TRANSIENT ANALYSIS OF A MARKOV MODULATED FLUID QUEUE WITH LINEAR SERVICE RATE

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Abstract: We consider an infinite capacity fluid queue governed by a continuous time Markov chain and with linear service rate. The transient behavior of this fluid flow model is described by a linear differential equation. We study the transient distribution of the fluid level in the queue and we derive a partial differential equation satisfied by the cumulative distribution function of the fluid level. Using this partial differential equation, we obtain a simple expression of the moments of this transient distribution, as well as its Laplace transform.

Keywords: Stochastic Models, Fluid Queue, Markov Models, Queueing Systems.

1 INTRODUCTION

We consider in this paper an infinite capacity fluid queue of which level at time t is denoted by $Q(t)$. Fluid arrives in this queue according to a non decreasing process $A(t)$ and leaves the buffer at a rate $\tau(X(t), Q(t))$, where $X(t)$ is a continuous time Markov chain and τ is a non negative function. This is a generalization of standard fluid queues driven by a superposition of On-Off sources (see [Anick et al, 1982]), since here the service rate depends on the queue level, in addition to the state of the underlying Markov chain.
The fluid level in the queue $Q(t)$ then satisfies the following differential equation reflected at 0

$$dQ(t) = \quad dA(t) - \tau(X(t), Q(t))dt + dL_t$$

where L_t is a non decreasing process, (called the *regulator*), interfering only when $Q(t) = 0$ and preventing it from being negative.
We consider in this paper the *linear* model which corresponds to the case where $\tau(X(t), Q(t))$ is linear in $Q(t)$, that is $\tau(X(t), Q(t)) = \mu(X(t))Q(t)$, where μ is a positive function which depends only on the Markov chain X. This model has been studied in [Asmussen and Kella, 1996], [Kella and Stadje, 2002] and [Kella and Whitt, 1999], where $A(t)$ is a Lévy process or a Markov modulated Lévy process. These papers mainly focus on the limiting distribution of $Q(t)$. The authors identify a functional equation satisfied by the Laplace-Stieltjes transform of the limiting distribution, which can be used to evaluate the two first moments of the buffer level distribution.
We consider in this paper the case where $dA(t) = \lambda(X(t))dt$, where λ is a non negative function of the Markov chain. The evolution of $Q(t)$ is then described by the following equation.

$$dQ(t) = \lambda(X(t))dt - \mu(X(t))Q(t)dt, \qquad (1)$$

the term L_t having disappeared because 0 is an impenetrable barrier for $Q(t)$ (see [Asmussen and Kella, 1996]. Note that $(X(t), Q(t))$ is then a Markov process. In the literature, the differential equations governing this process are generally obtained from backward analysis. In this paper we use a forward argument to obtain them, this yields an easier study of the moments of $Q(t)$.
Throughout this paper, X denotes a stationary ergodic Markov chain evolving on a finite state space $S = \{1, ..., N\}$. We denote by $A = (a_{i,j})_{(i,j)\in S\times S}$ its infinitesimal generator and by $\pi = (\pi_1, ..., \pi_N)$ its stationary distribution. We also suppose that X is a two sided process, i.e. indexed by $\mathbb{R}$.
The paper is organized as follows. In Section 2, we give an explicit expression of the fluid level $Q(t)$ and we describe the jumps of its distribution. In Section 3, we derive a partial differential equation satisfied by the cumulative distribution function of $Q(t)$

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and we obtain in Section 4 an expression of the moments of $Q(t)$, as well as its Laplace transform.

2 PRELIMINARIES

Let us denote by $Q^y(t)$ the fluid level in the queue at time t with the initial condition $Q^y(0) = y$. For $t \geq 0$, $Q^y(t)$ satisfies the equation (1) and it can be easily checked that $Q^y(t)$ is given by

$$\begin{aligned} Q^y(t) &= y \exp\left(-\int_0^t \mu(X(s))ds\right) \\ &+ \int_0^t \exp\left(-\int_s^t \mu(X(v))dv\right) \lambda(X(s))ds. \end{aligned}$$

We also have the following relation between $Q^y(t)$ and $Q^y(t')$ for $t \geq t' \geq 0$,

$$\begin{aligned} Q^y(t) &= Q^y(t') \exp\left(-\int_{t'}^t \mu(X(s))ds\right) \\ &+ \int_{t'}^t \exp\left(-\int_s^t \mu(X(v))dv\right) \lambda(X(s))ds. \end{aligned}$$

