INVARIANT PROPERTIES OF CONTRAST PARAMETERS OF PLANE ELASTIC COMPOSITES

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ABSTRACT

Contrast dimensionless parameters of two-phase fibrous composites are investigated. The local stress and strain fields in composites and the effective constants can be described by means of two contrast parameters. The importance of the contrast parameters lies in their role in representing complex potentials and corresponding local fields through power series expansions. A set of transformations under which the contrast parameters are invariant is described. This set extends the shift transformations introduced by Cherkaev, Lurie, and Milton (CLM transformation). The relation between contrast parameters and Dundurs' constant is established.

1 Introduction

Reducing the number of parameters for elastic composites is essential in computer simulations for optimal design problems when an engineer is looking for an optimal set of elastic constants satisfying some requirements.

Invariant properties of plane elastic composites and equivalence classes of composites were discussed in the seminal papers Cherkaev et al. (1992); Cherkaev and Gibiansky (1993). It was established that a constant shift in the bulk modulus simultaneously with the opposite shift in the shear modulus yields the equivalent elastic local fields. Moreover, their effective constants are shifted in the same way. The method holds for general anisotropic media and spatially varying elastic constants. This result is closely related to the earlier paper Dundurs (1967) concerning the reduction of parameters and discussion Dundurs and Jasiuk (1997). Moreover, the invariant properties established in Cherkaev et al. (1992); Cherkaev and Gibiansky (1993) lead to the translation method Cherkaev (2012); Milton (2002) for the bounds of the effective properties of composites.

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In the present paper, the contrast parameters introduced in Drygaś and Mityushev (2017)Mityushev and Drygas (2019); Drygaś et al. (2019) are explored to investigate the invariant properties of composites. The shift transformations Cherkaev et al. (1992) discussed above are extended to functional dependencies for two-phase plane composites. First, the contrast parameter similar to the contrast parameter for conductivity for locally incompressible elastic media was introduced by Drygaś and Mityushev (2017) and by Czaplinski et al. (2018) for general elastic materials.

The complex variable $z = x_1 + ix_2 \in \mathbb{C}$ is used with complex potentials introduced below by Muskhelishvili (1953). Following the homogenization theory Bakhvalov and Panasenko (2012), consider a double periodicity composite represented by a parallelogram Q. Two pairs of opposite sides of Q are identified in a topological sense. Hence, Q can be considered as a fundamental domain of the plane torus. Consider non-overlapping domains D_k $(k = 1, 2, \ldots, N)$ on the torus Q with smooth boundaries ∂D_k; $D := Q \setminus (\cup_{k=1}^{n} D_k \cup \partial D_k)$.

We consider locally isotropic elastic materials determined by two elastic constants, the shear modulus μ and the bulk modulus k. Let the domains D and D_k $(k = 1, 2, \ldots, N)$ be occupied by elastic materials with the constants μ, k and μ_1, k_1, respectively. Let the contact between the components of the composite be perfect. Let the area of Q be normalized to unity, $|Q| = 1$. The main result presented in Section 4 concerns the invariant transformation of contrast elastic parameters for two-phase composites.

2 Boundary value problem for elastic plane composites

Let E, ν be the Young modulus and Poisson's ratio. The following useful relations between the elastic constants from Dundurs and Jasiuk (1997) take place

$$k = \frac{2\mu}{\kappa - 1} \iff \kappa = \frac{2\mu}{k} + 1, \tag{1}$$

$$k = \frac{E}{1-\nu}, \quad \nu = \frac{k-\mu}{k+\mu}, \tag{2}$$

$$E = 2\mu(1+\nu) = \frac{4k\mu}{k+\mu}, \tag{3}$$

where Muskhelishvili's constant $\kappa = 3 - 4\nu$ for plane strain and $\kappa = \frac{3-\nu}{1+\nu}$ for plane stress are used.

