TRANSIENT ANALYSIS OF A DISCRETE-TIME PRIORITY QUEUE

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KEYWORDS

Stochastic Models, Performance Modelling

ABSTRACT

In this paper, we present a transient analysis of the system contents in a two-class priority queue with single-slot service times. In particular, we derive an expression for the generating function of the transient system contents of both classes. Performance measures are calculated from this generating function. Finally, by means of a numerical example we investigate the impact of the traffic parameters on the transient performance measures.

INTRODUCTION

In this paper, we focus on the effect of a Head-Of-Line (HOL) priority scheduling discipline on the transient behavior of a queue. We assume that one type of arriving traffic (type-1) has priority over another type of traffic (type-2), i.e., when the server becomes idle, a packet of type-1 traffic, when available, will always be scheduled next. Only when there are no packets of type-1 present, a packet of type-2 (if any) can be scheduled for transmission. This type of scheduling is used when traffic of type-1 is more delay-sensitive than traffic of type 2. For example, voice traffic can be given priority over data traffic in multimedia networks.

Although the steady-state behavior of discrete-time HOL priority queues is well understood (see (Sidi and Segall 1983), (Khamisy and Sidi 1992), (Laevens and Bruneel 1998) and (Walraevens et al. 2003)), results on the transient behavior of priority queueing systems are scarce (e.g. (Krinik et al. 2002)). Examples of a transient analysis in buffers with a FIFO queueing discipline can be found in (Bruneel 1988) and (Bruneel 1991).

We perform an analysis based on generating functions for assessing the time-dependent performance of discrete-time buffers with a priority scheduling discipline in this paper. The numbers of per-slot arrivals are i.i.d. and the service times are deterministically equal to one slot. From the obtained generating functions, we then calculate expressions for some interesting performance measures, such as the time-dependent probability of having an empty buffer and the time-dependent mean system contents of both priority classes.

The outline is as follows. In the following section, we introduce the mathematical model. We then investigate the time-dependent system contents. We first relate the generating function of the time-dependent system contents to some unknown boundary functions, which are investigated in the next section. We finally illustrate our approach by means of a numerical example and formulate some conclusions.

MATHEMATICAL MODEL

We consider a single-server queueing system with infinite buffer space. Time is assumed to be slotted and one server transmits packets at the rate of one packet per slot. Two types of traffic are arriving to the system, namely packets of type-1 and packets of type-2. We denote the number of arrivals of type-j during slot k by $a_j^{(k)}$ ($j = 1, 2$). The numbers of arrivals are assumed to be i.i.d. from slot-to-slot and are characterized by the joint probability generating function (pgf)

$$A(z_1, z_2) = \mathrm{E}\left[z_1^{a_1^{(k)}} z_2^{a_2^{(k)}}\right].$$

Notice that the number of arrivals from different types can be correlated (within a slot). Furthermore, we denote the total number of arriving packets during slot k by $a_T^{(k)} \triangleq a_1^{(k)} + a_2^{(k)}$ and its pgf is defined as

$$A_T(z) \triangleq \mathrm{E}\left[z^{a_T^{(k)}}\right] = A(z, z).$$

Proceedings 12th International Conference ASMTA 2005
Khalid Al-Begain, Gunter Bolch, Miklos Telek © ECMS, 2005
ISBN 1-84233-112-4 (Set) / ISBN 1-84233-113-2 (CD)

We define the pgfs of the number of arrivals of type-1 and type-2 by

$$A_1(z) \triangleq \mathrm{E}\left[z^{a_1^{(k)}}\right] = A(z,1)$$

and

$$A_2(z) \triangleq \mathrm{E}\left[z^{a_2^{(k)}}\right] = A(1,z)$$

respectively. The arrival rate of type-j packets ($j = 1, 2$) is given by $\lambda_j = A_j'(1)$, while the total arrival rate is given by $\lambda_T = A_T'(1)$.

ANALYSIS

We denote the type-j system contents at the beginning of slot $k+1$ by $u_j^{(k)} (j = 1, 2)$. Furthermore, the joint pgf of $u_1^{(k)}$ and $u_2^{(k)}$ is denoted by $U^{(k)}(z_1, z_2)$, i.e.,

$$U^{(k)}(z_1, z_2) = \mathrm{E}\left[z_1^{u_1^{(k)}} z_2^{u_2^{(k)}}\right].$$

The system contents are characterized by the following set of system equations:

$$\begin{aligned} u_1^{(k)} &= [u_1^{(k-1)} - 1]^+ + a_1^{(k)} \\ u_2^{(k)} &= \begin{cases} [u_2^{(k-1)} - 1]^+ + a_2^{(k)} & \text{if } u_1^{(k-1)} = 0 \\ u_2^{(k-1)} + a_2^{(k)} & \text{if } u_1^{(k-1)} > 0 \end{cases}, \end{aligned}$$

