METHOD FOR BOUNDING THE RATE OF CONVERGENCE FOR ONE CLASS OF FINITE-CAPACITY MARKOVIAN TIME-DEPENDENT QUEUES WITH BATCH ARRIVALS WHEN EMPTY

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KEYWORDS

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ABSTRACT

Consideration is given to one class of inhomogeneous birth and death processes with finite state space and additional transitions from the origin, which may be used to study the queue-length process in finite-capacity Markovian time-dependent queues with possible batch arrivals when empty. The latter means that customers may arrive in batches only during the periods when the system is idle. All possible transition intensities are allowed to be state-dependent non-random functions of time. Method based on Lyapunov functions, which allows one to obtain ergodicity bounds, is presented. Short numerical example is given.

INTRODUCTION

In this short note we revisit the problem of bounding the rate of convergence to the limiting regime (whenever it exists) of the queue-length process in finite-capacity queues of type $M_n(t)/M_n(t)/1/(S-1)$ (and some other queues, for example, $M_n(t)/M_n(t)/S/0$) with possible batch arrivals when empty (see section 2). The queue-length process in such a queue, further denoted by $X(t)$, can be described by one subclass of continuous-time Markov chains – inhomogeneous birth and death processes with additional transitions from and to origin[1]. Under the assumption that the state space is countable, this subclass was studied in Zeifman et al. (2016, 2017); Zeifman, Korotysheva et al. (2017). It was shown (particularly in (Zeifman et al., 2017, Eq. (15))) that under the presence of disasters and some other conditions on batch arrival intensities, it is possible to obtain the ergodicity bounds[2] using the method based on the logarithmic norm of linear operators and special transformations of the intensity matrix. In order to obtain the ergodicity bounds for the considered finite-capacity queue with possible batch arrivals, one could try to the same approach. But in the case of finite state spate (and the state space of $X(t)$ is finite) and in the absence of disasters, the ergodicity bounds cannot[3] be obtained using the logarithmic norm method. Thus a different approach is needed. In section 3 it is shown, that the method based on Lyapunov functions can be used[4] to find the upper bound on the rate of convergence. When the transition intensities are periodic the method yields the constant decay parameter (spectral gap). Section 4 concludes the paper with a short numerical example. Since for Markov chains with a finite state space apparently no general method for the construction of Luyapunov functions can be suggested, the results of section 3 may be of independent interest.

BIRTH AND DEATH PROCESS

Let $X(t)$ be an inhomogeneous continuous-time Markov chain with the state space $\mathcal{X} = \{0, 1, 2, \ldots, S\}$. A transition, whenever it occurs from state 0, can be to any state $i > 0$, $i \in \mathcal{X}$, and has intensity $q_{0i}(t)$. A transition from state $i > 0$ can be only to neighbouring states, i.e. either to state $(i-1)$ with intensity $\mu_i(t)$ or to state $(i+1)$ with intensity $\lambda_i(t)$. All transition intensities $q_{0i}(t)$, $\mu_i(t)$ and $\lambda_i(t)$ are allowed to be non-random functions of time and, if so, are required to be continuous in t for $t \geq 0$.

Denote by $p_{ij}(s,t) = \Pr\{X(t) = j | X(s) = i\}$, $i, j \geq 0$, $0 \leq q, s \leq t$ transition probabilities of $X(t)$ and by $p_i(t) = \Pr\{X(t) = i\}$ probability that Markov chain $X(t)$ is in state i at time t. Let $\mathbf{p}(t) = (p_0(t), p_1(t), \ldots, p_S(t))^T$ be probability distribution vector at time t.

[1] In order to keep the connection with the past research, we note that in some papers, which consider related Markov chains, transitions "to the origin" are called "mass exodus" and "from the origin" – "resurrection" Li and Zhang (2017) and "mass arrivals" Chen and Renshaw (1997); Zhang and Li (2015).

[2] And also uniform in time error bounds of truncation that allow the (approximate) calculation of the limiting performance characteristics, whenever the limiting regime exists, see (Zeifman et al., 2017, Theorem 4.1). It is also worth noticing that the obtained results allow one to perform the analysis of the "inhomogeneous generalization" of the queues considered in Chen and Renshaw (1997, 2004); Li and Zhang (2017); Pakes (1997); Zhang and Li (2015).

