GLOBAL STABILITY OF FRACTIONAL POSITIVE NONLINEAR FEEDBACK SYSTEMS WITH INTERVAL STATE MATRICES

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ABSTRACT

The global stability of fractional positive continuous-time nonlinear systems with positive linear parts, positive feedbacks and interval state matrices is investigated. New sufficient conditions for the global stability of the classes of fractional positive nonlinear systems are established. The new stability conditions are demonstrated on simple examples of fractional positive nonlinear systems with interval state matrices.

1. INTRODUCTION

In positive systems inputs, state variables and outputs take only nonnegative values for any nonnegative inputs and nonnegative initial conditions (Kaczorek 2002, 2019a, Berman et.al. 1994). Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollutions models. A variety of models having positive behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive systems theory is given in the monographs (Berman et.al. 1994, Farina, et. al. 2000, Kaczorek 2002, 2011b, Kaczorek, et. al. 2015).

Descriptor positive systems have been analyzed in (Borawski 2017, Kaczorek 2012). Linear positive electrical circuits with state feedbacks have been addressed in (Borawski 2017, Kaczorek et.al. 2012). The superstabilization of positive linear electrical circuits by state feedbacks have been analyzed in (Kaczorek 2017) and the stability of nonlinear systems in (Kaczorek, et. al. 2017). The global stability of nonlinear systems with negative feedbacks and positive not necessary asymptotically stable linear parts has been investigated in (Kaczorek 2015a, 2019b). The global stability of positive standard and fractional nonlinear feedback systems has been analyzed in (Kaczorek 2020).

In this paper the global stability of fractional nonlinear feedbacks systems with positive linear parts with interval state matrices will be addressed.

The paper is organized as follows. In section 2 the basic definitions and theorems concerning the fractional positive linear systems are recalled. The stability of fractional positive systems with interval state matrices is addressed in section 3. New sufficient conditions for the global positive nonlinear systems with interval state matrices are established in section 4. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix.

2. FRACTIONAL POSITIVE LINEAR SYSTEMS

Consider the fractional continuous-time linear system

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t),\ 0<\alpha<1 \tag{1a}$$

$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$,

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha}d\tau,\ \dot{x}(\tau) = \frac{dx(\tau)}{d\tau} \tag{1c}$$

is the Caputo fractional derivative and

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt,\ \mathrm{Re}(z) > 0 \tag{1d}$$

is the gamma function (Kaczorek 2011b, Kaczorek, et. al. 2015).

Definition 1. (Kaczorek 2011b, Kaczorek, et. al. 2015) The fractional system (1) is called (internally) positive if $x(t)\in\Re_+^n$ and $y(t)\in\Re_+^p$, $t\ge 0$ for any initial conditions $x(0)\in\Re_+^n$ and all inputs $u(t)\in\Re_+^m$, $t\ge 0$.

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A real matrix $A=[a_{ij}]\in\Re^{n\times n}$ is called Metzler matrix if its off-diagonal entries are nonnegative, i.e. $a_{ij}\geq 0$ for $i\neq j$. The set of $n\times n$ Metzler matrices will be denoted by M_n.

Theorem 1. (Kaczorek 2011b, Kaczorek, et. al. 2015) The fractional system (1) is positive if and only if

$$A\in M_n,\ B\in\Re_+^{n\times m},\ C\in\Re_+^{p\times n},\ D\in\Re_+^{p\times m}. \quad (2)$$

Definition 2. (Kaczorek 2011b, Kaczorek, et. al. 2015) The positive fractional system (1) (for $u(t)=0$) is called asymptotically stable (the matrix A is Hurwitz) if

$$\lim_{t\to\infty} x(t)=0 \text{ for any } x(0)\in\Re_+^n. \quad (3)$$

Theorem 2. (Kaczorek 2011b, Kaczorek, et. al. 2015) The positive system (1) is asymptotically stable if and only if one of the equivalent conditions is satisfied:

1) All coefficient of the characteristic polynomial

$$\det[I_n s-A]=s^n+a_{n-1}s^{n-1}+...+a_1 s+a_0 \quad (4)$$

are positive, i.e. $a_k>0$ for $k=0,1,...,n-1$.

