of time series, so that this combination is a process with stationary distribution function. Besides that there are several heuristic methods such as Winters model (Winters 1960) and others econometric models (Johnston and Dinardo, 1997), using for forecasting of nonstationary behavior of time series.

In this paper we construct a new theoretical scheme for statistical analyzing and numerical modeling of nonstationary time series. As far as we know the evolution equations for the sample distribution function (SDF) were not derived in the frame of time series analysis.

The main problem is to predict the SDF at some horizon τ with the use of a given sample data $\{x_1,...,x_n\}$. In practice this problem arises under investigation of the evolution of statistical properties of any complex system with many degrees of freedom. For example, in the works (Orlov et al. 2017; Ivchenko et al. 2017) the quality metrics of wireless connection were modeling for the case of non-stationary random walk of subscribers. For this purpose the method of kinetic evolution equation for SDF was proposed. In this method there is a problem of mathematical correctness of construction of nonstationary empirical statistics, such as drift velocity of the sample distribution function density (SDFD). If we have a sample of data $\{x_1,...,x_n\}$, where n is sufficiently large number, we can construct SDF $F_n(x)$ as an empirical probability, corresponding to random variable with the values x_k. If we suppose further, that this function approximates an objective general distribution function $F(x)$ with density $f(x)$, then numerical difference $\Delta F_n(x) = F_n(x+\Delta x) - F_n(x)$ can be approximately interpreted in terms of SDFD $f_n(x)$, so that $\Delta F_n(x) = f_n(x)\Delta x + o(\Delta x)$. But if the analizing data $\{x_1,...,x_n\}$ correspond to nonstationary random process, then we need to introduce a time dependence of SDF $F_n(x,t)$ and SDFD $f_n(x,t)$, where time t is measured in terms of numbers k of values x_k. So it is naturally to construct the evolution equation for SDFD $f_n(x,t)$ as an approximation procedure for some differentiating function of two variables $f(x,t)$. We suggest a following scheme.

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Let $\Phi_n(x,v,t)$ is a joint SDFD of values x and differences $v = \Delta x$ with unit time step, so that the value $v_k = x_{k+1} - x_k$ numerically approximates velocity dx/dt. According to construction of joint probability density we have

$$f_n(x,t) = \int \Phi_n(x,v,t)dv, \qquad (1)$$

where the integral in (1) represents in fact the summation over the differences v_k. As it is known from traditional statistical mechanics, if we have a dynamical system $v(x,t) = dx/dt$ and phase coordinates x, v are independent, the joint DFD $\Phi(x,v,t)$ in phase space obeys to Liouville equation, which express the phase space volume conservation law:

$$\frac{\partial \Phi(x,v,t)}{\partial t} + v\frac{\partial \Phi(x,v,t)}{\partial x} = 0. \qquad (2)$$

If we assume, that the boundary conditions for $\Phi(x,v,t)$ correspond to zero density, we obtaine from (2):

$$\frac{\partial f(x,t)}{\partial t} = \int \frac{\partial \Phi(x,v,t)}{\partial t}dv = -\int v\frac{\partial \Phi(x,v,t)}{\partial x}dv = \\ = -\frac{\partial}{\partial x}\int v\Phi(x,v,t)dv = -\frac{\partial\big(u(x,t)f(x,t)\big)}{\partial x}. \qquad (3)$$

