STATIONARY ANALYSIS OF TANDEM FLUID QUEUES FED BY HOMOGENEOUS ON-OFF SOURCES

N. BARBOT and B. SERICOLA

IRISA-INRIA
Campus universitaire de Beaulieu
35042 Rennes cedex, France
Email : {Nelly.Barbot, Bruno.Sericola}@irisa.fr,

Abstract: We consider a fluid system composed of multiple buffers in series. The first buffer receives fluid from a finite superposition of independent identical on-off sources. The active and silent periods of sources are exponentially distributed. The ith buffer releases fluid in the $(i+1)$th buffer. Assuming that the input rate of one source is greater than the service rate of the first buffer, the output process of each buffer can be modeled by an on-off source with the active period distributed as the busy period of an M/M/1 queue. For $i \geq 2$, the stationary content distribution of the ith buffer is obtained by the use of generating functions which are explicitly inverted.
Keywords: Tandem fluid queues, output process, generating functions.

1 INTRODUCTION

We consider tandem fluid queues fed by a finite number of identical on-off sources. It is assumed that silent and active periods of the sources are independent and exponentially distributed. Tandem fluid queues are composed of consecutive infinite capacity buffers. The stationary behavior of the first buffer is explicitly derived in [Anick et al., 1982], using spectral decomposition arguments. As far as the other buffers are concerned, the output processes need to be characterized. In [Aalto, 1998] and [Boxma and Dumas, 1998], the authors consider a fluid queue driven by a superposition of on-off sources, with exponentially distributed silent periods and generally distributed active periods. Assuming that the input rate of one source is greater than the constant service rate of the buffer, they prove that the output process behaves as an on-off source with exponentially distributed silent periods and active periods distributed like the busy periods of a M/G/1 queue. In this paper, we consider the stationary behavior of each buffer level in the tandem fluid queues, apart from the first one. Using results of [Aalto, 1998] and [Boxma and Dumas, 1998], the output processes look like on-off sources with active periods distributed as busy periods of an M/M/1 queue. This tandem of fluid queues has been studied in [Aalto, 1998], where the output processes have been considered as alternating renewal processes. The authors obtained the stationary fluid level distribution of each buffer in terms of a Bessel function integral. Here, we derive a new analytic expression of these distributions. By using the method developped in [Leguesdron et al, 1991] and [Barbot and Sericola, 2002], we write the solutions in terms of a matrix exponential and then via generating functions that are explicitly inverted. Nevertheless, as shown in the next section, we deal here with a more general setting than the one of [Barbot and Sericola, 2002].

2 MODEL FORMULATION

We consider M infinite capacity fluid queues in series. The first one is fed by the superposition of N independent identical on-off sources with exponentially distributed on-off periods with parameters μ and λ respectively. During the on period, a source emits fluid at a constant rate c_0. The first buffer empties in the second one at the rate c_1. For $i \geq 2$, the input of the ith buffer is the output from the buffer $i-1$ and its service rate is denoted by c_i. It is assumed that $Nc_0 > c_1 > \ldots > c_M > 0$ in order

Proceedings 17th European Simulation Multiconference
 ISBN 3-936150-25-7

to avoid the trivial case where one or more buffers remain empty. Moreover, we make the restrictive assumption $c_0 \geq c_1$ which permits the output process of the first buffer to be simply derived.

Definition 1 *An on-off source is called an MM1(β,a,b,r) source if the off periods are exponentially distributed with rate β and the on periods are distributed as the busy periods of an M/M/1 queue with arrival rate a and service rate b. During the on periods, the source emits fluid at rate r.*

The infinitesimal generator associated with such a source is denoted by A. Its non-zero entries are

$$A_{0,0} = -\beta,\ A_{0,1} = \beta,\ A_{j,j-1} = b,$$

$$A_{j,j} = -(a+b) \text{ and } A_{j,j+1} = a \text{ for } j \geq 1. \quad (1)$$

Note that in [Barbot and Sericola, 2002], we considered a single fluid queue fed by a classical M/M/1 queue, which is, our definition, a fluid queue fed by an MM1(a,a,b,r) source. Here we have to deal with MM1(β,a,b,r) sources, where $\beta \neq a$, which generalizes the results of [Barbot and Sericola, 2002].

The following lemmas are proved in [Aalto, 1998] and [Boxma and Dumas, 1998].