The component of the stress tensor can be found in the Kolosov-Muskhelishvili formulas (Muskhelishvili, 1953)

$$\sigma_{11} + \sigma_{22} = \begin{cases} 4\mathrm{Re}\,\varphi_k'(z), & z \in D_k \cup \partial D_k, \\ 4\mathrm{Re}\,\varphi_0'(z), & z \in D \cup \partial D, \end{cases} \quad (4)$$

$$\sigma_{11} - \sigma_{22} + 2i\sigma_{12} =$$

$$\begin{cases} -2\left[z\overline{\varphi_k''(z)} + \overline{\psi_k'(z)}\right], & z \in D_k \cup \partial D_k, \\ -2\left[z\overline{\varphi_0''(z)} + \overline{\psi_0'(z)}\right], & z \in D \cup \partial D, \end{cases}$$

where Re denotes the real part, and the bar denotes the complex conjugation. The functions $\varphi_0(z)$, $\psi_0(z)$ and $\varphi_k(z)$, $\psi_k(z)$ are analytic in the domains D and D_k, respectively, and continuously differentiable two and one times in their closures.

The strain tensor and displacement are expressed by the complex potentials by analogous formulas

$$\epsilon_{11} + \epsilon_{22} = \begin{cases} \frac{\kappa_1 - 1}{\mu_1}\,\mathrm{Re}\,\varphi_k'(z), & D_k \cup \partial D_k, \\ \frac{\kappa - 1}{\mu}\,\mathrm{Re}\,\varphi_0'(z), & D \cup \partial D,, \end{cases} \quad (5)$$

$$\epsilon_{11} - \epsilon_{22} + 2i\epsilon_{12} =$$

$$\begin{cases} -\frac{1}{\mu_1}\left[z\overline{\varphi_k''(z)} + \overline{\psi_k'(z)}\right], & D_k \cup \partial D_k, \\ -\frac{1}{\mu}\left[z\overline{\varphi_0''(z)} + \overline{\psi_0'(z)}\right], & D \cup \partial D. \end{cases}$$

The perfect bonding at the matrix-inclusion interface yields the following conditions

$$\begin{aligned} &\varphi_k(t) + t\overline{\varphi_k'(t)} + \overline{\psi_k(t)} \\ &= \varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)}, \\ &\kappa_1\varphi_k(t) - t\overline{\varphi_k'(t)} - \overline{\psi_k(t)} \\ &= \frac{\mu_1}{\mu}\left[\kappa\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)}\right]. \end{aligned} \quad (6)$$

The functions $\varphi_0(z)$ and $\psi_0(z)$ satisfy conditions of quasi-periodicity Grigolyuk and Fil'shtinskii (1970, 1994); Andrianov et al. (2022).

3 Contrast elastic parameters

It is clear that the solution of the boundary value problem (6) with the corresponding quasi-periodicity conditions depends on three parameters, say $\frac{\mu_1}{\mu}$, κ and κ_1. Therefore, the effective properties also depend on these three parameters. It was proved that the number of parameters can be reduced to two Dundurs (1967)

$$\alpha = \frac{\frac{\mu_1}{\mu}(\kappa+1) - (\kappa_1+1)}{\frac{\mu_1}{\mu}(\kappa+1) + (\kappa_1+1)}, \ \beta = \frac{\frac{\mu_1}{\mu}(\kappa-1) - (\kappa_1-1)}{\frac{\mu_1}{\mu}(\kappa+1) + (\kappa_1+1)}. \quad (7)$$

Introduce the contrast parameters Mityushev and Drygas (2019); Drygaś et al. (2019)

$$\varrho_1 = \frac{\frac{\mu_1}{\mu} - 1}{\frac{\mu_1}{\mu} + \kappa_1}, \ \varrho_2 = \frac{\kappa\frac{\mu_1}{\mu} - \kappa_1}{\kappa\frac{\mu_1}{\mu} + 1}, \ \varrho_3 = \frac{\frac{\mu_1}{\mu} - 1}{\kappa\frac{\mu_1}{\mu} + 1}. \quad (8)$$

Two of them are independent, and the third parameter can be expressed by two ones. For instance,

$$\varrho_2 = 1 + \varrho_3 - \frac{\varrho_3}{\varrho_1}. \quad (9)$$

One can see that $\varrho_1 = \varrho_3 = 0$ and $\varrho_2 = \frac{\kappa - \kappa_1}{\kappa + 1}$ for $\mu_1 = \mu$. All the contrast parameters are proportional to $\frac{\mu_1}{\mu} - 1$ for $\kappa = \kappa_1,$.