Here the value $u(x,t)$ is introduced as an average velocity in phase space:

$$u(x,t)f(x,t) = \int v\Phi(x,v,t)dv \qquad (4)$$

Following to this logic, we can consider the appropriate phase space, associated with two time series samples $\{x_1,...,x_n\}$ and $\{v_1,...,v_{n-1}\}$, and construct for *n*-1 joint combinations $\{x_k, v_k\}$ the corresponding SDFD and empirical velocity $u_n(x,t)$, so that in the sense of numerical procedure we obtain Liouville equation

$$\frac{\partial f_n(x,t)}{\partial t} = -\frac{\partial\big(u_n(x,t)f_n(x,t)\big)}{\partial x} \qquad (5)$$

for the case, when the process $x(t)$ has independent differences.
If we know the empirical velocity $u_n(x,t)$, we can solve the equation (5) under given initial conditions and analyze the statistical properties of ensemble of forecasting trajectories of this process. If we need to construct average trajectory, the velocity $u_n(x,t)$ is taken to be a continuous analog of the corresponding discrete dynamical system, based on the sample of size *n* and defined by the equation

$$x_{t+1} = x_t + u_n(x_t, t). \qquad (6)$$

The main problem is that the empirical velocity $u_n(x,t)$ is unknown for time moments $t > n-1$ and $f_n(x,t)$ can not be forecasted without additional suppositions. For example, if we put, that estimated phase velocity $u_n(x,t)$ dose not depend on time, the corresponding value $u_n(x)$ might be used for forecasting over some horizon τ from $t = n+1$ to $t = n+\tau$. Initial SDFD is given for the time moment $t_0 = n$. But in fact it is a very rough approximation.
One approach to solve this problem is to derive evolution equation for velocity itself, using the equations (2), (4) and (5). We have in one-dimensional case the following sequence of equalities, which leads to infinite chain of moment equations:

$$\frac{\partial\big(u_n(x,t)f_n(x,t)\big)}{\partial t} = \frac{\partial}{\partial t}\int v\Phi_n(x,v,t)dv = \\ = \int v\frac{\partial}{\partial t}\Phi_n(x,v,t)dv = -\int v^2\frac{\partial \Phi_n(x,v,t)}{\partial x}dv. \qquad (7)$$

If we introduce the average square of velocity $e_n(x,t)$, so that

$$e_n(x,t)f_n(x,t) = \int v^2\Phi_n(x,v,t)dv, \qquad (8)$$

we obtaine from (6), that

$$\frac{\partial\big(u_n(x,t)f_n(x,t)\big)}{\partial t} = -\frac{\partial\big(e_n(x,t)f_n(x,t)\big)}{\partial x}. \qquad (9)$$

But the problem is not solved yet: we need now to suppose the definite dependence on time for the value $e_n(x,t)$ and so on. In practice we can stop, if $e_n(x,t)$ or moment of higher order is sufficiently small. But it is not very conveniently, because higher order moments are usually increased with the order number.
Another approach is to calculate some velocity $\tilde{u}_n(x,t)$ for several samples as an average value over the samples. If we have several (s) samples $\{x_1,...,x_n\}$, $\{x_{n+1},...,x_{2n}\}$ and so on, we can construct the value

$$\tilde{u}_n(x,t) = p_1u_n(x,n) + p_2u_n(x,2n) + ... + \\ + p_su_n(x,sn). \qquad (10)$$

Here $sn = t$ and sum of positive weight coefficients p_k is equal to unit:

$$\sum_{k=1}^{s} p_k = 1, \quad p_k > 0. \qquad (11)$$

Unfortunately, the value of $\widetilde{u}_n(x,t)$ has no mechanical sense as appropriate velocity in phase space, because in general case the average value of group (or semigroup) is not a group. In this paper below we present an iteration mathematical procedure, which enables us to use the formulas (10-11) in correct mathematical sense.