where $[.]^+$ denotes the maximum of the argument and 0. The first equation can be understood by noticing that service of type-1 packets is not influenced by type-2 packets. A type-2 packet on the other hand can only be served, if there are no type-1 packets waiting in the queue at the end of the previous slot (because of the priority scheduling). This leads to the second equation. Using these system equations, we form the following relation between $U^{(k)}(z_1, z_2)$ and $U^{(k-1)}(z_1, z_2)$:

$$\begin{aligned} U^{(k)}(z_1, z_2) =& A(z_1, z_2) \left\{ \frac{z_2 - 1}{z_2} U^{(k-1)}(0, 0) \right. \\ &\left. + \frac{z_1 - z_2}{z_1 z_2} U^{(k-1)}(0, z_2) + \frac{1}{z_1} U^{(k-1)}(z_1, z_2) \right\}. \end{aligned} \quad (1)$$

Define $U(x, z_1, z_2)$ as the z-transform of the sequence $U^{(k)}(z_1, z_2)$ with respect to the time parameter k, i.e.,

$$U(x, z_1, z_2) \triangleq \sum_{k=0}^{\infty} U^{(k)}(z_1, z_2) x^k.$$

By summing both sides of equation (1) over all $k = 1, .., \infty$, we find the following expression for $U(x, z_1, z_2)$:

$$\begin{aligned} U(x, z_1, z_2) =& \frac{1}{z_1 - xA(z_1, z_2)} \left\{ z_1 U^{(0)}(z_1, z_2) \right. \\ &+ \frac{xA(z_1, z_2)}{z_2} [z_1(z_2 - 1)U(x, 0, 0) \\ &\left. + (z_1 - z_2)U(x, 0, z_2)] \right\}. \end{aligned} \quad (2)$$

Assuming that $U^{(0)}(z_1, z_2)$ - the joint pgf of the system contents of both types at the beginning (of the first slot) - is known, there are still two unknown functions in the right hand side of equation (2), namely $U(x, 0, z_2)$ and $U(x, 0, 0)$. $U(x, 0, 0)$ is the z-transform of the sequence $U_k(0, 0)$, i.e., $U(x, 0, 0)$ describes the evolution in time of the probability of an empty buffer. $U(x, 0, z_2)$ is the z-transform of the sequence $U_k(0, z_2)$ and is thus the evolution in time of the joint pgf of the class-1 and class-2 system contents given that there are no class-1 packets present. Once these quantities are calculated, the transient behavior of the type-1 and type-2 system contents can be found from equation (2). More specifically, we show in the following subsections how to calculate means (and higher moments) of the total system contents and of the system contents of type-1 and type-2 from expression (2). How to determine the unknown boundary functions ($U(x, 0, 0)$ and $U(x, 0, z_2)$) and/or the sequence of which they are the transform function of will be shown in the next section.

Total system contents

Substituting z_1 and z_2 by z in equation (2), we find:

$$U_T(x, z) = \frac{zU_T^{(0)}(z) + xA_T(z)(z-1)U_T(x, 0)}{z - xA_T(z)}, \quad (3)$$

with $U_T(x, z) = U(x, z, z)$ and $U_T^{(k)}(z) = U^{(k)}(z, z)$. So, concerning the transient total system contents, the only unknown is $U_T(x, 0)$, or equivalently, the transient probabilities of an empty system $U_T^{(k)}(0)$. For example, if $\mathrm{E}[u_T^{(k)}]$ denotes the mean total system contents at the beginning of slot k, or,

$$\mathrm{E}\left[u_T^{(k)}\right] \triangleq \left.\frac{dU_T^{(k)}(z)}{dz}\right|_{z=1},$$

then the transform function of the sequence $\left\{\mathrm{E}\left[u_T^{(k)}\right]\right\}$ can be found from $U_T(x, z)$ as:

$$\sum_{k=0}^{\infty} \mathrm{E}\left[u_T^{(k)}\right] x^k = \left.\frac{\partial U_T(x, z)}{\partial z}\right|_{z=1}. \quad (4)$$

Taking the first derivative in z of both sides of equation (3), substituting z by 1 and using equation (4), we obtain:

$$\sum_{k=0}^{\infty} \mathrm{E}\left[u_T^{(k)}\right] x^k = \frac{\mathrm{E}\left[u_T^{(0)}\right] + xU_T(x, 0)}{1 - x} - \frac{(1 - \lambda_T)x}{(1 - x)^2}. \quad (5)$$

Since the coefficients in x^k have to be equal in both sides of equation (5), this equation is equivalent with

$$\mathrm{E}\left[u_T^{(k)}\right] = \mathrm{E}\left[u_T^{(k-1)}\right] + \lambda_T - 1 + U_T^{(k-1)}(0), \quad (6)$$

for all $k \geq 1$. Equation (6) can thus be used to recursively calculate the mean total system contents $\mathrm{E}\left[u_T^{(k)}\right]$, once the sequence $\left\{U_T^{(k)}(0)\right\}$ is known. As already mentioned, we will show how to calculate this sequence (or its transform function) in the next section.