The contrast parameters are related to the Dundurs parameters through the simple equations

$$\varrho_1 = \frac{\alpha - \beta}{1 - \beta}, \ \varrho_2 = \frac{\alpha + \beta}{1 + \beta}, \ \varrho_3 = \frac{\alpha - \beta}{1 + \beta} \quad (10)$$

and

$$\alpha = \frac{\varrho_2 + \varrho_3}{2 - \varrho_2 + \varrho_3}, \quad \beta = \frac{\varrho_2 - \varrho_3}{2 - \varrho_2 + \varrho_3} \quad (11)$$

Two of the contrast parameters (8) will be considered an alternative pair to Dundurs' pair (7). The importance of the contrast parameters (8) resides in their role in representing complex potentials through power series expansions in ϱ_j, as detailed below.

Consider the case of equal circular inclusions

$$D_k = \{z \in : |z - a_k| < r\} \ (k = 1, 2, \ldots, N), \ (12)$$

where a_k denotes the center of kth inclusion, r their radius. In this case, it is convenient to introduce the complex potential in the disk $|z - a_k| \le r$

$$\Phi_k(z) = \left(\frac{r^2}{z - a_k} + \overline{a_k}\right)\varphi_k'(z) + \psi_k(z). \quad (13)$$

The function $\Phi_k(z)$ is considered instead of $\psi_k(z)$.

The following functions are introduced in the non-connected domain of inclusions $\cup_{k=1}^n (D_k \cup \partial D_k)$

$$\boldsymbol{\varphi}(z) = \varphi_k(z), \ \boldsymbol{\Phi}(z) = \Phi_k(z), \ z \in D_k \cup \partial D_k. \quad (14)$$

The boundary value problem (6) is reduced to functional equations

$$\boldsymbol{\varphi}(z) = \varrho_1(A_{11}\boldsymbol{\varphi})(z) + \varrho_1(A_{12}\boldsymbol{\Phi})(z) + F_1(z), \ (15)$$

$$\begin{aligned} &\boldsymbol{\Phi}(z) = \varrho_2(A_{21}\boldsymbol{\varphi})(z) + \varrho_3(A_{22}\boldsymbol{\varphi})(z) \\ &+\varrho_3(A_{23}\boldsymbol{\Phi})(z) + F_2(z), \ |z - a_k| \le r, \quad (16) \\ &k = 1, 2, \ldots, N. \end{aligned}$$

The compact operators A_{kl} depend analytically on the radius r^2. The operators A_{kl} and the given functions $F_{1,2}(z)$ are exactly written in Mityushev and Drygas (2019); Drygaś et al. (2019).

Let constant components of the symmetric stress tensor σ_{jl}^∞ be given at infinity. Application of successive approximations to equations (15)-(16) and expansion of the obtained series on r^2 yield the following expansions for the complex potentials in the inclusions

$$\begin{aligned} &\boldsymbol{\varphi}(z) = (1 + \varrho_3)\textstyle\sum_{l_1,l_2,l_3=0}^{\infty} r^{2l_1}\varrho_2^{l_2}\varrho_3^{l_3}\boldsymbol{\varphi}_{l_1 l_2 l_3}(z) \\ &+constant, \quad z \in \cup_{k=1}^n D_k \end{aligned} \quad (17)$$

in the case $\sigma_{12}^{\infty} = 0$. A constructive algorithm to analytically determine the functions $\boldsymbol{\varphi}_{l_1 l_2 l_3}(z)$ using the successive approximations is developed in Mityushev and Drygas (2019); Drygaś et al. (2019). The formula (17) can be considered as a decomposition of $\boldsymbol{\varphi}(z)$ onto the linear combination of terms $\boldsymbol{\varphi}_{l_1 l_2 l_3}(z)$ depending only on the locations of inclusions with the coefficients $\varrho_2^{l_2}\varrho_3^{l_3}$ depending only on the elastic constants multiplied by r^{2l_1}. The function $\boldsymbol{\Phi}(z)$ has an analogous structure. The complex potential in the domain D can be represented in the analogous form

$$\varphi_0(z) = Bz + \sum_{l_1,l_2,l_3=0}^{\infty} r^{2l_1}\varrho_2^{l_2}\varrho_3^{l_3+1}\widetilde{\varphi}_{l_1 l_2 l_3}(z)$$
$$+constant, \quad z \in D. \tag{18}$$

where $B = \frac{1}{2}(\sigma_{11}^{\infty} + \sigma_{22}^{\infty})$.