Using the stationarity of X, we have that $Q^y(t)$ and $\widetilde{Q}^y(t)$ have the same distribution, where $\widetilde{Q}^y(t)$ is given by

$$\begin{aligned} \widetilde{Q}^y(t) &= y \exp\left(-\int_{-t}^0 \mu(X(s))ds\right) \\ &+ \int_{-t}^0 \exp\left(-\int_s^0 \mu(X(v))dv\right) \lambda(X(s))ds. \end{aligned} \quad (2)$$

Let us denote by X^* the reversed process of X defined by $X^*(s) = X(-s)$. It is standard that X^* is a continuous time Markov chain with infinitesimal generator $A^* = \Pi^{-1} A^T \Pi$, where T denotes the transpose operator and Π is the diagonal matrix containing the vector π, which is also the stationary distribution of X^*. The variable changes $s := -s$ and $v := -v$ in Relation (2) leads to the following expression for $\widetilde{Q}^y(t)$

$$\begin{aligned} \widetilde{Q}^y(t) &= y \exp\left(-\int_0^t \mu(X^*(s))ds\right) \\ &+ \int_0^t \exp\left(-\int_0^s \mu(X^*(v))dv\right) \lambda(X^*(s))ds. \end{aligned} \quad (3)$$

Because the initial buffer level y is fixed throughout this paper, and for readability purpose, we simply use the notation $Q(t)$ instead of $\widetilde{Q}^y(t)$.
It is easy to check that for a fixed $t > 0$, the distribution of $Q(t)$ has jumps which correspond to the fact that the Markov chain X^* stays during the whole interval $[0, t]$ in a subset of states having the same values for $\lambda(i)$ and $\mu(i)$. More precisely, let m be the number of distinct pairs $(\lambda(i), \mu(i))$ for $i \in S$. If we denote these m different pairs by $(u(1), v(1)), \ldots, (u(m), v(m))$, we obtain the partition $B_1, \ldots, B_m$ of the state space S by defining B_l as

$$B_l = \{i \in S \mid (\lambda(i), \mu(i)) = (u(l), v(l))\}.$$

For $l = 1, \ldots, m$ and $t > 0$, we denote by $s_l(t)$ the quantities

$$s_l(t) = ye^{-v(l)t} + \frac{u(l)(1 - e^{-v(l)t})}{v(l)}.$$

We then have from Relation (3)

$$Q(t) = s_l(t) \Longleftrightarrow X^*(s) \in B_l, \ \forall s \in [0, t)$$

It follows that

$$\Pr\{Q(t) = s_l(t)\} = \pi_{B_l} e^{A_{B_l B_l} t} \mathbb{1},$$

where $A_{B_l B_l}$ is the sub-infinitesimal generator of dimension $|B_l|$ obtained from A by considering only the internal transitions of the subset B_l and π_{B_l} is the subvector of dimension $|B_l|$ obtained from vector π by considering the stationary probabilities of the subset B_l. The vector $\mathbb{1}$ is the column vector with all its entries equal 1, its dimension being given by the context.

3 DISTRIBUTION OF THE FLUID LEVEL IN THE QUEUE

We denote by $F_i(t, x)$ the cumulative distribution function of $Q(t)$ given that $X^*(0) = i$, that is, $F_i(t, x) = \Pr\{Q(t) \leq x | X^*(0) = i\}$. We denote by $F(t, x)$ the column vector $(F_i(t, x))_{i \in S}$, by D the diagonal matrix containing the $\mu(i)$'s and by Λ the diagonal matrix containing the $\lambda(i)$'s.
The distribution $F(t, x)$ of $Q(t)$ verifies the following differential equation.

Theorem 3.1 *For every (t, x) such that $ye^{-\mu(i)t} + \lambda(i)(1 - e^{-\mu(i)t})/\mu(i) \neq x$ for all $i \in S$ we have*

$$\partial_t F(t, x) = A^* F(t, x) + (Dx - \Lambda)\partial_x F(t, x). \quad (4)$$