TIME SERIES GENERATION METHOD

Let us suppose, that we solve the kinetic equation (5) for SDFD. Then we have this function in a discrete moments of time $f_n(x,t+1),...,f_n(x,t+\tau)$ as a set of real non-negative values in definite class intervals. Thereafter, for each time $k=1,2,...,\tau$ a random value $\widetilde{x}_k$ is generated from corresponding distribution function, which is represented as

$$F_n(x,t)=\int_0^x f_n(y,t)dy\ . \qquad (12)$$

In practice we deal with limited values, so that it is convenient to normalize the sample on the unite interval. In particular, if the solution of equation (5) is represented as a histogram $f_n(j,t)$, where j is a number of class interval which divided the area of integration, than the continuous strictly monotonic DF has the form

$$\begin{gathered} F_n(x,t)=(Jx-j)\cdot f_n(j+1,t)+\sum_{k=1}^{j} f_n(k,t), \\ x\in[(j-1)/J;\, j/J], \quad j=1\div J\ . \end{gathered} \qquad (13)$$

The computational step on the variable x is equal to $1/J$, where J is a number of class intervals in (13). So if we know the value of drift velocity $\widetilde{u}_n(j,t)$ for each class interval, we can construct the SDFD in the form:

$$\begin{gathered} f_n(j,t+1)=f_n(j,t)-J\widetilde{u}_n(j+1,t)f_n(j+1,t)+ \\ +J\widetilde{u}_n(j,t)f_n(j,t). \end{gathered} \qquad (14)$$

The trajectory of random process $x(t)$ can be generated as follows. At first we generate stationary series of numbers $\{\xi_k\}$ of length τ, which is uniformly distributed at $[0;1]$. The corresponding series with distribution $F_n(x,t)$ from equations (12-14) is based on the inverse function to the distribution function, that moving in a sliding window of length τ:

$$\xi_k=F_n(\widetilde{x}_k,k)\,. \qquad (15)$$

By generating a set of N uniformly distributed samples, denoted by the superscript $i=1,2,...,N$, we obtain the corresponding set of trajectories that can be considered as an ensemble of solutions of the kinetic equation (5): $\{\widetilde{x}_1^i,...,\widetilde{x}_\tau^i\}$. Each of the trajectories has SDFD, which depends on time according to Liouville equation (5).

The principal distinction from traditional generation method is that SDF in the formula (15) explicitly depends on time through the variable *k*.

To examine the accuracy of this method we need to estimate the non-stationafy level of SDF. This level is introduced following by (Orlov, 2014). It is called as self-consistent significance level (SCSL). This value represents a stationary point of significance level for distribution function of the values of non-parametric Kolmogorov-Smirnov criterion (Kolmogoroff, 1933; Durbin, 1972). This distribution is a distribution of the distances between sample distribution functions.

SELF-CONSISTENT SIGNIFICANCE LEVEL

The problem of belonging of two SDF $F_{1,n}(x)$ and $F_{2,n}(x)$ to the same general set can be solved with the use of non-parametric statistics of Kolmogorov-Smirnov test

$$S_n=\sup_x\left|F_{1,n}(x)-F_{2,n}(x)\right|, \qquad (16)$$

for which the following asymptotic representation is valid:

$$\lim_{n\to\infty} P\left\{0<\sqrt{\frac{n}{2}}\,S_n<z\right\}=K(z), \qquad (17)$$

where $K(z)$ is a tabulated Kolmogorov function (see e.g. Durbin, 1972) and n is a sample length. In formula (16) the significance level Q is usually approximated by the value of $1-K(z)$. More precisely, if we shall be given a significance level α of the distribution function of criterion (16), we must calculate the corresponding α-quantile. Let us denote $\varepsilon_n(\alpha)$ the value of probability of exceeding of a given value z, that is $P(S_n\ge z)=\alpha$. The critical set for this criterion is defined by the condition $S_N\ge\varepsilon_N(\alpha)$. For this function in the work (Bol'shev, 1963) the asymptotic representation was obtained:

$$\varepsilon_n(\alpha)=\sqrt{\frac{z}{2n}}-\frac{1}{6n}+o(1/n)\,, \qquad (18)$$

where z is a root of the equation

$$1-K\left(\sqrt{\frac{z}{2}}\right)=\alpha\,. \qquad (19)$$

It is more conveniently to use the stationary point of the function (19). Namely, let ε be a distance between samples in the supremum norm, defined by (16). Then SCSL for the sample of length n is defined as a solution of the equation

$$1-K\left(\sqrt{\frac{n}{2}}\varepsilon\right)=\varepsilon . \qquad (20)$$

In the stationary case the corresponding SCSL $\varepsilon=\varepsilon_0(n)$ does not depend on the distribution function *F*. This solution is unique because of the monotonicity of the function $K(z)$. Tabulated values of $\varepsilon_0(n)$ can be found in (Orlov, 2014).