In the case $\sigma_{11}^{\infty} = \sigma_{22}^{\infty} = 0$, a similar linear combination holds for $\varphi(z)$ in D but with the coefficients ρ_1 and ρ_3. Therefore, application of successive approximations in this case and the corresponding expansions yields the same representations (17)-(18) but with other basic functions $\varphi_{l_1 l_2 l_3}(z)$ and $\widetilde{\varphi}_{l_1 l_2 l_3}(z)$.

Analogous expansions were derived in Drygaś et al. (2019) for a set of disks D_k of different radii occupied by other locally isotropic composites.

A disk packing can approximate any smooth domain D_k. This observation concludes that the expansions on contrast parameters take place for various shapes of inclusions. The radius r is replaced to shape parameters of inclusions similar to the conductivity problem Mityushev and Rylko (2022).

Some restrictions must be imposed for the proper limit transition from disks to other shapes. They can be illustrated in Figures 1-3. The elastic constants are called not critical if $m_1 < \mu, \mu_1 < M_1$ and $m_2 < k, k_1 < M_1$ for fixed positive constants $m_1 < M_1$, $m_2 < M_2$. The elastic constants of inclusions (and inclusions) are called critical in the following four cases: $\mu_1 = 0$, $k_1 = 0$, $\mu_1 = \infty$, $k_1 = \infty$. The critical constants correspond to holes and perfectly rigid inclusions. Let Figure 1 represent the inclusions without critical constants and Figure 2 with critical constants. This scheme of disk packing with critical constants changes the concentration of inclusions since the domain bounded by four disks and occupied by non-critical materials is isolated and, hence, refers to the critical elastic domain. Such a case in Figure 2 has to be considered by a separate packing. The case in Figure 3 is proper since the critical inclusions are separated.

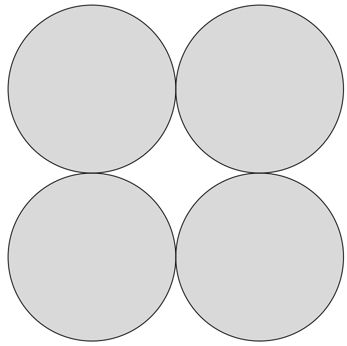

Figure 1: Four touching disks form a proper packing.

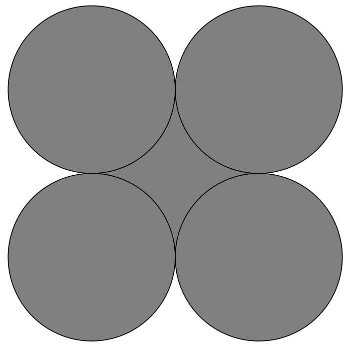

Figure 2: Four rigid touching disks with the extreme parameter $\frac{\mu_1}{\mu} = \infty$ do not form a proper packing.

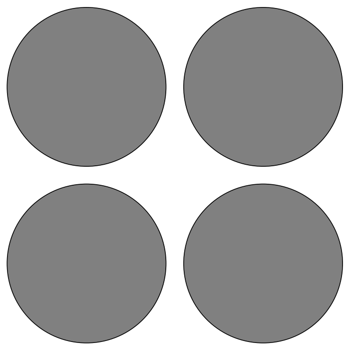

Figure 3: Four separated disks with the extreme parameter form a proper packing.

4 Invariant transformation of contrast elastic parameters for two-phase composites

Theorem 1 *Let $g(x)$ be a function continuous for $0 \le x \le +\infty$. Introduce the function*

$$h(x) = \frac{x\left(\frac{\mu_1}{\mu} - 1\right)}{\frac{x+\mu_1}{g(\mu)} - \frac{\mu_1}{\mu}\frac{x+\mu}{g(\mu_1)}} \tag{19}$$

and the transformation of four elastic constants to four other elastic constants

$$\{\mu, k, \mu_1, k_1\} \mapsto \{\mu', k', \mu_1', k_1'\} \equiv$$
$$\{g(\mu), h(k), g(\mu_1), h(k_1)\} \tag{20}$$