[3] Even though the logarithmic norm does exist, it does not allow one to obtain any meaningful ergodicity bounds.

[4] This method was also successfully applied to a different process in Zeifman et al. (2020).

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Given any proper initial condition $\mathbf{p}(0)$, the probabilistic dynamics of $X(t)$ is described by the forward Kolmogorov system of differential equations

$$\frac{d\mathbf{p}(t)}{dt} = A(t)\mathbf{p}(t), \tag{1}$$

where $A(t)$ denotes the transposed intensity matrix, i.e.

$$A(t) = \begin{pmatrix} a_{00}(t) & \mu_1(t) & 0 & 0 & \dots & 0 & 0 \\ a_{10}(t) & a_{11}(t) & \mu_2(t) & 0 & \dots & 0 & 0 \\ a_{20}(t) & \lambda_1(t) & a_{22}(t) & \mu_3(t) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ a_{S0}(t) & 0 & 0 & 0 & \dots & \lambda_{S-1}(t) & -\mu_S(t) \end{pmatrix}.$$

Note that all column sums of $A(t)$ are equal to zero for $t \geq 0$ and thus $A(t)$ is essentially non-negative i.e. all its off-diagonal elements are non-negative for any $t \geq 0$.

ERGODICITY BOUNDS

Throughout the paper by $\|\cdot\|$ we denote the Euclidean norm, i. e., $\|\mathbf{p}(t)\| = \sqrt{\sum_{i\in\mathcal{X}} p_i(t)^2}$.

Recall that a Markov chain $X(t)$ is called weakly ergodic, if $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $\mathbf{p}^*(0)$ and $\mathbf{p}^{**}(0)$, where $\mathbf{p}^*(t)$ and $\mathbf{p}^{**}(t)$ are the corresponding solutions of (1). The rate at which this difference tends to zero is called the rate of convergence.

As it was mentioned in the Introduction the method based on the logarithmic norm (Zeifman et al. (2018)) does not allow one to obtain ergodicity bounds for the considered $X(t)$. In the rest of the section we show that it is possible to find the upper bound on the rate of convergence using Lyapunov functions.

Using the normalization condition it can be verified that the first equation of (1) i.e. $p_0'(t) = a_{00}(t)p_0(t) + \mu_1(t)p_1(t)$ is identical to

$$\frac{dp_0(t)}{dt} = (a_{00}(t) - \mu_1(t))\, p_0(t) + \mu_1(t) - \mu_1(t)\sum_{i=2}^{S} p_i(t).$$

Thus the system (1) can be rewritten as

$$\frac{d\mathbf{p}(t)}{dt} = A^*(t)\mathbf{p}(t) + \mathbf{f}(t), \tag{2}$$

where $\mathbf{f}(t) = (\mu_1(t), 0, ..., 0)^T$.

Let $\mathbf{p}^*(t)$ and $\mathbf{p}^{**}(t)$ be the solutions of (2) corresponding to different initial conditions $\mathbf{p}^*(0)$ and $\mathbf{p}^{**}(0)$. Then for the vector $\mathbf{z}(t) = \mathbf{p}^*(t) - \mathbf{p}^{**}(t) = (z_1(t), z_2(t), \dots, z_{S+1}(t))^T$ we have

$$\frac{d\mathbf{z}(t)}{dt} = A^*(t)\mathbf{z}(t). \tag{3}$$

Fix $S+1$ positive numbers, say $d_1, \dots, d_{S+1}$ and put $w_i(t) = d_i z_{i-1}(t)$, $1 \leq i \leq S+1$. Multiply the previous equation from the right by D^{-1} and from the left by D, where $D = diag\,(d_1, d_2, \dots d_{S+1})$. Then (3) in terms of the vector $\mathbf{w}(t) = (w_1(t), w_2(t), \dots, w_{S+1}(t))^T$ has the form:

$$\frac{d\mathbf{w}(t)}{dt} = A^{**}(t)\mathbf{w}(t). \tag{4}$$

where $A^{**}(t) = DA^*(t)D^{-1}$. It is important to notice that the coordinates of $\mathbf{w}(t)$ can be of arbitrary signs i.e. they have no probabilistic meaning.