For nonstationary SDF the distribution of distances between samples with defined length *n* differs from the statistics (17). We can construct the empirical distribution function $G_n(S)$ for the distances S_n between two independent samples with length *n*. The numerical solution of the equation

$$G_n(S)=1-S \qquad (21)$$

is called as self-consistent stationary level $S^*(n)$. It will be the probability that distance between independent samples of length *n* is more than $S^*(n)$. If it happens, that $S^*(n)>\varepsilon_0(n)$, then this time series is nonstationary. Especially for this case we suggest the above procedure of kinetic forecasting of nonstationary distribution function. The characteristic feature of this procedure is that the SCSL for the various samples from the forecasting ensemble of trajectories is more, then stationary level $\varepsilon_0(n)$ and less, then self-consistent stationary level $S^*(n)$ between two independent SDF $F_n(x,t)$ and $F_n(x,t+n)$. It means, that the modeling process is nonstationary and our forecasting method is more precise, then stationary approximation.

CHERNOFF EQUIVALENCE

Chernoff theorem (Chernoff, 1968) is concerned to iteration process of semigroup construction as a solution of Cauchy problem for differential equation in partial derivatives. In terms of solver operators this theorem is following.

Let *X* is Banach Space and *B*(*X*) is Banach Space of bounded linear operators acting in X. Let also the function $\varphi: R_+ \to B(X)$ satisfies to a condition $\varphi(0)=I$, continuous in strong operator topology and $\|\varphi\|_{B(X)} \le e^{at}, t\ge 0$, where a is some finite value. If also operator $\varphi'(0)$ being closed and its closure is a generator of strong continuous semigroup *U*(*t*), then there exists a following limit:

$$\lim_{n\to\infty}\sup_{t\in[0;T]}\left\|\left(U(t)-\left(\varphi\left(\frac{t}{n}\right)\right)^n\right)g\right\|_X=0 \qquad (22)$$

for all $T>0$ and for all $g\in X$.

This theorem is generalization of well-known mathematical limit for differentiable functions: if function $\varphi(x)$ satisfies to the conditions, that $\varphi(0)=1$ and $\varphi'(0)=a<\infty$, then there exists the limit $\lim_{n\to\infty}\left(\varphi(t/n)\right)^n=e^{at}$.

This theorem enables us to construct dynamical systems as an asymptotical state of some chaotical process after iteration procedure (22). Finite iteration procedure corresponds to approximation of the semigroup with definite accuracy. The examples of this method for quantum mechanics problems were presented in the work (Smolyanov et al., 2007).

In the work (Orlov et al., 2014) the definition of Chernoff-equivalent operator functions was introduced. The operator-valued functions φ and ψ, acting from a right half-neighborhood of the origin of the axis of real numbers to the Banach Space *B*(*X*) of bounded linear operators acting in a Banach Spase *X* are said to be Chernoff-equivalent, if the following condition is valid:

$$\lim_{n\to\infty}\sup_{t\in[0;T]}\left\|\left(\left(\varphi\left(\frac{t}{n}\right)\right)^n-\left(\psi\left(\frac{t}{n}\right)\right)^n\right)g\right\|_X=0 \qquad (23)$$

for all $T>0$ and for all $g\in X$. We shall write in this case $\varphi \overset{Ch}{\approx} \psi$.

A random variable ξ taking values in the set *C*(*X*) of strongly continuous one-parameter semigroups of operators acting in a Banach space *X* is called a random semigroup of operators in the Banach space X.

