ACOUSTIC MANIFESTATATIONS OF SYMMETERY BREAKING IN SELF-SIMILAR SIGNALS

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ABSTRACT

A gradual construction of signal based on the Weierstraß-Mandelbrot function starting from a single pure tone up to a fully developed self-similar waveform allows one to aurally perceive gain/loss of self-similarity symmetry. A numerically synthesized example involves a scaling transformation with the value of scaling factor increasing continuously from unity to the value defining the self-similarity of the full Weierstraß-Mandelbrot function. In the case of partially self-similar signals the sequence of a sequential repetition of this transformation is readily perceptible, whereas for the fully self-similar signal the repetitions manifest themselves as an ever-ascending glissando specific to Shepard tones. The usefulness of the concept of self-similarity in the condensed matter physics is discussed.

1 WEIERSTRAß-MANDELBROT FUNCTION

Self-similar objects remain unchanged (invariant) under a rescaling of their coordinates. The rescaling of a coordinate, or just scaling transformation, is meant here as multiplication of the coordinate with a numerical factor. The multiplication of numbers having all the properties of group: closure, associativity, identity and inverses (except for 0) (Dresselhaus, 2008) the self-similarity can be regarded as a kind of symmetry group. A known example is the 2D Sierpinski's triangle gasket (Zhao, 2016) which preserves its shape under multiplication of its both coordinates with 1, 2,1/2, 4, 1/4, 8, 1/8…i.e. all the integer powers of 2, negative powers included.

A prototypic self-similar function of one variable is the generalized Weierstraß function (K. Weierstraß, 1895). Originally presented in a lecture for the Prussian Academy of Sciences the 18th of July 1872, the function was an example of a curve which was everywhere continuous, but nowhere differentiable. G. H. Hardy remarked that the range of parameters b might be extended compared to the original Weierstraß concept (Hardy, 1916). In turn, B. Mandelbrot found a self-similar nature of the Weierstraß function by extending the summation to minus infinity (B. Mandelbrot, 1983). The formula for the so generalized Weierstraß-Mandelbrot function thus reads

$$W(t) = \sum_{k=0\ (-\infty)}^{\infty} \frac{\cos(b^k 2\pi f_0 t)}{b^{k(2-D)}}, \qquad (1)$$

where the parameter b should be any real number $b > 1$ and D is the box-counting dimension of the graph of the function, conjectured to be equal to the Hausdorff dimension of the graph. (Zaleski, 2012). The frequency f_0 is an arbitrary real number, One can easily see that upon rescaling the argument t by multiplying it with the factor b the function becomes multiplied with $1/b^{D-2}$. This property marks so called self-affine objects, i.e. those which upon scaling become multiplied with a factor. In the limit $D \rightarrow 2$ the function is scaling invariant, i.e. self-similar. For $D < 2$ one should stop the summation on the side of negative powers k to avoid divergence of the factors $\frac{1}{b^{k(2-D)}}$. However, in the application to time series the limit $k \rightarrow -\infty$ corresponds to long-range trends that can be often removed without loosing essential features of the signal (see T. Stanisz et al., 2024 and references given therein). When used to construct acoustic signals the low-frequency partials are immaterial since they fall outside the audible range.

2 SHEPARD TONES

The formula of Eq. (1) represents a trigonometric series, which becomes a Fourier series for integer parameters b. The corresponding acoustic signal, i.e. time-dependent acoustic pressure, then is a harmonic polyton characteristic of signals with well-defined pitch, at least pitch class (Lerdahl, 2001). In particular, for $b = 2$ all the neighbouring partial tones are an octave apart. For irrational values of the parameter b the signal is intrinsically aperiodic although its spectrum is still discrete. But what holds for any value of b is that the frequency ratios between the neighbouring partials are equal so that the signal is a stack of pure tones equidistant in the pitch scale. An example is schematized in Figure 1.

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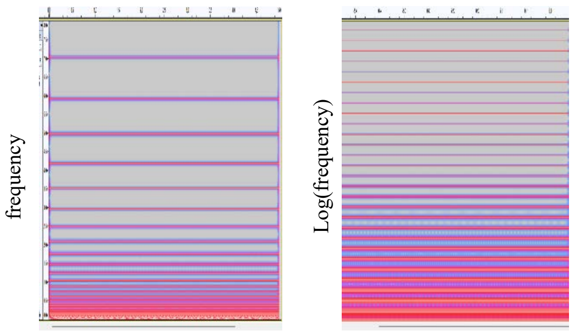

Figure 1. Spectrogram of signal synthesized according to Eq. (1) in linear scale (left) and in logarithmic i.e. pitch scale (right, differences in quality of lines result from graphical resolution).

Looking at the right panel of Figure 1 one realizes that the self-similarity of the signal is equivalent to translational discrete invariance (periodicity) in the pitch scale. Of course it is very analogous to spatial periodicity characteristic of crystals. Here, however, the translations concern pitch. A rescaling of time with any scaling factor translates the ladder of the right panel of Figure 1 by an amount proportional to a logarithm of the factor. When, however, the factor is a power of the parameter b of Eq. (1) the signal overlaps with itself. This underlies the infinite glissando or circularity of pitch discovered in 1964 by R. Shepard for the case of $b = 2$ (Shepard, 1964). It will be shown that the circularity occurs in a wider range of the parameter b. One should note that the signal given by the time series of Eq. (1) is entirely deterministic in contrast with many natural time series resulting from random fluctuations and/or chaotic phenomena.