2) All principal minors $\overline{M}_i$, $i=1,...,n$ of the matrix $-A$ are positive, i.e.

$$\overline{M}_1=|-a_{11}|>0,\ \overline{M}_2=\begin{vmatrix}-a_{11} & -a_{12}\\ -a_{21} & -a_{22}\end{vmatrix}>0,$$
$$\ldots,\overline{M}_n=\det[-A]>0. \quad (5)$$

3) There exists strictly positive vector $\lambda^T=[\lambda_1\ \cdots\ \lambda_n]^T$, $\lambda_k>0$, $k=1,...,n$ such that

$$A\lambda<0 \text{ or } A^T\lambda<0. \quad (6)$$

3. STABILITY OF FRACTIONAL INTERVAL POSITIVE LINEAR SYSTEMS

Consider the fractional interval positive linear continuous-time system

$$\frac{d^\alpha x}{dt^\alpha}=Ax,\ 0<\alpha<1, \quad (7)$$

where $x=x(t)\in\Re^n$ is the state vector and the matrix $A\in M_n$ is defined by

$$A_1\leq A\leq A_2 \text{ or equivalently } A\in[A_1,A_2]. \quad (8)$$

Definition 3. The interval positive system (7) is called asymptotically stable if the system is asymptotically stable for all matrices $A\in M_n$ satisfying the condition (8).

By condition (6) of Theorem 2 the positive system (7) is asymptotically stable if there exists strictly positive vector $\lambda>0$ such that the condition (6) is satisfied.

For two fractional positive linear systems

$$\frac{d^\alpha x_1}{dt^\alpha}=A_1x_1,\ A_1\in M_n \quad (9)$$

and

$$\frac{d^\alpha x_2}{dt^\alpha}=A_2x_2,\ A_2\in M_n \quad (10)$$

there exists a strictly positive vector $\lambda\in\Re_+^n$ such that

$$A_1\lambda<0 \text{ and } A_2\lambda<0 \quad (11)$$

if and only if the systems (9), (10) are asymptotically stable.

Theorem 3. If the matrices A_1 and A_2 of fractional positive systems (9), (10) are asymptotically stable then their convex linear combination

$$A=(1-k)A_1+kA_2 \text{ for } 0\leq k\leq 1 \quad (12)$$

is also asymptotically stable.

Proof. By condition (6) of Theorem 2 if the fractional positive linear systems (9), (10) are asymptotically stable then there exists strictly positive vector $\lambda\in\Re_+^n$ such that

$$A_1\lambda<0 \text{ and } A_2\lambda<0. \quad (13a)$$

Using (6) and (13) we obtain

$$A\lambda=[(1-k)A_1+kA_2]\lambda=(1-k)A_1\lambda+kA_2\lambda<0 \quad (13b)$$

for $0\leq k\leq 1$. Therefore, if the positive linear systems (9), (10) are asymptotically stable then their convex linear combination (12) is also asymptotically stable. □

Theorem 4. The interval positive systems (7) are asymptotically stable if and only if the positive linear systems (9), (10) are asymptotically stable.

Proof. By condition (6) of Theorem 2 if the matrices $A_1\in M_n$, $A_2\in M_n$ are asymptotically stable then there exists a strictly positive vector $\lambda\in\Re_+^n$ such that (6) holds. The convex linear combination (12) satisfies the condition $A\lambda<0$ if and only if (13) holds. Therefore, the interval system (8) is asymptotically stable if and only if the positive linear system is asymptotically stable. □

Example 1. Consider the fractional interval positive linear continuous-time system (8) with the matrices

$$A_1 = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}, \ A_2 = \begin{bmatrix} -3 & 2 \\ 4 & -4 \end{bmatrix}. \qquad (14)$$

Using the condition (6) of Theorem 2 we choose for A_1 $\lambda_1 = [1 \ \ 1]^T$ and we obtain

$$A_1\lambda_1 = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} < 0, \qquad (15a)$$

and for A_2, $\lambda_2 = [0.8 \ \ 1]^T$

$$A_2\lambda_2 = \begin{bmatrix} -3 & 2 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} 0.8 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.4 \\ -0.8 \end{bmatrix} < 0. \qquad (15b)$$

Therefore, the matrices (14) are Hurwitz. Note that

$$A_1\lambda_2 = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 0.8 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.6 \\ -1.4 \end{bmatrix} < 0. \qquad (16)$$

Therefore, for both matrices (14) we may choose $\lambda = \lambda_1 = \lambda_2 = [0.8 \ \ 1]^T$ and by Theorem 4 the fractional interval positive system (7) with (14) is asymptotically stable.

