THE $MAP/MSP/1/r$ QUEUEING SYSTEM WITH BACKGROUND CUSTOMERS

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ABSTRACT

A single server queueing system with Markov flow of principal customers and background customers arriving from infinite reserve source is studied. The Markov service processes of customers of both types are considered. The system has a finite buffer for principal customers. The service discipline is characterized by non-preemptive priority of principal customers respect to background customers. An effective matrix-recurrent algorithm for the calculation of the stationary state probabilities of the underlying Markov process is derived.

1 SYSTEM DESCRIPTION

Let us consider a single server queueing system. We assume arrivals of two types: principal customers called 1-customers and background customers called 2-customers.

The arrivals of 1-customers form a Markov flow (MAP) defined by the matrices Λ and N of order l. The matrix Λ sets the intensities of the generation phase transitions of the process (which controls 1-customers' generation) without customers appearance in the system, while the matrix N defines the transitions with appearance of a new 1-customer.

There is a buffer with r $(0 < r < \infty)$ waiting places for 1-customers. The 1-customers are lost if they arrive in the system at the instant when the buffer is full.

The source of the background customers is able to generate a new customer at any time instant, i.e. possesses an infinite reserve of 2-customers.

The service process of i-customers is a Markov one (MSP — Markov service process) defined by the matrices M_i, S_i of order m_i, $i = 1, 2$. Here the matrix M_i sets the intensities of the service phase transitions of the process (which controls i-customers service) without service completion, while the matrix S_i defines the transitions with completion of customer service. When the customer type in the server is changed to i then the initial service phase of the new customer beginning the service is defined by the vector $\boldsymbol{\beta}_i$ of initial probabilities of the service process of i-customers.

The principal customers are served with non-preemptive priority respect to background customers. This means that after the service completion of an 1-customer a new 1-customer from the queue of principal customers in the buffer goes into the server. If there are no principal customers in the queue then the service process of background customers starts, i.e. a new 2-customer enters immediately the server. If during the service of a 2-customer a new 1-customer enters the system he joins the buffer's queue without interrupting the service process, and gets service immediately after the 2-customer service is completed.

The service systems with background customers are widely used in modern information transmission technologies. For instance, the use of the signalization system in a node of the telephone network with common signal channel supposes that the signalling traffic is a background one unlike the principal voice traffic (Samuilov 2004).

The $PH/PH/1/r$ queueing system with two types of customers and non-preemptive priority was studied in (Bocharov 1989) where a matrix algorithm for the calculation of the stationary state probabilities of the system was obtained.

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In this paper the results of (Bocharov 1989) are extended to the case of an input flow of principal customers of MAP-type and service of principal customers and background ones of MSP-type. A computation algorithm for the stationary state probabilities of the $MAP/MSP/1/r$ queueing system is derived. This algorithm allows to compute performance measures of the considered system for large values of buffer capacity and it is far more efficient in this sense than the algorithm derived in (Bocharov 1989).

2 STOCHASTIC MODEL

The queueing system under consideration is described by a Markov process $\{X(t), \quad t \geq 0\}$ with the state set

$$\mathcal{X} = \bigcup_{i=1}^{2} \bigcup_{k=2-i}^{r+u(2-i)} \mathcal{X}_{ki},$$

where

$$\mathcal{X}_{ki} = \{(i,j,k,s), \quad i=1,2, \quad j=\overline{1,l},$$
$$k=\overline{2-i, r+u(2-i)}, \quad s=\overline{1,m_i}\};$$

here $u(y)$ is the Heaviside function.
For a time instant t, the state $X(t) = (i,j,k,s)$ of the process $X(t)$ means that at this instant an i-customer is attended by the server, the generation process of an 1-customer is in phase j, there are k 1-customers in the system, and the service process of the i-customer is in phase s.

The total number $k(i)$ of customers in the system standing in the state (i,j,k,s) is determined by the following relations:

$$k(i) = \begin{cases} k, & \text{if } i=1, \\ k+1, & \text{if } i=2, \end{cases}$$

where $k(i) = \overline{1,R}$, $i=1,2$, and $R = r+1$ is the queueing system capacity.
In such a way, if an 1-customer is attended by the server then the maximal number of 1-customers who can be placed in the system simultaneously is equal to R; but if a 2-customer is served then no more than r 1-customers can be placed in the system.
The total number $|\mathcal{X}|$ of the states of the process $X(t)$ is

$$|\mathcal{X}| = lR(m_1+m_2).$$

It is easily to show that the process $\{X(t), \quad t \geq 0\}$ is ergodic, and its final probabilities $\lim_{t\to\infty} \mathcal{P}\{X(t) = x\} = p_x > 0, \quad x \in \mathcal{X}$, do not depend on the initial state $X(0)$ of the process and coincide with stationary state probabilities.