A strongly continuous one-parameter semigroup *U* of bounded linear transformations of the Banach space *X* is called a generalized mean value of a random semigroup ξ if the semigroup *U* is Chernoff equivalent in the sense (23) to the mathematical expectation $E[\xi]$.

The following theorem of averaging semigroups is valid (Orlov et al., 2014).

Let $\{H_n\}$ be a sequence of generators of strongly continuous semigroups in a Banach space X. Let $\{p_n\}$ be a sequence of nonnegative numbers such that $\sum_{n=1}^{\infty}p_n=1$. Suppose also that there exits a linear manifold $D\subset X$ that is an essential domain for each generator H_n, and is such that the series $\sum_{n=1}^{\infty}p_n\|H_n g\|_X$ converges for every $g\in D$. If the operator *H*, defined on the linear manifold D as

$Hg = \sum_{n=1}^{\infty} p_n H_n g$ for $g \in D$, is closable and its closure is a generator of a strongly continuous semigroup $U(t) = e^{tH}$, t ≥ 0, then the mean value $\varphi = E[U]$ of the random semigroup $U_n(t) = e^{tH_n}$, defined by the formula $\varphi(t) = \sum_{n=1}^{\infty} p_n U_n(t) = \sum_{n=1}^{\infty} p_n e^{tH_n}$, t ≥ 0, is Chernoff equivalent to the unitary semigroup $U(t) = e^{tH}$.

So this theorem helds, that

$$\sum_{n=1}^{\infty} p_n e^{tH_n} \overset{Ch}{\approx} e^{tH}, \quad H = \sum_{n=1}^{\infty} p_n H_n. \tag{24}$$

For the problem of random process approximation this result means, that the random value $u_n(x_t, t)$ from (6) can be averaged in the sense of (10-11) and represented the finite approximation of the semigroup, solving Cauchy problem for Liouville equation (5) for the sample distribution function density.

CHERNOFF EQUIVALENCE FOR SIGNIFICANCE LEVEL

The Chernoff equivalence notion is very useful for optimization of numerical experiments in the area of non-parametric nonstationary statistics. Suppose that we have self-consistent significance level $S_n(L_{tot})$ for the time-series of length L_{tot} as a solution of equation (21) for two nearest independent samples from the main data set. It should be taken into account that $L_{tot} >> 2n$ and it is possible to fit enough numbers of segments with length *n*, so that we can actually collect statistics for building of the distribution $G_n(S)$. For this length L_{tot} in some cases the distance between two nearest samples from the main data set is greater than $S_n(L_{tot})$. If now in a certain sliding window of length *L* the proportion of events where distances between two nearest samples from the main data set of length *n* turned out to be larger than the value $S_n(L_{tot})$, then in this window we can register a disorder situation. To achieve such conclusion, it is required that the SCSL of the nonstationary time series must be a stochastic variable with a stationary distribution.

Studying the local value of the SCSL – that is, the value obtained over the whole interval *L*, which is substantially smaller than the original data set L_{tot}, the question can arise about the fluctuations of this local SCSL relative to the base self-consistent stationary level of the whole data set. It is important to understand that the SCSL of the whole data set $S_n(L_{tot})$ is not the average value of the sequence of SCSL $S_n(k, L)$, where *k* is a number of subset. Each element of the sequence $S_n(k, L)$ is constructed for a certain set of two nearest samples of length *n* from the main data set in windows of length *L*. Let us consider the function

$$\Psi_n(S) = 1 - G_n(S). \tag{25}$$

Suppose that empirical distribution function $G_n(S)$ approximates the differentiable function $G(S)$ of the corresponding general data set. Then the function $\Psi_n(S)$ has the following properties: $\Psi_n(0) = 1, \Psi_n'(0) = -a_n \le 0$. In this case there exist a limit

$$\lim_{n\to\infty}\left(\Psi_n\left(\frac{S}{n}\right)\right)^n = e^{-Sa_n} \equiv \Phi_n(S). \tag{26}$$

Following the above definitions for operator functions, the limit function $\Phi_n(S)$ is called Chernoff equivalent function to $\Psi_n(S)$ or, that is the same, to the significance level of the distribution $G_n(S)$. So

$$\Phi_n(S) \overset{Ch}{\approx} \Psi_n(S). \tag{27}$$

Obviously, $|\Phi_N(\rho) - \Psi_N(\rho)| = o(\rho)$.