Proof. Let us denote by P^* the transition probability matrix of the uniformized discrete-time Markov chain associated with X^*. We then have $P^* = I + A^*/\nu$, where I is the identity matrix and ν is the uniformization rate satisfying $\nu \geq \max\{-a^*_{i,i}, \ i \in S\}$. Let T_1 be the first instant of jump of X^*. T_1 is linked to the uniformized Markov chain via the equality

$$d\Pr\{X^*(T_1) = j, T_1 = u \mid X^*(0) = i\} = p^*_{i,j} \nu e^{-\nu u} du.$$

Defining $G_{i,j}(t,u,x) = \Pr\{Q(t) \le x \mid X^*(T_1) = j, T_1 = u, X^*(0) = i\}$, we obtain

$$\begin{aligned} F_i(t,x) &= \sum_{j \in S} \int_0^\infty G_{i,j}(t,u,x) \\ &\quad d\Pr\{X^*(T_1) = j, T_1 = u \mid X^*(0) = i\} \\ &= \nu \sum_{j \in S} p^*_{i,j} \int_0^\infty G_{i,j}(t,u,x) e^{-\nu u} du. \quad (5) \end{aligned}$$

Let us define for all $t \ge 0$

$$I_1(i,t,x) = \nu \sum_{j \in S} p^*_{i,j} \int_0^t G_{i,j}(t,u,x) e^{-\nu u} du$$

$$\text{and} \quad I_2(i,t,x) = \nu \sum_{j \in S} p^*_{i,j} \int_t^\infty G_{i,j}(t,u,x) e^{-\nu u} du.$$

We first consider $I_1(i,t,x)$, where we integrate $G_{i,j}(t,u,x)e^{-\nu u}$ for $u \in [0,t]$. Then, when u lies in that interval and $T_1 = u$ and $X^*(0) = i$, we have

$$\begin{aligned} Q(t) &= y \exp\left(-\int_0^t \mu(X^*(s))ds\right) \\ &+ \int_0^t \exp\left(-\int_0^s \mu(X^*(v))dv\right) \lambda(X^*(s))ds \\ &= y \exp\left(-\int_0^u \mu(X^*(s))ds - \int_u^t \mu(X^*(s))ds\right) \\ &+ \int_0^u \exp\left(-\int_0^s \mu(X^*(v))dv\right) \lambda(X^*(s))ds \\ &+ \int_u^t \exp\left(-\int_0^s \mu(X^*(v))dv\right) \lambda(X^*(s))ds \\ &= ye^{-\mu(i)u} \exp\left(-\int_u^t \mu(X^*(s))ds\right) \\ &+ \frac{\lambda(i)(1-e^{-\mu(i)u})}{\mu(i)} \\ &+ \int_u^t \exp\left(-\int_0^s \mu(X^*(v))dv\right) \lambda(X^*(s))ds \\ &= ye^{-\mu(i)u} \exp\left(-\int_u^t \mu(X^*(s))ds\right) \\ &+ \frac{\lambda(i)(1-e^{-\mu(i)u})}{\mu(i)} \\ &+ e^{-\mu(i)u} \int_u^t \exp\left(-\int_u^s \mu(X^*(v))dv\right) \\ &\quad \lambda(X^*(s))ds \\ &= \frac{\lambda(i)(1-e^{-\mu(i)u})}{\mu(i)} + e^{-\mu(i)u} Q_u(t), \end{aligned}$$

where

$$Q_u(t) = y \exp\left(-\int_u^t \mu(X^*(s))ds\right)$$
$$+ \int_u^t \exp\left(-\int_u^s \mu(X^*(v))dv\right) \lambda(X^*(s))ds.$$

Hence for $u \in [0,t]$, we have

$$\begin{aligned} &G_{i,j}(t,u,x) \\ &= \Pr\{Q(t) \le x \mid X^*(T_1) = j, T_1 = u, X^*(0) = i\} \\ &= \Pr\{\frac{\lambda(i)(1-e^{-\mu(i)u})}{\mu(i)} + e^{-\mu(i)u} Q_u(t) \le x \\ &\mid X^*(T_1) = j, T_1 = u, X^*(0) = i\}. \end{aligned}$$

Now from the Markov property and the homogeneity of X^* we get that the distribution of $Q_u(t)$ given $X^*(u)$ is the same as the distribution of $Q(t-u)$ given $X^*(0)$, thus

$$\begin{aligned} &G_{i,j}(t,u,x) \\ &= \Pr\{\frac{\lambda(i)(1-e^{-\mu(i)u})}{\mu(i)} \\ &+ e^{-\mu(i)u} Q(t-u) \le x \mid X^*(0) = j\} \\ &= \Pr\{Q(t-u) \le e^{\mu(i)u} \\ &\left(x - \frac{\lambda(i)(1-e^{-\mu(i)u})}{\mu(i)}\right) \mid X^*(0) = j\} \\ &= F_j(t-u, xe^{\mu(i)u} + \lambda(i)(1-e^{\mu(i)u})/\mu(i)) (6) \end{aligned}$$