4. GLOBAL STABILITY OF FRACTIONAL NONLINEAR FEEDBACK SYSTEMS WITH POSITIVE LINEAR PARTS

In this section sufficient conditions for the global stability of fractional nonlinear systems with interval state matrices of asymptotically stable positive linear parts and positive state feedbacks with gain h will be proposed.

Consider the fractional nonlinear system shown in Fig. 1 which consists of the positive linear part with interval asymptotically stable state matrix, the nonlinear element with characteristic $u = f(e)$ and positive feedback with gain h.

The linear part is described by the equations

$$\frac{d^\alpha x}{dt^\alpha} = Ax + Bu, \qquad y = Cx, \qquad (17)$$

where $x = x(t) \in \Re_+^n$, $u = u(t) \in \Re_+$, $y = y(t) \in \Re_+$ is the state vector, input and output and $A \in M_n$, $B \in \Re_+^{n\times 1}$, $C \in \Re_+^{1\times n}$.

It is assumed that the positive linear part is asymptotically stable (the matrix $A \in M_n$ is Hurwitz) for A belonging to the interval (8).

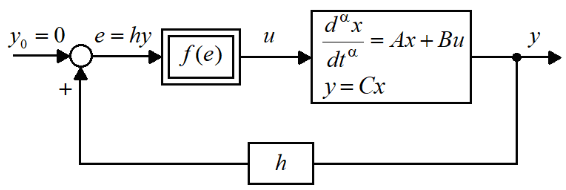

Figure 1: The nonlinear feedback system

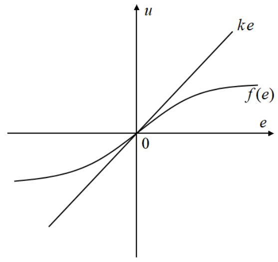

Figure 2: Characteristic of the nonlinear element

The characteristic of the nonlinear element is shown in Fig. 2 and it satisfies the condition

$$0 \le \frac{f(e)}{e} \le k < \infty. \qquad (18)$$

Definition 4. The nonlinear positive system with interval state matrix $A \in [A_1, A_2] \in M_n$ is called globally stable if it is asymptotically stable for all nonnegative initial conditions $x(0) \in \Re_+$.

The following theorem gives sufficient conditions for the global stability of the positive nonlinear feedbacks systems.

Theorem 5. The nonlinear system consisting of the positive asymptotically stable linear part with interval state matrix $A \in [A_1, A_2]$, the nonlinear element satisfying the condition (18) and positive feedback with gain h is globally stable if

$$(1-q)A_1 + qA_2 + khBC = \begin{cases} A_1 + khBC \in M_n & \text{for} \quad q = 0 \\ A_2 + khBC \in M_n & \text{for} \quad q = 1 \end{cases} \qquad (19)$$

is the Hurwitz Metzler matrix.

Proof. The proof will be accomplished by the use of the Lyapunov method (Lyapunov 1963, Leipholz 1970). As the Lyapunov function $V(x)$ we choose

$$V(x) = \lambda^T x \ge 0 \ \text{ for } \ x \in \Re_+^n, \qquad (20)$$

where λ is strictly positive vector, i.e. $\lambda_k > 0$, $k = 1, \ldots, n$.

Using (20) and (17) we obtain

$$\frac{d^\alpha}{dt^\alpha}V(x) = \lambda^T \frac{d^\alpha x}{dt^\alpha} = \lambda^T (Ax + Bu) \\ = \lambda^T (Ax + Bf(e)) \le \lambda^T (A + khBC)x \tag{21}$$

since $u = f(e) \le ke = khCx$ and $A = (1-q)A_1 + qA_2$ for $q \in [0, 1]$.

From (20) it follows that $\frac{d^\alpha}{dt^\alpha}V(t) < 0$ if the condition (19) is satisfied and the nonlinear system is globally stable. □

To find the maximal value of k_1 for which the nonlinear system is globally stable the following procedure can be used.

Procedure 1

Step 1. Find the value of k_1 for which the matrix

$$A_1 + k_1 hBC \in M_n \tag{22}$$

is asymptotically stable.

Step 2. Find the value of k_2 for which the matrix

$$A_2 + k_2 hBC \in M_n \tag{23}$$

is asymptotically stable.