Let $\mathbf{w}(t)$ be the solution of (4). By differentiating $V(t) = \sum_{k=1}^{S+1} w_k^2(t)$, we obtain

$$\frac{dV(t)}{dt} = \sum_{k=1}^{S+1} 2w_k(t)\frac{dw_k(t)}{dt} = $$
$$= -2\sum_{i=1}^{S+1}\sum_{j=1}^{S+1}\left(-a^{**}_{i-1,j-1}(t)\right) w_i(t) w_j(t). \tag{5}$$

From (5) it follows that if one finds a set of positive numbers $\{d_i, 1 \leq i \leq S+1\}$ and a function $\beta^*(t)$ satisfying

$$\frac{dV(t)}{dt} \leq -2\beta^*(t)V(t), \tag{6}$$

for any $\mathbf{w}(t)$ being the solution of (4), then

$$\|\mathbf{w}(t)\| \leq e^{-\int_0^t \beta^*(\tau)\,d\tau}\|\mathbf{w}(0)\|,$$

for any initial condition $\mathbf{w}(0)$.

Assume that $q_{0i}(t)$, $\mu_i(t)$ and $\lambda_i(t)$ do not depend on t. Thus the matrix $A^{**}(t) = A^{**} = (a^{**})_{i,j=0}^{S}$ is equal to

$$A^{**} = \begin{pmatrix} a_{00} - \mu_1 & 0 & -\mu_1\frac{d_1}{d_3} & -\mu_1\frac{d_1}{d_4} & \cdots & -\mu_1\frac{d_1}{d_{S+1}} \\ a_{10}\frac{d_2}{d_1} & a_{11} & \mu_2\frac{d_2}{d_3} & 0 & \cdots & 0 \\ a_{20}\frac{d_3}{d_1} & \lambda_1\frac{d_3}{d_2} & a_{22} & \mu_3\frac{d_3}{d_4} & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ a_{(S-1)0}\frac{d_S}{d_1} & 0 & 0 & 0 & \cdots & \mu_S\frac{d_S}{d_{S+1}} \\ a_{S0}\frac{d_{S+1}}{d_1} & 0 & 0 & 0 & \cdots & -\mu_S \end{pmatrix}.$$

By plugging the explicit values of the entries of the matrix A^{**} in (5) , we get

$$\frac{1}{2}\frac{dV(t)}{dt} = (a_{00} - \mu_1)\, w_1^2(t) + \sum_{i=1}^{S-1} a_{ii} w_{i+1}^2(t) +$$
$$+ a_{10}\frac{d_2}{d_1} w_1(t) w_2(t) - \mu_S w_{S+1}^2(t) +$$
$$+ \sum_{i=2}^{S}\left(a_{i0}\frac{d_{i+1}}{d_1} - \mu_1\frac{d_1}{d_{i+1}}\right) w_1(t) w_{i+1}(t) +$$
$$+ \sum_{i=2}^{S}\left(\lambda_{i-1}\frac{d_{i+1}}{d_i} + \mu_i\frac{d_i}{d_{i+1}}\right) w_i(t) w_{i+1}(t).$$

Put $d_i = d_1\sqrt{\frac{\mu_1}{a_{i0}}}$ for $1 \leq i \leq S+1$. Thus each term with the coefficient $(a_{i0}\frac{d_{i+1}}{d_1} - \mu_1\frac{d_1}{d_{i+1}})$ is equal to zero and the previous equality can be rewritten in the form

$$\frac{1}{2}\frac{dV(t)}{dt} = (a_{00} - \mu_1)\, w_1^2(t) + \sum_{i=1}^{S-1} a_{ii} w_{i+1}^2(t) +$$
$$+ a_{10}\frac{d_2}{d_1} w_1(t) w_2(t) - \mu_S w_{S+1}^2(t) +$$
$$+ \left(\lambda_1\sqrt{\frac{\mu_1}{a_{20}}}\frac{d_1}{d_2} + \mu_2\sqrt{\frac{a_{20}}{\mu_1}}\frac{d_2}{d_1}\right) w_2(t) w_3(t) +$$
$$+ \sum_{i=3}^{S}\left(\lambda_{i-1}\sqrt{\frac{a_{i-1,0}}{a_{i0}}} + \mu_i\sqrt{\frac{a_{i0}}{a_{i-1,0}}}\right) w_i(t) w_{i+1}(t).$$