3 SYSTEM OF EQUILIBRIUM EQUATIONS

Let us set $\boldsymbol{\lambda} = -\Lambda\mathbf{1}, \quad \boldsymbol{\mu}_i = -M_i\mathbf{1}$; here $\mathbf{1}$ is the vector consisting of ones which dimension depends on the context. The component λ_n of the vector $\boldsymbol{\lambda}$ is the arrival intensity of an 1-customer from phase n, and the component μ_{iv} of the vector $\boldsymbol{\mu}_i$ is the departure intensity from the server of an i-customer, who completed his service at the phase v. Let us note that the following equalities hold: $\boldsymbol{\lambda} = N\mathbf{1}$ and $\boldsymbol{\mu}_i = S_i\mathbf{1}$.

Further, we introduce the vectors

$$\mathbf{p}_{ki}^T = (p_{1,1,k,i}, \ldots, p_{1,m_i,k,i},$$
$$\ldots, p_{l_1,1,k,i}, \ldots, p_{l_1,m_i,k,i}),$$
$$\mathbf{p}^T = (\mathbf{p}_{0,2}^T, \mathbf{p}_{1,1}^T, \mathbf{p}_{1,2}^T, \mathbf{p}_{2,1}^T, \mathbf{p}_{2,2}^T, \cdots, \mathbf{p}_{r,2}^T, \mathbf{p}_{R,1}^T). \quad (1)$$

The vector $\mathbf{p}$ of stationary state probabilities satisfies the system of equilibrium equations (SEE)

$$\mathbf{p}^T A = \mathbf{0}^T, \quad (2)$$

where A is the generator of the process $\{X(t), t \geq 0\}$; here $\mathbf{0}$ is the vector of zeros which dimension depends on the context.
Taking into consideration the states' order which is defined, according to the formula (1), by vectors $\mathbf{p}_{ki}$ and $\mathbf{p}$, we write the SEE (2) in explicit form:

$$\mathbf{p}_{0,2}^T(\Lambda \oplus M_2 + I_l \otimes S_2) + \mathbf{p}_{1,1}^T(I_l \otimes \boldsymbol{\mu}_1\boldsymbol{\beta}_2^T) = \mathbf{0}_{lm_2}^T,$$
$$\mathbf{p}_{1,1}^T(\Lambda \oplus M_1) + \mathbf{p}_{2,1}^T(I_l \otimes S_1) + \mathbf{p}_{1,2}^T(I_l \otimes \boldsymbol{\mu}_2\boldsymbol{\beta}_1^T) = \mathbf{0}_{lm_1}^T,$$
$$\mathbf{p}_{k-1,1}^T(N \otimes I_{m_1}) + \mathbf{p}_{k,1}^T(\Lambda \oplus M_1$$
$$+ u(k-r)(N \otimes I_{m_1})) + u(r-k+1)(\mathbf{p}_{k+1,1}^T(I_l \otimes S_1)$$
$$+ \mathbf{p}_{k,2}^T(I_l \otimes \boldsymbol{\mu}_2\boldsymbol{\beta}_1^T)) = \mathbf{0}_{lm_1}^T, \quad k=\overline{2,r+1},$$
$$\mathbf{p}_{k-1,2}^T(N \otimes I_{m_2}) + \mathbf{p}_{k,2}^T(\Lambda \oplus M_2$$
$$+ u(k-r+1)(N \otimes I_{m_2})) = \mathbf{0}_{lm_2}^T, \quad k=\overline{1,r}, \quad (3)$$

where $U \otimes V$ is the Kronecker product of the matrices U and V, $U \oplus V = U \otimes I_n + I_m \otimes V$, and m, n are orders of the matrices U, V, respectively.
Besides, the following normalizing condition holds:

$$\mathbf{p}^T\mathbf{1} = 1. \quad (4)$$

4 SOLUTION OF EQUILIBRIUM EQUATION

Let a be the maximal absolute value of elements of the principal diagonal of the matrix A. Let us also put

$$P = I + \frac{1}{a}A. \quad (5)$$

Then we can write the SEE (2) in the form

$$\mathbf{p}^T P = \mathbf{p}^T. \tag{6}$$

whereby the matrix P presented, according to the formula (3), in explicit form can be written as follows:

$$\begin{pmatrix} P_{em} & 0 & B_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ C_0 & P_{od} & 0 & B_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & C & P_{ev} & 0 & B_2 & \cdots & 0 & 0 & 0 \\ 0 & D & 0 & P_{od} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & C & P_{ev} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & P_{od} & 0 & B_1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & C & P_{2r} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & D & 0 & P_{2r+1} \end{pmatrix}. \tag{7}$$