The contrast parameters (8) *are invariant under the transformation* (20)*. Therefore, the local elastic fields are expressed in terms of the same complex potentials.*

Proof. In order to prove the theorem, we first express the contrast parameters by the shear and bulk modulus using the relation

$$\varrho_1 = \frac{k_1\left(\frac{1}{\mu} - \frac{1}{\mu_1}\right)}{2 + k_1\left(\frac{1}{\mu} + \frac{1}{\mu_1}\right)}, \quad \varrho_2 = \frac{\frac{2}{k} - \frac{2}{k_1} + \frac{1}{\mu} + \frac{1}{\mu_1}}{\frac{2}{k} + \frac{1}{\mu} + \frac{1}{\mu_1}},$$
$$\varrho_3 = \frac{\frac{1}{\mu} - \frac{1}{\mu_1}}{\frac{2}{k} + \frac{1}{\mu} + \frac{1}{\mu_1}}. \tag{21}$$

Substituting the constants (20) to equations (21), we arrive at the same contrast parameters after calculations. The calculation can be performed by $Mathematica^{®}$ as shown in Appendix.

The theorem is proved.

Remark 1 *Though the function $g(x)$ is arbitrary and $h(x)$ includes $g(x)$, some conditions related to the mechanical restrictions for the elastic constants must be satisfied.*

Theorem 1 can be formulated in the following equivalent way

Theorem 2 *Consider two-phase dispersed composites with the same geometry. Let the elastic constants μ, k and μ_1, k_1 be assigned to the host and inclusions, respectively, in one composite; μ', k' and μ_1', k_1' to another composite. Let*

$$k' = \frac{k\left(\frac{\mu_1}{\mu}-1\right)}{\frac{k+\mu_1}{\mu'} - \frac{\mu_1}{\mu}\frac{k+\mu}{\mu_1'}},$$
$$k_1' = \frac{k_1\left(\frac{\mu_1}{\mu}-1\right)}{\frac{k_1+\mu_1}{\mu'} - \frac{\mu_1}{\mu}\frac{k_1+\mu}{\mu_1'}}. \quad (22)$$

Then, the contrast parameters (8) *are the same for the elastic constants $\{\mu, k, \mu_1, k_1\}$ and $\{\mu', k', \mu_1', k_1'\}$.*

Consider the function

$$g(x) = \left(\frac{1}{x} - \frac{1}{\lambda}\right)^{-1}. \quad (23)$$

where λ is an arbitrary constant parameter. Substituting it into (19) we obtain

$$h(x) = \left(\frac{1}{x} + \frac{1}{\lambda}\right)^{-1}. \quad (24)$$

This is the shift transformation Cherkaev et al. (1992) discussed in the Introduction for general composites.

We now repeat some arguments from Cherkaev et al. (1992) on the bounds of the effective constants, replacing the special transformation (23)-(24) by the general transformation (20) for two-phase composites. For definiteness, it is assumed that $\mu_1 \geq \mu$ and $k_1 \geq k$. Let the considered composite be macroscopically isotropic. Hence, its behavior is described by two constants k_e and μ_e. These constants change in the same way according to the transformation (20)

$$g(\mu_e') = g(\mu_e) \text{ and } h(k_e') = h(k_e) \quad (25)$$

The Voight bound on k_e' has the form

$$k_e' \leq \langle k' \rangle \equiv f k_1' + (1-f)k', \quad (26)$$

where f denotes the concentration of the phase with k_1 in the composite. Substitution of (20) into (26) yields the inequality

$$h(k_e) \leq f h(k_1) + (1-f)h(k). \quad (27)$$

Substitute (19) into (27) assuming that $\frac{1}{g(\mu_1)} = 0$. After simplifications, we arrive at the upper Hashin-Shtrikman bound

$$k_e \leq k_1 + \frac{1-f}{\frac{1}{k-k_1} + \frac{f}{k_1+\mu_1}}. \quad (28)$$

5 Conclusion and discussion

The local stress and strain fields in composites and the effective constants depend on many parameters. The problem of reducing the number of independent and dimensionless variables is an essential problem of scaling analysis on elasticity. The number of material constants used for the local fields and the effective constants for two-phase composites was first reduced to two Dundurs (1967). We propose to use other pairs of contrast parameters.