Type-1 system contents

Substituting z_2 by 1 (and z_1 by z for ease of notation) in equation (2) yields

$$U_1(x,z) = \frac{zU_1^{(0)}(z) + xA_1(z)(z-1)U_1(x,0)}{z - xA_1(z)}, \qquad (7)$$

with $U_1(x,z) \triangleq U(x,z,1)$ and $U_1^{(k)}(z) = U^{(k)}(z,1)$. So, this equation is very similar to equation (3) and thus we get a similar recursive expression as expression (6) for the mean type-1 system contents:

$$\mathrm{E}\left[u_1^{(k)}\right] = \mathrm{E}\left[u_1^{(k-1)}\right] + \lambda_1 - 1 + U_1^{(k-1)}(0), \qquad (8)$$

for all $k \geq 1$. Equation (8) can thus be used to recursively calculate the mean type-1 system contents $\mathrm{E}\left[u_1^{(k)}\right]$, once the sequence $\left\{U_1^{(k)}(0)\right\}$ is known.

Type-2 system contents

By substituting z_1 by 1 (and z_2 by z) in equation (2), we find

$$U_2(x,z) = \frac{zU_2^{(0)}(z) + xA_2(z)(1-z)[U(x,0,z) - U_T(x,0)]}{z(1 - xA_2(z))}, \qquad (9)$$

with $U_2(x,z) = U(x,1,z)$ and $U_2^{(k)}(z) = U^{(k)}(1,z)$. So, concerning the transient system contents of type-2, there are still two unknowns, $U(x,0,z)$ and $U_T(x,0)$, or equivalently, the sequences $\left\{U^{(k)}(0,z)\right\}$ and $\left\{U_T^{(k)}(0)\right\}$. If $\mathrm{E}\left[u_2^{(k)}\right]$ denotes the mean system contents of type-2 at the beginning of slot k, or,

$$\mathrm{E}\left[u_2^{(k)}\right] \triangleq \left.\frac{dU_2^{(k)}(z)}{dz}\right|_{z=1},$$

then the transform function of the sequence $\left\{\mathrm{E}\left[u_2^{(k)}\right]\right\}$ can be found from $U_2(x,z)$ as:

$$\sum_{k=0}^{\infty} \mathrm{E}\left[u_2^{(k)}\right] x^k = \left.\frac{\partial U_2(x,z)}{\partial z}\right|_{z=1}.$$

Taking the first derivative in z of both sides of equation (9), and substituting z by 1, we obtain:

$$\sum_{k=0}^{\infty} \mathrm{E}\left[u_2^{(k)}\right] x^k = \frac{\mathrm{E}\left[u_2^{(0)}\right] + x[U_T(x,0) - U_1(x,0)]}{1-x} + \frac{\lambda_2 x}{(1-x)^2}. \qquad (10)$$

Since the coefficients in x^k have to be equal in both sides of equation (10), this equation is equivalent with

$$\mathrm{E}\left[u_2^{(k)}\right] = \mathrm{E}\left[u_2^{(k-1)}\right] + \lambda_2 + U_T^{(k-1)}(0) - U_1^{(k-1)}(0), \qquad (11)$$

for all $k \geq 1$. From equation (11), the mean type-2 system contents $\mathrm{E}\left[u_2^{(k)}\right]$ can be recursively calculated once the $\left\{U_T^{(k)}(0)\right\}$'s and $\left\{U_1^{(k)}(0)\right\}$'s are known.

Notice that equations (6), (8) and (11) fulfill the relation

$$\mathrm{E}\left[u_T^{(k)}\right] = \mathrm{E}\left[u_1^{(k)}\right] + \mathrm{E}\left[u_2^{(k)}\right],$$

as expected. Furthermore, higher moments and cross moments (such as the covariance) of type-1 and type-2 system contents can also be recursively calculated by computing higher order derivatives of $U(x,z_1,z_2)$ (expression (2)). Notice that - although for the *mean* type-2 system contents it suffices to calculate $U_T(x,0)$ and $U_1(x,0)$ - $U(x,0,z)$ will have to be determined if one wants to calculate *higher* moments of the type-2 system contents and/or *cross*-moments of the system contents of both types. As an example, we show the expression of the variance of $u_2^{(k)}$:

$$\begin{aligned}\mathrm{Var}\left[u_2^{(k)}\right] = &\mathrm{Var}\left[u_2^{(k-1)}\right] + \lambda_2(1-\lambda_2) + \mathrm{Var}[a_2^{(k-1)}] \\ &+ 2\lambda_2 \mathrm{E}\left[u_2^{(k)}\right] - \left(\mathrm{E}\left[u_2^{(k)}\right] - \mathrm{E}\left[u_2^{(k-1)}\right]\right) \\ &- \left(\left(\mathrm{E}\left[u_2^{(k)}\right]\right)^2 - \left(\mathrm{E}\left[u_2^{(k-1)}\right]\right)^2\right) \\ &- 2\mathrm{E}\left[u_2^{(k-1)} \middle| u_1^{(k-1)} = 0\right] \mathrm{Prob}\left[u_1^{(k-1)} = 0\right].\end{aligned} \qquad (12)$$

Since

$$\mathrm{E}\left[u_2^{(k-1)} \middle| u_1^{(k-1)} = 0\right] \mathrm{Prob}\left[u_1^{(k-1)} = 0\right] = \left.\frac{\partial U^{(k-1)}(0,z_2)}{\partial z_2}\right|_{z_2=1} \qquad (13)$$

the variance of the transient class-2 system contents can be recursively calculated from expression (12) when $U(x,0,z_2)$ or the $\left\{U^{(k)}(0,z_2)\right\}$'s are known.