Finally, by applying the same reasoning as in (Zeifman et al., 2020, Theorem 4), one can show that there exists a positive number β^* and a set of numbers $\{\alpha_i, 1 \le i \le S+1\}$ such that

$$\frac{dV(t)}{dt} = -2\beta^* \sum_{k=1}^{S+1} w_k^2(t) - -2\sum_{k=1}^{S}(\alpha_k w_k(t) - \alpha_{k+1} w_{k+1}(t))^2. \quad (7)$$

Thus the following bound on the rate of convergence holds:

$$\|\mathbf{w}(t)\| \le e^{-\beta^* t}\|\mathbf{w}(0)\|, \quad (8)$$

where β^* is the decay parameter (spectral gap) of $X(t)$. Unfortunately the closed-form expression for β^* cannot be obtained. For a given matrix A^{**} it is computed algorithmically-wise (see (Zeifman et al., 2020, Section 4))). Note that since the state space of $X(t)$ is finite, (8) can be rewritten in the form:

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \le Me^{-\beta^* t}, \quad (9)$$

for some positive M and thus β^* is the decay parameter of the original Markov chain $X(t)$.

Now assume that all the transition intensities $q_{0i}(t)$, $\mu_i(t)$ and $\lambda_i(t)$ are non-random functions of time t. For simplicity we also assume that the transition rates are periodic with period 1 and thus $X(t)$ may have a periodic limiting regime. Let $a_{i0}(t) = 2 + \cos(2\pi t)$, $\lambda_i(t) = \lambda(t) = 2 + \sin(2\pi t)$, $\mu_i(t) = i\,(2 + \cos(2\pi t)) = i\mu(t)$ for $1 \le i \le S$.

Then the non-zero entries of the matrix $A(t)$ are:

$$\begin{aligned} a_{i0}(t) &= 2 + \cos(2\pi t) = \mu(t), \\ a_{i-1,i}(t) &= i\mu(t), \\ a_{i+1,i}(t) &= \lambda(t), \\ a_{00}(t) &= -S\mu(t), \\ a_{i,i}(t) &= -(\lambda(t) + i\mu(t)). \end{aligned}$$

Put $d_i = 1$ for all $1 \le i \le S+1$. By computing the entries of the matrix $A^{**}(t)$ and plugging them into (5), we get

$$-\frac{1}{2}\frac{dV(t)}{dt} = (S+1)\mu(t)w_1^2(t) - \mu(t)w_1(t)w_2(t) + + \sum_{i=2}^{S}(\lambda(t) + (i-1)\mu(t))\, w_i^2(t) + S\mu(t)w_{S+1}^2(t) - - \sum_{i=2}^{S}(\lambda(t) + i\mu(t))\, w_i(t)w_{i+1}(t).$$

The right part of the previous relation can be bounded from below i.e. it can be shown that

$$-\frac{1}{2}\frac{dV(t)}{dt} \ge S\mu(t)w_1^2(t) + + \sum_{i=2}^{S} k_i(t)\,(w_i(t) - w_{i+1}(t))^2 + +\mu(t)\left(\frac{3}{2}\left(w_1^2(t) + w_2^2(t)\right) + \left(\frac{w_1(t)}{\sqrt{2}} - \frac{w_2(t)}{\sqrt{2}}\right)^2\right).$$

where $k_i(t) = \min\{\lambda(t), i\mu(t)\}$.

From here, using the same arguments as in the proof of (Zeifman et al., 2020, Theorem 4) and and using the fact that the intensities are periodic functions, one can establish the existence of a positive constant β^* such that the bounds (8) and (9) on the rate of convergence hold. Just like in the homogeneous case the closed-form expression for β^* cannot be obtained and for a given matrix A^{**} the value of β^* is computed algorithmically-wise.