Lemma 2 *In the stationary regime, the output process of the first buffer is equivalent to an MM1($N\lambda$,λ_1,μ_1,c_1) source where $\lambda_1 = \lambda(N - c_1/c_0)$ and $\mu_1 = \mu c_1/c_0$.*

Lemma 3 *In the stationary regime, the output process of a buffer with service rate c and fed by an MM1(β,a,b,r) source is equivalent to an MM1(β,a',b',c) source where $a' = ac/r + \beta(1-c/r)$ and $b' = bc/r$.*

Using Lemmas 2 and 3, the output process of the ith buffer, for $1 \leq i \leq M$, is equivalent to an MM1($N\lambda$,λ_i,μ_i,c_i) source where $\lambda_i = \lambda(N - c_i/c_0)$ and $\mu_i = \mu c_i/c_0$. Therefore, the traffic intensity in the ith buffer is given by $\rho_i = c_0 N\lambda/(c_i(\lambda+\mu))$ and the stability condition of the tandem fluid queues is $\rho_M < 1$.

3 A BUFFER FED BY AN MM1(β,a,b,r) SOURCE

We consider a single fluid buffer fed by an MM1(β,a,b,r) source. The service rate of the buffer is denoted by c, $c < r$. We derive an expression of the stationary buffer content distribution in terms of a series whose coefficients correspond to the successive powers of a *key matrix* G. The generating function of G is expressed as a function of the known generating function of a *key matrix* T and is explicitly inverted.

The continuous time birth and death process associated with the MM1(β,a,b,r) source is denoted by $\{X_t,\ t \geq 0\}$ and its infinitesimal generator A is described by (1). We assume that $a \leq \beta$.

The drifts of that fluid queue represent the difference between the input and the service rates. Let d_j be the drift when X_t is in the state j. We thus have $d_0 = -c$ and $d_j = r - c$, for every $j \geq 1$. The diagonal matrix containing these drifts is denoted by D. Since we are concerned by the stationary behavior of that fluid queue, we suppose that $a < b$ and that the stability condition is satisfied. Since the mean duration of on periods is $1/(b-a)$, we have

$$\rho_0 = \frac{r\beta}{c(b-a+\beta)} < 1.$$

The stationary state of the Markov chain $\{X_t, t \geq 0\}$ and the stationary amount of fluid in the buffer are denoted X and Q respectively.

Let $F_j(x) = \Pr\{X = j, Q \leq x\}$. It is easy to see that for $j \geq 1$, we have $F_j(0) = 0$ and it has been shown in [Sericola and Tuffin, 1999] that $F_0(0) = 1 - \rho_0$. It is well-known, see e.g. [Mitra, 1988], that the functions F_j satisfy, for $x > 0$, the following system of differential equations $F'(x) = F(x)AD^{-1}$ where $F(x)$ denotes the infinite row vector containing the $F_j(x)$ and $F'(x)$ the derivative of $F(x)$ with respect to x. Its solution is given by $F(x) = F(0)e^{AD^{-1}x}$. Using a method similar to the uniformization technique, we introduce the *key matrix* G defined by $G = I + AD^{-1}/\theta$, where $\theta = (a+b)/(r-c)$ and I is the identity matrix. We then have, for every $j \geq 0$,

$$F_j(x) = (1-\rho_0)\sum_{n=0}^{\infty} e^{-\theta x}\frac{(\theta x)^n}{n!}\, G^n_{0,j}\,, \quad (2)$$

where $G^n_{0,j}$ denotes the $(0,j)$ entry of matrix G^n. In what follows, we focus on the calculation of $G^n_{0,j}$ using generating functions.

3.1 Generating Functions

Let us consider the complex matrices M indexed on $\mathbb{N} \times \mathbb{N}$. We define

$$\nu(M) = \sup_{i \in \mathbb{N}} \sum_{j=0}^{\infty} |M_{ij}|$$

and denote by $\mathcal{M}$ the set of infinite complex matrices M such that $\nu(M)$ is finite. ν is a norm on

$\mathcal{M}$ and $(\mathcal{M}, \nu)$ is a Banach algebra. With each $M \in \mathcal{M}$, we associate the complex function Φ_M, called potential kernel of M or generating function, defined by

$$\Phi_M(z) = \sum_{k=0}^{\infty} M^k z^k$$

for every z such that $|z| < 1/\nu(M)$. Note that for $M \in \mathcal{M}$ and z such that $|z| < 1/\nu(M)$, we have $\Phi_M(z) \in \mathcal{M}$ since $\nu(\Phi_M(z)) \leq 1/(1 - |z|\nu(M)) < +\infty$.
The following lemma is a classical straightforward result, so we give it without proof.

Lemma 4 *For every matrix H, $H\Phi_M$ is the only solution to the matrix equation*

$$X(z) = H + zX(z)M$$

for every z such that $|z| < 1/\nu(M)$.

We shall also need the following result, due to [Leguesdron et al., 1991], which will be used along with Lemma 4.