DETERMINATION OF THE UNKNOWNS

In this section we show how to calculate the boundary functions $U(x,0,0)$ and $U(x,0,z_2)$, or alternatively, the sequences $\left\{U_T^{(k)}(0)\right\}$ and $\left\{U^{(k)}(0,z_2)\right\}$ respectively. We will apply two different methods. The first method will only be useful for specific arrival processes, while the second one is more generally applicable.

Method 1: Calculation of the boundary functions

Applying Rouché's theorem, it can be proven that the denominator of the right hand side of equation (2) has one zero in the unit circle for z_1 for given values of x and z_2 ($|x| < 1$, $|z_2| < 1$), namely $X(x,z_2) \triangleq xA(X(x,z_2),z_2)$. One easily shows that $U(x,z_1,z_2)$ is analytic for all x and z_j with $|x| < 1$ and $|z_j| < 1$ ($j = 1,2$). Therefore $X(x,z_2)$ must also be a zero of the numerator of the right hand side of (2), yielding

$$U(x,0,z_2) = \frac{1}{z_2 - X(x,z_2)} \left\{ z_2 U^{(0)}(X(x,z_2),z_2) \right.$$
$$\left. + X(x,z_2)(z_2-1)U(x,0,0) \right\}. \qquad (14)$$

Applying Rouché's theorem once more, it can be proven that the denominator of the right hand side of equation (14) has one zero in the unit circle for z_2 for given value of x ($|x| < 1$), namely $Y(x) \triangleq X(x,Y(x))$. Since $U(x,0,z_2)$ is finite in the unit circle, $Y(x)$ must also be a zero of the numerator of the right hand side of (14), yielding

$$U(x,0,0) = \frac{U^{(0)}(Y(x),Y(x))}{1-Y(x)}. \qquad (15)$$

From (2), (14) and (15), it is obvious that the two boundary functions $U(x,0,z_2)$ and $U(x,0,0)$ and thus also $U(x,z_1,z_2)$ itself are fully determined, once the functions $X(x,z_2)$ and $Y(x)$ are explicitly known. Since these functions are only implicitly defined, this proves to be a hard task in general. However for specific choices of $A(z_1,z_2)$, this is rather straightforward.

Special case: $A(0,z_2) = 0$*:* If - for a certain z_2 - $A(0,z_2) = 0$, $X(x,z_2) = 0$ irrespective of x. Expression (14) then becomes

$$U(x,0,z_2) = U^{(0)}(0,z_2).$$

So in this case $U(x,z_1,z_2)$ is always explicitly found by this first method and we will thus exclude the case $A(0,z_2) = 0$ in the second method.

Method 2: Calculation of the transient sequences

Since the determination of $z_1 = X(x,z_2)$ as a solution of

$$z_1 - xA(z_1,z_2) = 0, \qquad (16)$$

for given values of x and z_2, may be hard, we will show an alternative way to calculate the unknowns. Note that solving equation (16) for the unknown x, for given values z_1 and z_2 is straightforward. This solution is given by

$$x = f(z_1,z_2) \triangleq \frac{z_1}{A(z_1,z_2)}. \qquad (17)$$

It is clear from equation (17), that if $|z_1| < 1$ is chosen sufficiently small - say $|z_1| < \varepsilon$ - and $|z_2| < 1$, $|x| = |f(z_1,z_2)|$ will also be smaller than unity as $|f(z_1,z_2)| \approx |z_1|/|A(0,z_2)|$ for small $|z_1|$ and since we have assumed $|A(0,z_2)| > 0$ (see also the special case in the previous subsection).

In the remainder of this subsection, we will first show in detail how to calculate the $U_T^{(k)}(0)$ from (3). $U_T(x,z)$ must be analytic for $(x,z) = (f(z,z),z)$, if $|z| < \varepsilon$. Since the denominator of (3) is zero for $(x,z) = (f(z,z),z)$, the numerator must vanish too, leading to

$$U_T(f(z,z),0) = \frac{U_T^{(0)}(z)}{1-z}, \qquad (18)$$

for $|z| < \varepsilon$. We now show that the former expression is sufficient to derive the probabilities $U_T^{(k)}(0)$. $U_T^{(k)}(0)$ is by definition the coefficient of x^k in the expansion of $U_T(x,0)$ about $x = 0$. In other words, using complex analysis, $U_T^{(k)}(0)$ can be obtained as the residue of the function $x^{-k-1}U_T(x,0)$ in $x = 0$. This residue can be calculated as follows:

$$U_T^{(k)}(0) = \frac{1}{2\pi i} \oint_{C_0} U_T(x,0) x^{-k-1} dx, \qquad (19)$$

with $i = \sqrt{-1}$ and C_0 a small contour which surrounds $x = 0$, but no singularities of $U_T(x,0)$. Changing the variable x by the variable z with $x = f(z,z)$ - $f(z_1,z_2)$ is defined in (17) - this expression transforms in

$$U_T^{(k)}(0) = \frac{1}{2\pi i} \oint_{C_1} U_T(f(z,z),0) f(z,z)^{-k-1} \frac{df(z,z)}{dz} dz,$$

with C_1 the transformation of C_0 in the z-plane, which circles the point $z = 0$ exactly once. We may choose C_0 such that $|z| < \varepsilon$ on contour C_1. Therefore, we can use expression (18), leading to

$$U_T^{(k)}(0) = \frac{1}{2\pi i} \oint_{C_1} \frac{U_T^{(0)}(z) A_T(z)^{k-1} [A_T(z) - zA'_T(z)]}{1-z}$$
$$\times z^{-k-1} dz. \qquad (20)$$

Since this expression only contains known functions and quantities, it leads to the determination of $U_T^{(k)}(0)$. Performing the contour integration in expression (20) - by means of any method - yields the $U_T^{(k)}(0)$'s. A practical interpretation of (20) is the following: $U_T^{(k)}(0)$ is the coefficient of z^k in the expansion about $z = 0$ of

$$P_T^{(k)}(z) = \frac{U_T^{(0)}(z) A_T(z)^{k-1} [A_T(z) - zA'_T(z)]}{1-z}. \qquad (21)$$

The calculation of this coefficient, using any method - depending on the particular form of $U_T^{(0)}(z)$ and $A_T(z)$ - thus gives $U_T^{(k)}(0)$, for this specific k.

A similar technique can also be used to calculate the sequence $U^{(k)}(0,z_2)$. $U(x,0,z_2)$ must be analytic for $(x,z_2) = (f(z_1,z_2),z_2)$, if $|z_2| < 1$ and if $|z_1| < \varepsilon$. Since the denominator of (2) is zero for $(x,z_2) = (f(z_1,z_2),z_2)$, the numerator must vanish too, leading to

$$U(f(z_1,z_2),0,z_2) = \frac{1}{z_2 - z_1} \left\{ z_2 U^{(0)}(z_1,z_2) \right.$$
$$\left. + z_1(z_2-1)U_T(f(z_1,z_2),0) \right\}, \qquad (22)$$

for $|z_1| < \varepsilon$ and $|z_2| < 1$. $U^{(k)}(0,z_2)$ is by definition the coefficient of x^k in the expansion of $U(x,0,z_2)$ about $x = 0$. We thus have

$$U^{(k)}(0,z_2) = \frac{1}{2\pi i} \oint_{C_0} U(x,0,z_2) x^{-k-1} dx,$$

with C_0 a small contour which surrounds $x = 0$, but no singularities of $U(x, 0, z_2)$. Changing the variable x by the variable z_1 with $x = f(z_1, z_2)$ this expression transforms in

$$U^{(k)}(0, z_2) = \frac{1}{2\pi i} \oint_{C_1'} U(f(z_1, z_2), 0, z_2) f(z_1, z_2)^{-k-1} \times \frac{\partial f(z_1, z_2)}{\partial z_1} dz_1,$$

with C_1' the transformation of C_0 in the z_1-plane, which circles the point $z_1 = 0$ exactly once. Choosing C_0 such that $|z| < \varepsilon$ on contour C_1', we can use expression (22), leading to

$$U^{(k)}(0, z_2) = \frac{1}{2\pi i} \oint_{C_1'} \frac{1}{z_2 - z_1} \left\{ z_2 U^{(0)}(z_1, z_2) + z_1(z_2 - 1) U_T(z_1/A(z_1, z_2), 0) \right\} \times \frac{A(z_1, z_2) - z_1 \dfrac{\partial A(z_1, z_2)}{\partial z_1}}{A(z_1, z_2)^{-k+1}} z_1^{-k-1} dz_1. \tag{23}$$

Since this expression only contains known functions and quantities, it leads to the determination of $U^{(k)}(0, z_2)$. E.g., $U^{(k)}(0, z_2)$ is the coefficient of z_1^k in the expansion about $z_1 = 0$ of

$$P^{(k)}(z_1, z_2) = \frac{z_2 U^{(0)}(z_1, z_2) + z_1(z_2 - 1) U_T(z_1/A(z_1, z_2), 0)}{z_2 - z_1} \times \frac{A(z_1, z_2) - z_1 \dfrac{\partial A(z_1, z_2)}{\partial z_1}}{A(z_1, z_2)^{-k+1}}. \tag{24}$$

Note that although $U_T(z, 0)$ - which still appears in this expression - is not explicitly calculated, the expansion of $U_T(z, 0)$ about $z = 0$ was the topic of the first part of this section. Note further that by substituting z_2 by 1 in expression (23) the sequence $\left\{ U_1^{(k)}(0) \right\}$ is calculated.