Let us now consider $I_2(i,t,x)$. For $u \ge t$, the expression of $G_{i,j}(t,u,x)$ is given by

$$\begin{aligned} &G_{i,j}(t,u,x) \\ &= \Pr\{ye^{-\mu(i)t} + \lambda(i)(1-e^{-\mu(i)t})/\mu(i) \le x \\ &\mid X^*(T_1) = j, T_1 = u, X^*(0) = i\} \\ &= 1_{\{ye^{-\mu(i)t} + \lambda(i)(1-e^{-\mu(i)t})/\mu(i) \le x\}}. \quad (7) \end{aligned}$$

We denote this indicator function by $\eta(i,t,x)$. $\eta(i,t,x)$ is differentiable in t and x for all (t,x) in the domain $\{(t,x) | \; ye^{-\mu(i)t} + \lambda(i)(1-e^{-\mu(i)t})/\mu(i) \neq x\}$. Since $\eta(i,t,x)$ is a constant equal to 0 or 1 in this domain, its derivatives in t and x are both equal to 0. Hence from (6) and (7) we have

$$\begin{aligned} I_1(i,t,x) &= \nu \sum_{j \in S} p^*_{i,j} \int_0^t F_j(t-u, xe^{\mu(i)u} \\ &+ \lambda(i)(1-e^{\mu(i)u})/\mu(i)) e^{-\nu u} du \\ &= \nu e^{-\nu t} \sum_{j \in S} p^*_{i,j} \int_0^t F_j(u, xe^{\mu(i)(t-u)} \\ &+ \lambda(i)(1-e^{\mu(i)(t-u)})/\mu(i)) e^{\nu u} du, \\ I_2(i,t,x) &= \sum_{j \in S} p^*_{i,j} \eta(i,t,x) e^{-\nu t} \\ &= \eta(i,t,x) e^{-\nu t} \end{aligned}$$

where we made the change variable $u := t-u$ in the expression of $I_1(i,t,x)$.

Let us now derive (5) with respect to t. We have $\partial_t F_i(t,x) = \partial_t I_1(i,t,x) + \partial_t I_2(i,t,x)$, and direct calculation yields

$$\partial_t I_1(i,t,x) = -\nu I_1(i,t,x) + \nu \sum_{j\in S} p^*_{i,j} F_j(t,x)$$
$$+ \nu e^{-\nu t}(\mu(i)x - \lambda(i)) \sum_{j\in S} p^*_{i,j} \int_0^t \partial_x F_j(u,$$
$$xe^{\mu(i)(t-u)} + \lambda(i)(1-e^{\mu(i)(t-u)})/\mu(i))e^{\nu u} du$$
$$= -\nu I_1(i,t,x) + \nu \sum_{j\in S} p^*_{i,j} F_j(t,x)$$
$$+ (\mu(i)x - \lambda(i))\partial_x \Big[\nu e^{-\nu t} \sum_{j\in S} p^*_{i,j} \int_0^t F_j(u,$$
$$xe^{\mu(i)(t-u)} + \lambda(i)(1-e^{\mu(i)(t-u)})/\mu(i))e^{\nu u} du\Big]$$
$$= -\nu I_1(i,t,x) + \nu \sum_{j\in S} p^*_{i,j} F_j(t,x)$$
$$+(\mu(i)x - \lambda(i))\partial_x I_1(i,t,x).$$

and

$$\begin{aligned} \partial_t I_2(i,t,x) &= -\nu \eta(i,t,x) e^{-\nu t} \\ &= -\nu I_2(i,t,x). \end{aligned}$$

By adding these terms, we get

$$\partial_t F_i(t,x) = -\nu F_i(t,x)$$
$$+\nu \sum_{j\in S} p^*_{i,j} F_j(t,x) + (\mu(i)x - \lambda(i))\partial_x I_1(i,t,x).$$

But since $\partial_x I_2(i,t,x) = 0$, we obtain

$$\partial_t F_i(t,x) = -\nu F_i(t,x)$$
$$+\nu \sum_{j\in S} p^*_{i,j} F_j(t,x) + (\mu(i)x - \lambda(i))\partial_x F_i(t,x),$$

and the results follow by using the relation $P^* = I + A^*/\nu$. ■

3.1 Moments evaluation

We consider in this section the moments of the transient buffer level $Q(t)$. Let us note that, since the jumps of the cumulative distribution function of $Q(t)$ are known, the equation (4) has a unique solution provided that the initial conditions are fixed. However, we succeed in finding an expression of the moments of $Q(t)$ and an expression of its Laplace transform, without solving this equation.
We first recall the following well-known result