3 SYMMETRY BREAKING

Phases transitions are often encountered in the realm of crystals. The phenomenon involves a change of symmetry controlled by variations of external parameters such as temperature, pressure, concentration of dopants etc., the temperature being the best commonly known. Usually a high-temperature, and at the same time high-symmetry, initial phase undergoes a reduction of symmetry so that the symmetry group of the low-symmetry phase becomes a subgroup of the symmetry group of the initial phase (G.Ya. Lyubarskii, 1960). This means that some symmetry elements, i.e. proper or improper rotations and/or translations belonging to the high-symmetry group are lost. The control parameters being most often spatially isotropic, the disappearance of some symmetry operations is an instance of spontaneous symmetry breaking (Boccara, 1976).

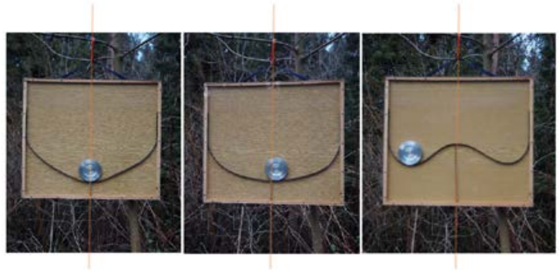

Figure 2. Model (made by Erazm M. Dutkiewicz) illustrating spontaneous symmetry breaking. Here the lost symmetry element is the mirror plane represented by the yellow trace.

Unless the crystal cracks or pulverises the loss of rotational symmetry elements may entail a ferroic texture, i.e. a pattern of domains each of which shows the lowered symmetry. Different domains are related by the lost symmetry elements. An example of ferroelastic texture can be found in (Moskwa et al., 2020). The loss of translational symmetry produces antiphase domains resulting from doubling, tripling etc. of the unit cell. A condensation of a periodicity incommensurate with respect to the initial lattice gives rise to theoretically infinite number of domains and a phason mode (T. Wasiutyński and H. Cailleu, 1992) which is an example of Goldstone mode. Noteworthy is that fluctuations occurring in the vicinity of the critical point, i.e. the very temperature separating both phases, in the case of continuous phases transitions are spatially self-similar that underlies the renormalization group technique of the quantitative description of critical phenomena (K. G. Wilson, 1971 and further publications on the subject).

4 GAIN/LOSS OF SELF-SIMILARITY SYMMETRY

Ferroic textures show sometimes patterns exhibiting a number of levels of self-similarity, as though the series of Eq. (1) contained only finite number of terms. The star patterns reported by (Bulesteix and Yangui, 1983) are the most convincing examples. This is, thus, an instance of broken self-similarity symmetry. It resembles rather a crystal growth or evaporation than a condensation of a wave encompassing a number of unit cells in an infinitely extended system. Crystal growth occurs by adding subsequent unit cells to an already existing seed. Translating this into the periodic system of pitches depicted in the right panel of Figure 1 one may imagine a gradual construction of the signal starting from a single pure tone and adding subsequent pure tones distant by the same pitch interval up to a fully developed self-similar waveform being represented by an infinite ladder. Of course, one can also realize a change of periodicity starting from an infinite ladder and imposing a breaking of translational symmetry by changing the tone amplitudes and/or their pitch intervals. This will be exemplified in the presentation.

The theory of self-similarity symmetry breaking and related phase transitions is not known to the author as yet. We can, nevertheless, represent the corresponding phenomena with acoustic synthesis. I have synthesized examples of different scenarios of such a symmetry breaking. In Figure 3 we can see a process of agglomeration of the terms in Eq. (1) i.e. a symmetry gain and loss analogous to crystal growth and evaporation. The thought, but also auditory perceptible, experiment depicted in Figure 3 is the following. We rescale the time coordinate continuously, by increasing the scaling factor from 1 to $b = 2^{3/12}$ i.e. to 3 tempered halftones which constitute an interval of equally tempered minor third and repeat this rescaling in sequence. We start from a "seed" of the Weierstraß-Mandelbrot function consisting of a single term in Eq. (1) but then we systematically add more and more neighbouring tones a minor third away from the preceding one as it is seen in Figure 3. After filling the whole audible region the number of partials is decreased back as it is seen in the right part of Figure 3. The listener notices the repetitions of the rescaling as long as the number of partials is low enough. However, the "suture" starts to be indiscernible for the high enough number of partials as it the case for the Shepard tones. Sonifications of other scenarios of the self-similarity gaining/loosing have been carried out.

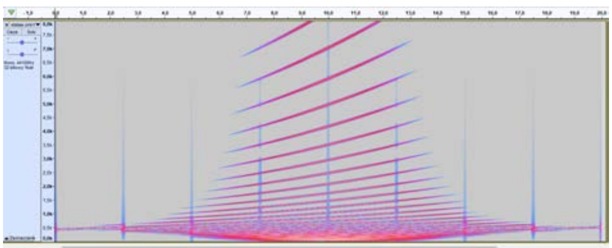

Figure 3. Spectrogram of thought experiment with increasing and decreasing number of partials in Weierstraß-Mandelbrot signal of Eq. (1). The vertical lines indicate the period of rescaling time variable from 1 to $b = 2^{3/12}$.

5 CONCLUSION AND PROSPECTIVES

An infinite glissando effect can be achieved in a wide range of the scaling factor b in the Weierstraß-Mandelbrot function signals given in Eq. (1). The signals are generalizations of Shepard tones. The gaining and loosing of self-similarity symmetry can be illustrated with the use of sound synthesis. Scenarios analogous to crystal growth/evaporation and multiplication/reduction of the size of unit cell are to be presented at the ECMS2024 symposium. An open question is whether similar procedures may be conceived for the partly self-similar ferroic textures and other condensed matter systems.

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