EXAMPLE

In this section, we apply our results from the former sections to an output-queueing switch in multimedia networks. We consider a non-blocking output-queueing switch with N inlets and N outlets (Figure 1). We assume two types of traffic. Traffic of type-1 is delay-sensitive (for instance voice) and traffic of type-2 is assumed to be delay-insensitive (for instance data). We investigate the effect of a HOL priority scheduling discipline on the transient system contents, as presented in the former of this paper.

The arrivals on each inlet are assumed to be i.i.d., and generated by a Bernoulli process with arrival rate λ_T. An arriving packet is assumed to be of class j with probability λ_j/λ_T ($j = 1, 2$) ($\lambda_1 + \lambda_2 = \lambda_T$). The incoming packets are then routed to the output queue corresponding to their destination, in an independent and uniform way. Therefore, the output queues behave identically and we can concentrate on the analysis of one output queue. In view of the previous, the arrivals of both types of packets to an output queue are generated according to a two-dimensional binomial process. It is fully characterized by the following joint pgf

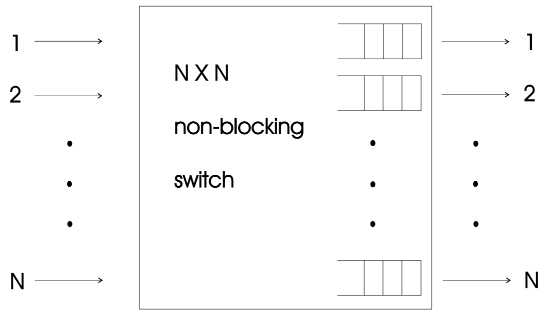

Figure 1: An NxN output queueing switch

$$A(z_1, z_2) = (1 - \frac{\lambda_1}{N}(1 - z_1) - \frac{\lambda_2}{N}(1 - z_2))^N. \tag{25}$$

N=2

In this case, explicit expressions for $X(x, z_2)$ and $Y(x)$ - as defined in the first method - can be found and are given by

$$X(x, z_2) = \frac{1}{\lambda_1^2 x} \Big\{ 2 - \lambda_1(2 - \lambda_1)x + \lambda_1\lambda_2 x(1 - z_2) - 2\sqrt{1 - \lambda_1(2 - \lambda_1)x + \lambda_1\lambda_2 x(1 - z_2)} \Big\}, \tag{26}$$

and

$$Y(x) = \frac{2 - \lambda_T(2 - \lambda_T)x - 2\sqrt{1 - \lambda_T(2 - \lambda_T)x}}{\lambda_T^2 x}, \tag{27}$$

respectively. One can calculate the performance measures in this case - e.g. the probability of the system contents being zero, the mean system contents, higher (cross-)moments of the system contents, ... - by first calculating the explicit expression of $U(x, z_1, z_2)$ (by substituting expressions (14)-(15) and (26)-(27) in expression (2)). Expanding this expression about $x = 0$ yields the $U^{(k)}(z_1, z_2)$ as coefficients of x^k. By taking derivatives and/or substituting z_1 and z_2 by 0 or 1 finally gives the required performance measure. Note that the order of the last two steps can be switched, i.e., taking the derivatives and/or the right substitutions can be done before the expansion about $x = 0$. Note further that we have already derived expressions for the mean (total, class-1 and class-2) system contents in a former section of this paper. Finally, we mark that the expansion about $x = 0$ is usually the most complicated step. For the specific arrival process discussed in this subsection, a number of square root functions will appear in the final expression of $U(x, z_1, z_2)$ (see expressions (26)-(27)), which have to be expanded about $x = 0$. As an example, we will show how to (recursively) calculate the coefficients of the square root in expression (26), i.e., of the function

$$H(x, z_2) \triangleq \sqrt{1 - \lambda_1(2 - \lambda_1)x + \lambda_1\lambda_2 x(1 - z_2)}. \qquad (28)$$

Taking the logarithmic derivative of this expression with respect to x, we obtain

$$\frac{\frac{\partial H(x,z_2)}{\partial x}}{H(x,z_2)} = \frac{-\lambda_1(2-\lambda_1) + \lambda_1\lambda_2(1-z_2)}{2[1-\lambda_1(2-\lambda_1)x + \lambda_1\lambda_2 x(1-z_2)]}.$$

Replacing $H(x, z_2)$ and $\partial H(x, z_2)/\partial x$ by their respective series expansions $\sum_{j=0}^{\infty} H_j(z_2)x^j$ and $\sum_{j=1}^{\infty} jH_j(z_2)x^{j-1}$, we find

$$2[1 - \lambda_1(2 - \lambda_1)x + \lambda_1\lambda_2 x(1 - z_2)] \sum_{j=1}^{\infty} jH_j(z_2)x^{j-1}$$
$$= [-\lambda_1(2 - \lambda_1) + \lambda_1\lambda_2(1 - z_2)] \sum_{j=0}^{\infty} H_j(z_2)x^j,$$

which leads to the recursive calculation of $H_j(z_2)$ by identification of the coefficients of x^j in both sides of this equation.