NUMERICAL EXAMPLE

Consider the $M_t/M_t/1/(S-1)$ queue with FIFO service and batch arrivals when empty. Let $X(t)$ be the queue-length process. If at time t there is at least one customer in the system then new customers arrive according to inhomogeneous Poisson process with intensity $\lambda(t)$. But if at time t the system is empty ordinary customers arrive in bulk (or groups) in accordance with a inhomogeneous Poisson process of intensity $S^{-1}\lambda(t)$ for a group of size n, $n = 1, 2, \ldots, S$. Whenever server becomes free customer from the queue (if there is any) enters server and its service time has exponential distribution with parameter $\mu(t)$.

Let the maximum number of customers in the system be equal to $S = 100$. Denote by $E(t,k) = E(X(t)|X(0) = k)$ the conditional expected number of customers in the queue (including server) at instant t, provided that initially (at instant $t = 0$) k customers were present. The probability of the empty queue $p_0(t)$ and the values of $E(t,k)$, computed using the ergodicity bounds obtained above, are shown in Fig. 1–4.

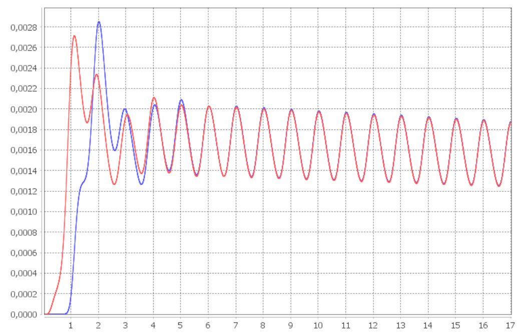

Figure 1: Probability of empty queue $p_0(t)$ for $t \in [0, 17]$ with initial conditions $X(0) = 35$ (blue) and $X(0) = 10$ (red).

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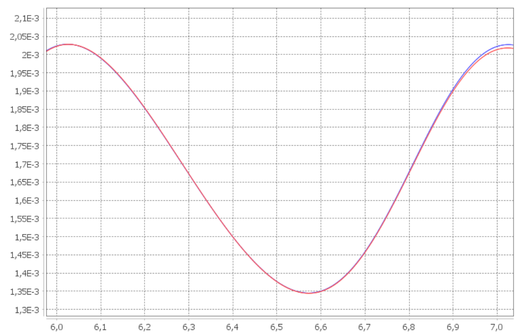

Figure 2: Probability of empty queue $p_0(t)$ for $t \in [6,7]$ with initial conditions $X(0) = 35$ (blue) and $X(0) = 10$ (red).

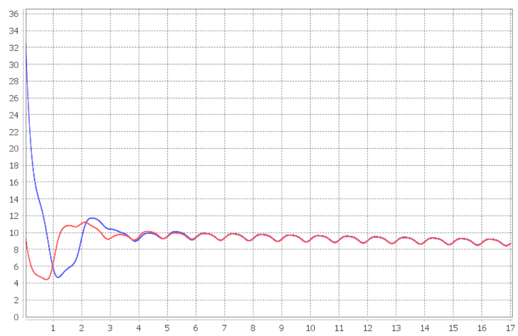

Figure 3: Expected number of customers in the system $E(t,k)$ for $t \in [0,17]$ with initial conditions $X(0) = 35$ (blue) and $X(0) = 10$ (red).

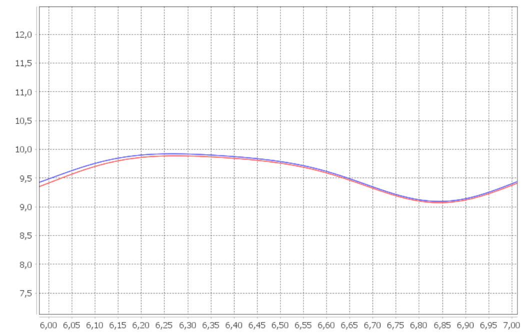

Figure 4: Expected number of customers in the system $E(t,k)$ for $t \in [6,7]$ with initial conditions $X(0) = 35$ (blue) and $X(0) = 10$ (red).

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