Lemma 5 *For every M and N in $\mathcal{M}$, we have* $\Phi_{M+N}(z) = \Phi_M(z) + z\Phi_{M+N}(z)N\Phi_M(z)$ *for every z such that $|z| < \min\{1/\nu(M), 1/\nu(M + N)\}$.*

Let us now introduce some notations. We define the infinite matrices V, W, R and S as

$$V_{i,j} = I_{i+1,j} \ , \ W_{i,j} = I_{i,j+1} \ , \ R_{i,j} = I_{i,0}I_{0,j}$$

and $S_{i,j} = I_{i,0}I_{1,j}$ for i and $j \in \mathbb{N}$. We studied in [Barbot and Sericola, 2002] the *key matrix* T associated to a fluid buffer fed by an M/M/1 queue with arrival rate a and service rate b. The input and service rates of the buffer are respectively r and c. Therefore, the non-zero entries of T are given by

$$T_{0,0} = q + pr/c \ , \ T_{0,1} = p \ , \ T_{1,0} = q - qr/c$$

$$T_{1,2} = p \ \text{ and for } i \geq 2, T_{i,i-1} = q \ , \ T_{i,i+1} = p$$

where p and q are defined by

$$p = a/(a + b) \ \text{ and } \ q = b/(a + b) .$$

Notice that the stability condition of the fluid model associated with T is satisfied, that is $\rho = ra/cb < 1$.
After some algebra, we easily obtain the following relation between matrices G and T.

Lemma 6 *We have $G = T + U$ where $U = (p_0 - p)((r/c - 1)R + S)$ and $p_0 = \beta/(a + b)$.*

Since $\beta \geq a$, we have $p_0 \geq p$ and so $\nu(G) \geq \nu(T)$. Using Lemma 5, we obtain

$$\Phi_G(z) = \Phi_T(z) + z\Phi_G(z)U\Phi_T(z) \qquad (3)$$

for every z such that $|z| < 1/\nu(G)$. We define the matrix $L(z)$ as

$$L(z) = U\Phi_T(z) .$$

For $|z| < 1/\nu(T)$, we have $\nu(L(z)) = \nu(U\Phi_T(z)) \leq \nu(U)/(1 - |z|\nu(T))$, and so for every z such as $|z| < 1/(\nu(T) + \nu(U))$, we have $|z| < 1/\nu(L(z))$ which proves that $L(z) \in \mathcal{M}$. Lemma 4 applied to Relation (3) with $X(z) = \Phi_G(z)$, $H = \Phi_T(z)$ and $M = L(z)$ leads to

$$\Phi_G(z) = \Phi_T(z)\Phi_{L(z)}(z) \qquad (4)$$

for $|z| < \min\{1/\nu(G), 1/(\nu(T) + \nu(U))\}$ where $\nu(U) = (p_0 - p)r/c$.
In order to derive an expression of the potential kernel Φ_G given in (4), we first recall in the next lemma the expression of Φ_T obtained in [Barbot and Sericola, 2002]. For that, we introduce, for z such that $|z| \leq 1/4$, the function $C(z) = (1 - \sqrt{1 - 4z})/2z$.

Lemma 7 *Let $|z| < 1$ and $\eta(z) = C(pqz^2)$. Let $X(z)$ and $Y(z)$ be the matrices defined by*

$$\begin{aligned} X_{i,j}(z) &= (qz\eta(z))^i(pz\eta(z))^j \\ Y(z) &= \sum_{k=0}^{\infty} W^k X(z) V^k . \end{aligned}$$

For every z such that $|z| < \min\{1/2, c/(qr + c)\}$, we have

$$\Phi_T(z) = \eta(z)Y(z) + qz\eta^2(z)\frac{(1 + \rho - \rho qz\eta(z))X(z) - \frac{r}{c}WX(z)}{(1 - qz\eta(z))(1 - \rho qz\eta(z))} . \qquad (5)$$

Theorem 8 *For every z such that $|z| < 1/2$, we have*

$$L(z) = u(z)RX(z) + \eta(z)(p_0 - p)RX(z)V , \qquad (6)$$

$$\Phi_{L(z)}(z) = I + \frac{z}{1 - zu(z)}L(z) , \qquad (7)$$

where $u(z) = (p_0 - p)(r/c - 1)\dfrac{\eta(z)}{1 - \rho qz\eta(z)}$.