Here we use the following notations:

$$P_{od} = \frac{1}{a}A_1 + I_{lm_1}, \quad P_{ev} = \frac{1}{a}A_2 + I_{lm_2},$$

$$P_{em} = P_{ev} + \frac{1}{a}I_l \otimes S_2, \quad P_{2r} = P_{ev} + \frac{1}{a}N \otimes I_{m_2},$$

$$P_{2r+1} = P_{od} + \frac{1}{a}N \otimes I_{m_1}, \quad B_i = \frac{1}{a}B_i^{abs},$$

$$C = \frac{1}{a}C^{abs}, \quad C_0 = \frac{1}{a}C_0^{abs}, \quad D = \frac{1}{a}D^{abs},$$

where, in turn, we use the notations

$$A_i = \Lambda \oplus M_i, \quad i = 1, 2, \quad A_{2r} = A_2 + N \otimes I_{m_2},$$

$$A_{2r+1} = A_1 + N \otimes I_{m_1}, \quad B_i^{abs} = N \otimes I_{m_i}, \quad i = 1, 2,$$

$$C^{abs} = I_l \otimes \boldsymbol{\mu}_2 \boldsymbol{\beta}_1^T, \quad C_0^{abs} = I_l \otimes \boldsymbol{\mu}_1 \boldsymbol{\beta}_2^T, \quad D^{abs} = I_l \otimes S_1.$$

We note that the matrix P defined by the formula (5) is a stochastic matrix, the system of equations (6) is the SEE for a Markov chain defined by the matrix P, and the vector $\mathbf{p}$ of stationary state probabilities of the Markov process $\{X(t), \quad t \geq 0\}$ defines also the stationary distribution of this Markov chain. Taking this fact into account we find the vector $\mathbf{p}$ from the SEE (6). To find it we use the method of reduction of state set $\mathcal{X}$ of the Markov chain with subsequent "gluing" of its trajectory parts remaining after the next reduction and corresponding re-calculation of transition probabilities (see, for instance, (Bocharov et al. 2004), ch. 1, § 6).

To apply this approach to obtain the solution of the SEE (6) we introduce the following macrostates:

$$\mathcal{X}_q = \mathcal{X}_{q-1,2} \bigcup \mathcal{X}_{q,1}.$$

$$\mathcal{X} = \bigcup_{q=1}^{R} \mathcal{X}_q,$$

i.e. the subset $\mathcal{X}_q$ contains all states of the process $X(t)$ with q customers in the system.

Further, we introduce the vectors

$$\mathbf{p}_q^T = (p_{1,1,q-1,2}, \ldots, p_{1,m_2,q-1,2}, \ldots,$$
$$p_{l_1,1,q-1,2}, \ldots, p_{l_1,m_2,q-1,2}, p_{1,1,q,1}, \ldots,$$
$$p_{1,m_1,q,1}, \ldots, p_{l_1,1,q,1}, \ldots, p_{l_1,m_1,q,1}).$$

Then the vector $\mathbf{p}$ defined by the formula (1) can be written in the form

$$\mathbf{p}^T = (\mathbf{p}_1^T, \mathbf{p}_2^T, \cdots, \mathbf{p}_r^T, \mathbf{p}_R^T).$$

In turn, the matrix P can be presented as follows:

$$P = \begin{pmatrix} F_1 & B & 0 & \cdots & 0 & 0 \\ G & F & B & \cdots & 0 & 0 \\ 0 & G & F & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & F & B \\ 0 & 0 & 0 & \cdots & G & F_R \end{pmatrix} \tag{8}$$

where we use the notations

$$F_1 = \begin{pmatrix} P_{em} & 0 \\ C_0 & P_{od} \end{pmatrix}, \quad F = \begin{pmatrix} P_{ev} & 0 \\ 0 & P_{od} \end{pmatrix},$$

$$F_R = \begin{pmatrix} P_{2r} & 0 \\ 0 & P_{2r+1} \end{pmatrix}, \quad B = \begin{pmatrix} B_2 & 0 \\ 0 & B_1 \end{pmatrix},$$

$$G = \begin{pmatrix} 0 & C \\ 0 & D \end{pmatrix}. \tag{9}$$

Now we formulate a Lemma which presents the results in (Bocharov et al. 2004), ch. 1, § 6 in a matrix form for the case when transition probability matrix is a block three-diagonal one.