In Figures 2 and 3, the transient probabilities of an empty total, class-1 and class-2 system respectively are plotted for $\lambda_1 = \lambda_2 = 0.4$. In Figure 2 the system is assumed empty at the beginning. In Figure 3 the system is started in the equilibrium state of a system with a FIFO-scheduling. The latter figure thus gives the transient probabilities of an empty (total, class-1 and class-2) system, when the scheduling discipline is switched from a FIFO scheduling discipline to a priority discipline. As can be seen, these probabilities go to their respective steady-state values for $k \to \infty$ (for the steady-state values, see (Walraevens et al. 2003)). Since the steady-state probabilities of an empty total system are identical when a FIFO and when a priority scheduling discipline are used, this leads to a horizontal line in Figure 3.

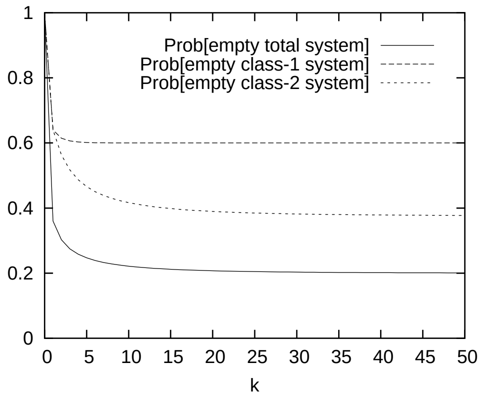

Figure 2: Transient probabilities of an empty total, class-1 and class-2 system starting with an empty system

In Figures 4 and 5, the transient mean system contents are shown for $\lambda_1 = \lambda_2 = 0.4$, starting with an empty system

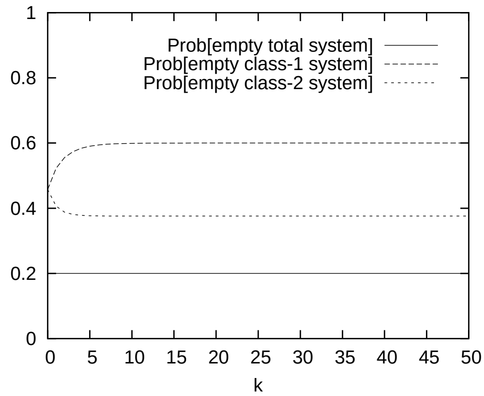

Figure 3: Transient probabilities of an empty total, class-1 and class-2 system starting with a FIFO system (in steady-state)

- Figure 4 - or with a system with a FIFO scheduling discipline (in steady-state) - Figure 5. We again see that the mean system contents go to their respective steady-state values (which again can be found in (Walraevens et al. 2003)).

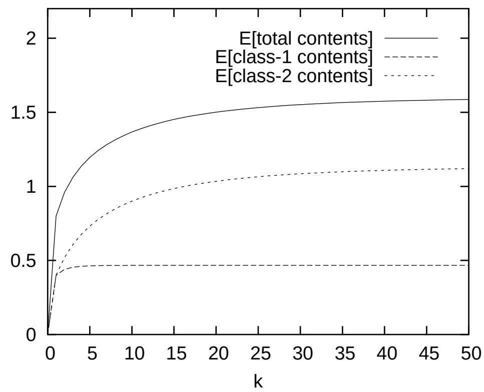

Figure 4: Transient mean total, class-1 and class-2 system contents starting with an empty system

In Figure 6, we show the variances of the transient total, class-1 and class-2 system contents starting with an empty system. In Figure 7, the correlation coefficient of the transient class-1 and class-2 system contents is shown, also starting with an empty system.

N=∞

In this case, equation (25) becomes

$$A(z_1, z_2) = e^{\lambda_1(z_1-1)}e^{\lambda_2(z_2-1)},$$

i.e., the number of per-slot arrivals of class-1 and class-2 are two mutually independent Poisson processes. Expression (16) is a transcendental equation and therefore we use the second method in this example. Obtaining the required performance measures - i.e., the transient probabilities of the total system, the class-1 and the class-2

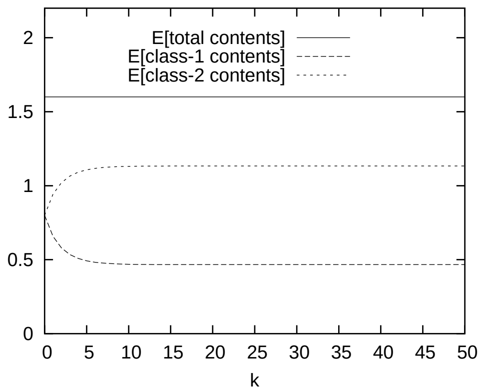

Figure 5: Transient mean total, class-1 and class-2 system contents starting with a FIFO system (in steady-state)