A class of invariant transformations for contrast parameters is introduced in Theorem 1. This transformation extends the CLM transformation Cherkaev et al. (1992) for two-phase composites.

It is interesting to extend the obtained results to multi-phase composites. The study of dispersed multiphase composites with inclusions D_m occupied by an elastic material with the constants μ_m and k_m can be easily extended. The contrast parameters for an N-phase composites are introduced similar to (8)

$$\varrho_{1m} = \frac{\frac{\mu_m}{\mu}-1}{\frac{\mu_m}{\mu}+\kappa_m}, \; \varrho_{2m} = \frac{\kappa\frac{\mu_m}{\mu}-\kappa_m}{\kappa\frac{\mu_m}{\mu}+1},$$
$$\varrho_{3m} = \frac{\frac{\mu_m}{\mu}-1}{\kappa\frac{\mu_m}{\mu}+1}, \; m = 1, 2, \ldots, N. \quad (29)$$

$2N$ parameters, for instance, ϱ_{1m} and ϱ_{3m} ($m = 1, 2, \ldots, N$) determine the local fields in composite Drygaś et al. (2019).

An extension of Theorem 1 to multi-phase composites is an open problem. It is equivalent to extensions of Dundurs' approach Dundurs (1967); Dundurs and Jasiuk (1997).

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Appendix. Mathematica® codes used in the proof of Theorem 1.

In[1]:= $$h[k\_] = \frac{k\,(-\mu + \mu1)\, f[\mu] \times f[\mu1]}{-((k+\mu)\,\mu1\, f[\mu]) + \mu\,(k + \mu1)\, f[\mu1]};$$

In[2]:= $$\rho1[\mu\_, \varkappa\_, \mu1\_, \varkappa1\_] = \frac{\frac{\mu1}{\mu} - 1}{\frac{\mu1}{\mu} + \varkappa1};$$

$$\rho2[\mu\_, \varkappa\_, \mu1\_, \varkappa1\_] = \frac{\varkappa\frac{\mu1}{\mu} - \varkappa1}{\varkappa\frac{\mu1}{\mu} + 1};$$

$$\rho3[\mu\_, \varkappa\_, \mu1\_, \varkappa1\_] = \frac{\frac{\mu1}{\mu} - 1}{\varkappa\frac{\mu1}{\mu} + 1};$$

In[3]:= $$\rho k1[\mu\_, k\_, \mu1\_, k1\_] = \rho1\left[\mu, \frac{2\mu}{k} + 1, \mu1, \frac{2\mu1}{k1} + 1\right] \text{ // Simplify}$$

Out[3]= $$\frac{k1\,(-\mu + \mu1)}{2\mu\,\mu1 + k1\,(\mu + \mu1)}$$

In[4]:= $$\rho k2[\mu\_, k\_, \mu1\_, k1\_] = \rho2\left[\mu, \frac{2\mu}{k} + 1, \mu1, \frac{2\mu1}{k1} + 1\right] \text{ // Simplify}$$

Out[4]= $$\frac{-2\,k\,\mu\,\mu1 + 2\,k1\,\mu\,\mu1 + k\,k1\,(-\mu + \mu1)}{k1\,(2\mu\,\mu1 + k\,(\mu + \mu1))}$$

In[5]:= $$\rho k3[\mu\_, k\_, \mu1\_, k1\_] = \rho3\left[\mu, \frac{2\mu}{k} + 1, \mu1, \frac{2\mu1}{k1} + 1\right] \text{ // Simplify}$$

Out[5]= $$\frac{k\,(-\mu + \mu1)}{2\mu\,\mu1 + k\,(\mu + \mu1)}$$

In[6]:= $$\rho k1[f[\mu], h[k], f[\mu1], h[k1]] == \rho k1[\mu, k, \mu1, k1] \text{ // Simplify}$$

Out[6]= True

In[7]:= $$\rho k2[f[\mu], h[k], f[\mu1], h[k1]] == \rho k2[\mu, k, \mu1, k1] \text{ // Simplify}$$

Out[7]= True

In[8]:= $$\rho k3[f[\mu], h[k], f[\mu1], h[k1]] == \rho k3[\mu, k, \mu1, k1] \text{ // Simplify}$$

Out[8]= True