Step 3. Find the desired value of k as

$$k = \min(k_1, k_2). \tag{24}$$

Example 2. Consider the nonlinear feedback system with the positive linear part with the interval matrix (14) and

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ C = \begin{bmatrix} 0.4 & 0.8 \end{bmatrix} \tag{28}$$

and the nonlinear element satisfying the condition (18) and $h = 0.5$. Find the maximal value of the coefficient for k for which the nonlinear system is globally stable.

Using Procedure 1 and (14), (28) we obtain

Step 1. Using (14), (28) and (22) we obtain

$$A_1 + k_1 hBC = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} + 0.5k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0.4 & 0.8 \end{bmatrix} \\ = \begin{bmatrix} -2 + 0.2k_1 & 1 + 0.4k_1 \\ 2 + 0.2k_1 & -3 + 0.4k_1 \end{bmatrix} \tag{29}$$

and the maximal value of k_1 for which the matrix (29) is Hurwitz is $k_1 < \frac{5}{3}$ since for this value the coefficients of the polynomial

$$\det[I_2 s - A_1 - k_1 hBC] \\ = s^2 + (5 - 0.6k_1)s + 4 - 2.4k_1 \tag{30}$$

are positive.

Step 2. Using (20), (28) and (23) we obtain

$$A_2 + k_2 hBC = \begin{bmatrix} -3 & 2 \\ 4 & -4 \end{bmatrix} + 0.5k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0.4 & 0.8 \end{bmatrix} \\ = \begin{bmatrix} -2 + 0.2k_1 & 1 + 0.4k_1 \\ 2 + 0.2k_1 & -3 + 0.4k_1 \end{bmatrix} \tag{31}$$

and the maximal value of k_2 for which the matrix (31) is Hurwitz is $k_2 < 1$ since for this value the coefficients of the polynomial

$$\det[I_2 s - A_2 - k_2 hBC] \\ = s^2 + (7 - 0.6k_2)s + 4 - 4k_2 \tag{32}$$

are positive.

Step 3. Using (24) and the results obtained in Steps 1 and 2 we obtain

$$k = \min(k_1, k_2) = \min\left(\frac{5}{3}, 1\right) = 1. \tag{33}$$

Therefore, the nonlinear system is globally stable for k less than given by (33).

5. CONCLUDING REMARKS

The global stability of positive continuous-time nonlinear feedback systems with interval state matrices has been investigated. New sufficient conditions for the global stability of this class of positive nonlinear systems are established. A procedure for computation of the value of the coefficient satisfying the condition (18) has been proposed. The new stability conditions are demonstrated on simple examples of positive nonlinear systems with interval state matrices. The considerations can be extended to fractional discrete-time nonlinear positive systems with all interval matrices of the linear parts.

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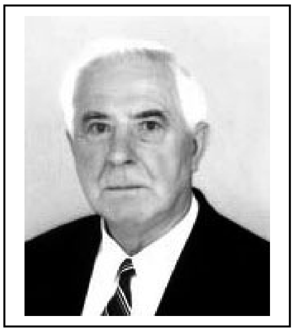

TADEUSZ KACZOREK received his MSc, PhD and DSc degrees in electrical engineering from the Warsaw University of Technology in 1956, 1962 and 1964, respectively. In the years 1968–1969 he was the dean of the Electrical Engineering Faculty, and in the period of 1970–1973 he was a deputy rector of the Warsaw University of Technology. In 1971 he became a professor and in 1974 a full professor at the same university. Since 2003 he has been a professor at the Białystok University of Technology. In 1986 he was elected a corresponding member and in 1996 a full member of the Polish Academy of Sciences. In the years 1988–1991 he was the director of the Research Center of the Polish Academy of Sciences in Rome. In 2004 he was elected an honorary member of the Hungarian Academy of Sciences. He has been granted honorary doctorates by 13 universities. His research interests cover systems theory, especially singular multidimensional systems, positive multidimensional systems, singular positive 1D and 2D systems, as well as positive fractional 1D and 2D systems. He initiated research in the field of singular 2D, positive 2D and positive fractional linear systems. He has published 28 books (8 in English) and over 1100 scientific papers. He has also supervised 69 PhD theses. He is the editor-in-chief of the Bulletin of the Polish Academy of Sciences: Technical Sciences and a member of editorial boards of ten international journals.