Lemma. *Let us set* $\mathbf{q}^T = (\mathbf{q}_1^T, \mathbf{q}_2^T, \ldots, \mathbf{q}_n^T)$, *where* $\mathbf{q}_i^T$ *is the vector of dimension* l_i, $i = \overline{0,n}$, *and let* $Q = (Q_{ij})_{i,j=\overline{0,n}}$ *be a stochastic matrix, where* Q_{ij} *is a matrix of order* $l_i \times l_j$ *and* $Q_{ij} = 0$ *if* $|i-j| > 1$. *Then the solution of the SEE*

$$\mathbf{q}^T = \mathbf{q}^T Q$$

is determined by the following algorithm:
at the first step the vector $\mathbf{q}_1$ *is found (accurate to a constant factor) from the system of equations*

$$\mathbf{q}_1^T = \mathbf{q}_1^T Q_{11}^{(n-1)},$$

where the matrix $Q_{11}^{(n-1)}$ *is determined by the following recurrent algorithm:*

$$Q_{n-k+1,n-k+1}^{(k)} = (I - QP_{n-k+1,n-k+1}^{(k-1)})^{-1},$$

$$Q_{n-k+1,n-k}^{(k)} = Q_{n-k+1,n-k+1}^{(k)} Q_{n-k+1,n-k},$$

$$Q^{(k)}_{n-k,n-k} = Q_{n-k,n-k}$$

$$+ Q_{n-k,n-k+1} Q^{(k)}_{n-k+1,n-k}, \quad k = \overline{1, n-1};$$

at the second step consequently (up to the constant factor too) the vectors $\mathbf{q}_i$, $i = \overline{2, n}$, *are determined by recurrent formula*

$$\mathbf{q}_i^T = \mathbf{q}_{i-1}^T Q_{i-1,i} Q_{ii}^{(n-i+1)};$$

at the last step the obtained probabilities are normalized in accordance to the condition

$$\mathbf{q}^T \mathbf{1} = 1.$$

This is a key Lemma to prove the following theorem.

Theorem. *Let* $\mathbf{p}$ *be a vector of the form (1) and the matrix* P *be of the form (7). Then the solution of the SEE (6) is defined by the following algorithm: at the first step the vector* $\mathbf{p}_{0,2}$ *is determined (up to a constant factor) from the system of equations*

$$\mathbf{p}_{0,2}^T P_0 = \mathbf{p}_{0,2}^T,$$

where the matrix P_0 *is determined by the following recurrent algorithm:*

$$P_{2r} = P_{ev} + N \otimes I_{m_2},$$

$$P_{2r+1} = P_{od} + N \otimes I_{m_1},$$

$$Q_{2r-1} = B_2 \left(I - P_{2r}\right)^{-1},$$

$$Q = B_2 \left(I - P_{ev}\right)^{-1},$$

$$\widetilde{P}_{2n+1} = \left(I - P_{2n+1}\right)^{-1}, \quad n = \overline{r, 0},$$

$$P_{2n-1} = P_{od} + B_1 \widetilde{P}_{2n+1} D, \quad n = \overline{r, 1},$$

$$P_0 = P_{em} + Q C_1 \widetilde{P}_1 C_0,$$

$$C_{2r-3} = C + Q_{2r-1} C \widetilde{P}_{2r-1} D,$$

$$C_{2n-1} = C + Q C_{2n+1} \widetilde{P}_{2n+1} D, \quad n = \overline{r-2, 1};$$

at the second step consequently the vectors $\mathbf{q}_i$, $i = \overline{2, n}$, *are determined (accurate to a constant too) by the formula*

$$\begin{cases} \mathbf{p}_{n,2}^T = \mathbf{p}_{n-1,2}^T Q, & n = \overline{1, r-1}, \\ \mathbf{p}_{r,2}^T = \mathbf{p}_{r-1,2}^T Q_{2r-1}, & \\ \mathbf{p}_{1,1}^T = \mathbf{p}_{1,2}^T C_1 \left(I - P_1\right)^{-1}, & \\ \mathbf{p}_{n,1}^T = \left(\mathbf{p}_{n-1,1}^T B_1 + \mathbf{p}_{n,2} C_{2n-1}\right) & \\ \quad \times \left(I - P_{2n-1}\right)^{-1}, & n = \overline{2, r}, \\ \mathbf{p}_{r+1,1}^T = \mathbf{p}_{r,1}^T B_1 \left(I - P_{2r+1}\right)^{-1}, & \end{cases}$$

at the last step the obtained probabilities are normalized in accordance to the condition (4).

The proof of the theorem follows after some algebra from the Lemma and formulas (7)–(9).

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