According to above theorem of semigroup averaging, if there is a set (finite or infinite) of functions $\Psi_n(k, S)$ in the form of (25), each of them is equivalent in the sense of (26-27) to the function $\Phi_n(k, S)$ with a coefficient $(-a_n^k)$ in the exponent. If we specify a set of corresponding non-negative coefficients p_k so, that $\sum_k p_k = 1$, then the average function $\overline{\Psi}_n(S) = \sum_k p_k \Psi_n(k, S)$ is Chernoff equivalent in the sense of (27) to the function $\Phi_n(S) = e^{-S\overline{a}_n}$, where

$$\overline{a}_n = \sum_k p_k a_n^k. \tag{28}$$

Consider a sequence of disjoint intervals of length L. For each interval k we construct an empirical distribution $G_n^k(S, L)$ of distances between two nearest samples of length *n* from the main data set. Let it be *m* of such intervals, so that $mL = L_{tot}$ is the total length of the sequence. Then the distribution of distances, constructed over the entire data, is the average distribution obtained by averaging over individual samples:

$$G_n(S,L_{tot}) = \frac{1}{m}\sum_{k=1}^{m} G_n^k(S,L),$$
$$\overline{\Psi}_n(\rho) = \sum_k p_k \Psi_n(k,S) = 1 - G_n(S,L_{tot}), \qquad (29)$$
$$p_k = \frac{1}{m}.$$

Using the results (23, 24), we can prove the following theorem about stationary points of distributions with continuous densities.

Let the distributions of random variables have continuous densities and let some non-negative measure be given on the set of these stochastic variables. Then the stationary point of the function will be Chernoff equivalent to the average level of significance of the given distributions, with accuracy up to second order of an infinitesimal, coincides with invers value of the mean value of the reciprocals of the stationary points of the functions Chernoff equivalent to levels of the significance of these distributions.

Namely, let $\widetilde{S}_n(k)$ is a stationary point of the function $\Phi_n(k,S)$, so that $\widetilde{S}_n(k,L)$ is a solution of the equation $\Phi_n(k,S) = S$. Let $\widetilde{S}_n(L_{tot})$ is an analogous solution for average function $\overline{\Phi}_n(S) = \sum_k p_k \Phi_n(k,S)$. Then

$$\frac{1}{\widetilde{S}_n(L_{tot})} = \sum_{k=1}^{m} \frac{p_k}{\widetilde{S}_n(k,L)}. \qquad (30)$$

The formula (30) is very important, because it allows to reduce considerably the number of calculations while studying the behavior of SCSL on unions of data sets in different tasks of Big Data analysis.

CONCLUSION

In this paper we proved the theorem, concerning statistical application of the special group theoretical notation, which is known as Chernoff equivalence. It allows us to significantly reduce the volume of calculation for analysis of complex statistical object such as variation between sample distribution function of the distances between sample distribution functions of non-linear functional of stochastic trajectories.

In particularly, it appears, that two sets of data with various non-stationary levels can be statistically separated in sufficiently narrow sliding window. The disorder indicator in the form of Chernoff equivalent self-consistency level has a stationary distribution function for each subset with the same non-stationary level, as a main sample of data. In other words, the type of non-stationary behavior of the data does not change during the period of time when the physical system is in the same condition.

This disorder indicator can be used for the practical solution of the problem of optimal non-stationary stochastic control.

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