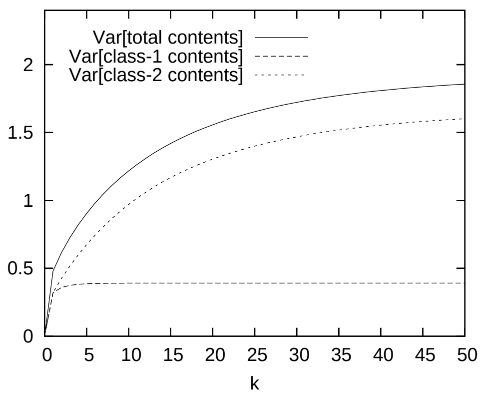

Figure 6: Variances of the transient total, class-1 and class-2 system contents starting with an empty system

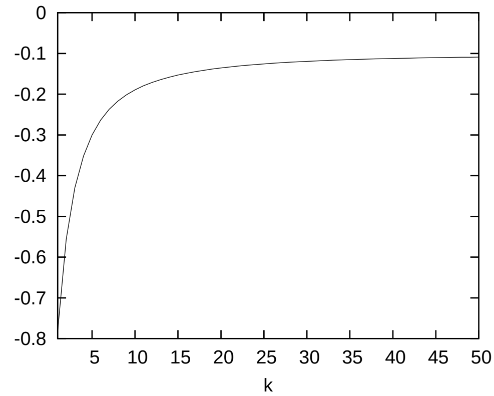

Figure 7: Correlation coefficient of the transient class-1 and class-2 system contents starting with an empty system

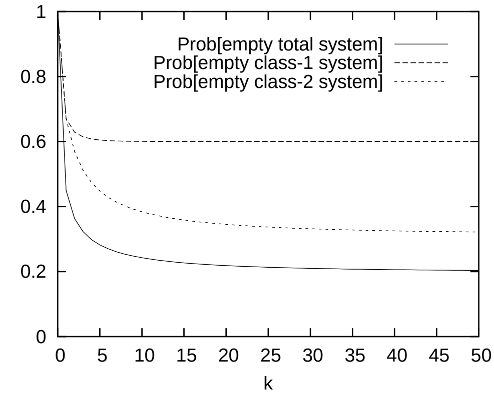

Figure 8: Transient probabilities of an empty total, class-1 and class-2 system starting with an empty system

buffer being empty, the mean transient system contents of both classes, ... - requires for each k the calculation of (21) and/or (24), expanding this expression about $z = 0$ ($z_1 = 0$ respectively) and calculating the coefficient of z^k (z_1^k respectively) in this expansion. Note that for the expansion of (24) it will be necessary to expand (21) first, since $U_T(x, 0)$ appears in the former expression.

The obtained figures are in this case similar as the ones obtained in the previous case ($N = 2$). We therefore show only two plots to demonstrate that the performance measures can also be calculated in this case. Figure 8 shows the transient probability of an empty total, class-1 and class-2 system respectively for $\lambda_1 = \lambda_2 = 0.4$. In Figure 9, we have plotted the mean transient total, class-1 and class-2 system contents respectively for $\lambda_1 = \lambda_2 = 0.4$. In both these figures, the system is assumed empty at the beginning.

CONCLUSIONS

In this paper, we studied the transient behavior of a priority queueing system. Packets of two types arrive in the system and packets of type-1 have priority over packets of type-2. Using generating functions, we analyzed the transient probabilities that the system is empty, that no packets of type-1 are present and that no packets of type-2 are present. Furthermore, we showed how to calculate the moments of the transient system contents of both types. We illustrated our approach by means of some numerical examples.

This work can be extended in several ways. Firstly, we can investigate whether the technique used in this paper can be extended to priority queueing systems with service times larger than one slot. Extending the arrival process to include correlated arrivals is a second potential direction for future work. However, since the steady-state analysis of a priority queue with correlated arrivals is not straigh-forward, this will a fortiori be the case for the transient analysis. Finally, we can look into the analysis of the (transient) delay.

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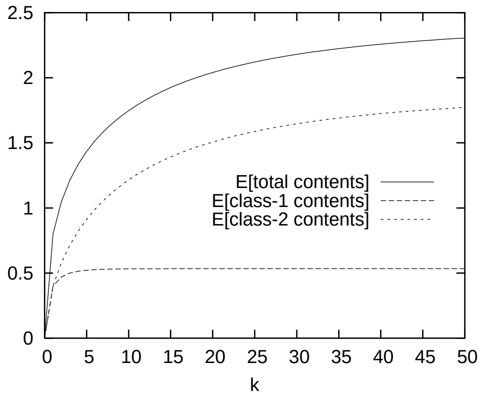

Figure 9: Transient probabilities of an empty total, class-1 and class-2 system starting with an empty system

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