Lemma 3.2 *Let H be the cumulative distribution function of a non negative random variable. For every $r \geq 1$, if the r-th order moment exists, we have*

$$\int_0^\infty x^r dH(x) = r \int_0^\infty x^{r-1}(1 - H(x))dx.$$

Proof. See for instance [Feller, 1957]. ■

Let us denote by $v_i(t,k)$ the kth moment of $Q(t)$ given that the initial state of the Markov chain X^* is i, that is

$$v_i(t,k) = E(Q(t)^k \mid X^*(0) = i).$$

We denote by $V(t,k)$ the column vector containing the $v_i(t,k)$. By definition, we have $V(t,0) = \mathbb{1}$. In the following corollary of Theorem 3.1, we give an expression for all the moments of the buffer level $Q(t)$.

Corollary 3.3 *For every $k \geq 1$, we have the following recursion for the process $\{V(t,k),\ t \geq 0\}$*

$$V(t,k) = e^{(A^*-kD)t} y^k \mathbb{1}$$
$$+e^{(A^*-kD)t} \int_0^t e^{-(A^*-kD)s} k\Lambda V(s,k-1)ds \quad (8)$$

Proof. Since $A^*\mathbb{1} = 0$, relation (4) can be written as

$$-\partial_t(\mathbb{1} - F(t,x)) = -A^*(\mathbb{1} - F(t,x))$$
$$+(Dx - \Lambda)\partial_x F(t,x).$$

Multiplying both sides by x^{k-1}, for $k \geq 1$, and after integration, we get

$$-\partial_t \int_0^\infty x^{k-1}(\mathbb{1} - F(t,x))dx$$
$$= -A^* \int_0^\infty x^{k-1}(\mathbb{1} - F(t,x))dx$$
$$+D \int_0^\infty x^k \partial_x F(t,x) dx$$
$$- \Lambda \int_0^\infty x^{k-1} \partial_x F(t,x) dx.$$

Using Lemma 3.2, we easily get

$$V'(t,k) = (A^* - kD)V(t,k) + k\Lambda V(t,k-1).$$

It is easily checked that the solution to this equation is (8), which completes the proof. ■

The kth moment of $Q(t)$ is then easily given by $E(Q(t)^k) = \pi V(t,k)$. Note that in the case $k = 1$ simple computation yields the following expression for $E(Q(t))$:

$$E(Q(t)) = y\pi e^{(A^*-D)t}\mathbb{1} + \pi(A^*-D)^{-1}[e^{(A^*-D)t} - I]\Lambda\mathbb{1}.$$

It is easy to verify that in the case $\mu(i) = \mu$ (i.e. $D = \mu I$) we have

$$E(Q(t)) = ye^{-\mu t} - \frac{1}{\mu}[e^{-\mu t} - 1]\pi\Lambda\mathbb{1}.$$

We now easily deduce the Laplace transform $\phi(t,\theta) = E(\exp(\theta Q(t)))$ of $Q(t)$ for all t:

Corollary 3.4 *For every $\theta \in \mathbb{R}$, we have*

$$\phi(t,\theta) = \pi \sum_{k=0}^{\infty} \frac{V(t,k)\theta^k}{k!}.$$

Proof. First note that it is easy to see that $Q(t)$ is upper bounded by the deterministic value $c(t,y) = y + t\sup_{i\in S}\lambda(i)$. Besides we have $\partial_\theta^k \phi(t,0) = E(Q(t)^k) = \pi V(t,k)$. Hence we have for all $N \geq 0$

$$\begin{aligned}
&\left|\phi(t,\theta) - \sum_{k=0}^{N} \frac{\partial_\theta^k \phi(t,0)\theta^k}{k!}\right| \\
&= \left|E\left(\exp(\theta Q(t)) - \sum_{k=0}^{N} \frac{Q(t)^k\theta^k}{k!}\right)\right| \\
&= \left|E\left(\sum_{k=N+1}^{\infty} \frac{Q(t)^k\theta^k}{k!}\right)\right| \\
&\leq \sum_{k=N+1}^{\infty} c(y,t)^k \frac{\theta^k}{k!} \\
&\longrightarrow 0 \quad \text{as } N \to \infty.
\end{aligned}$$

This completes the proof. ■

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