Proof. Let z be such that $|z| < 1/2$. Since $RW = 0$ and $SW = R$, we have by definition of $X(z)$ and $Y(z)$

$$RY(z) \;=\; RX(z)\;,\; SX(z) \;=\; qz\eta(z)RX(z)$$

and

$$SY(z) \;=\; qz\eta(z)RX(z) + RX(z)V\,.$$

Lemma 7 leads to

$$L(z) \;=\; \eta(z)(p_0 - p)\Big((r/c-1)R + S\Big)\Bigg(Y(z) + qz\eta(z)\frac{(1+\rho-\rho qz\eta(z))X(z) - \frac{r}{c}WX(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))}\Bigg)$$

and using the relations above, we obtain (6). Consider now the successive powers $L^k(z)$ of matrix $L(z)$. Observing that $VR = 0$ and

$$X(z)RX(z) \;=\; X(z)\,, \tag{8}$$

we easily get from (6) that $L^2(z) = u(z)L(z)$. It follows by induction that for every $k \geq 0$,

$$L^{k+1}(z) \;=\; u^k(z)L(z)\,.$$

Since $|z| < 1/2$, it is easy to check, from the definition of the function C, that $|\eta(z)| \leq 2$ and therefore $|qz\eta(z)| < 1$. Moreover, since $\rho_0 < 1$, we have $(p_0 - p)(r/c - 1) < q(1 - \rho)$ and so $|u(z)| < 1$. Thus, we obtain

$$\begin{aligned} \Phi_{L(z)}(z) &= I + z\sum_{k=0}^{\infty}(zu(z))^k L(z) \\ &= I + \frac{z}{1-zu(z)}L(z)\,. \end{aligned}$$

Theorem 9 *For $|z| < \min\{1/2, c/(qr+c), 1/(\nu(G)+\nu(U))\}$, we have*

$$\begin{aligned} \Phi_G(z) \;=\; &\eta(z)Y(z) \\ &+ \eta(z)\frac{qz\eta(z)(1+\rho-\rho qz\eta(z)) + \dfrac{zu(z)}{1-zu(z)}}{(1-qz\eta(z))(1-\rho qz\eta(z))}X(z) \\ &+ \left(\frac{c}{r-c}\right)\frac{zu(z)}{(1-qz\eta(z))(1-zu(z))}X(z)V \\ &- \left(\frac{r}{c}\right)\frac{qz\eta^2(z)}{(1-qz\eta(z))(1-zu(z))}WX(z) \\ &- \left(\frac{r}{r-c}\right)\frac{qz^2\eta^2(z)u(z)}{(1-qz\eta(z))(1-zu(z))}WX(z)V \end{aligned} \tag{9}$$

Proof. Let z be such that $|z| < \min\{1/2, c/(qr + c), 1/(\nu(G) + \nu(U))\}$. Replacing Relations (5) and (7) in (4), we obtain

$$\Phi_G(z) \;=\; \eta(z)\left(I + \frac{z}{1-zu(z)}L(z)\right)\Bigg(Y(z) + qz\eta(z)\frac{(1+\rho-\rho qz\eta(z))X(z) - \frac{r}{c}WX(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))}\Bigg). \tag{10}$$

Now, since $VR = 0$, we obtain from (8) that $Y(z)RX(z) = X(z)$. We get from (6),

$$Y(z)L(z) \;=\; u(z)X(z) + \eta(z)(p_0 - p)X(z)V$$

and using (8), $X(z)L(z) = Y(z)L(z)$. Putting these relations in (10), we obtain (9).

3.2 Explicit Solution For A Single Buffer

We obtain in this section a closed-form expression for $G^n_{0,j}$ and so for $\Pr\{Q \leq x\} = \sum_{j=0}^{\infty} F_j(x)$. For that purpose, we need the following well-known lemma which gives an analytical expression of the powers of $\eta(z)$. For the proof, see e.g. [Riordan, 1968] page 154.

Lemma 10 *For every $k \geq 1$ and $|z| \leq 1/4$, we have $C^k(z) = \sum_{n=0}^{\infty} s(k,n)z^n$ where $s(k,n)$ are the ballot defined by*

$$s(k,n) \;=\; k\frac{(2n+k-1)!}{n!(n+k)!}\,.$$

Theorem 11 *For every $x \geq 0$,*

$$\begin{aligned} \Pr\{Q \leq x\} \;=\; &(1-\rho_0)\sum_{n=0}^{\infty} e^{-\theta x}\frac{(\theta x)^n}{n!} \\ &\times \left(1 + (p_0 - p)\frac{\theta x}{n+1}\right)\sum_{j=0}^{n}\left(\frac{p}{q}\right)^j \sum_{m=0}^{n-j}\gamma^m \\ &\times \sum_{k=0}^{\lfloor\frac{n-j-m}{2}\rfloor} s(n-2k+1,k)p^k q^{n-m-k} \\ &\times \sum_{h=0}^{n-j-m-2k}\frac{(m+h)!}{h!}\rho^h \end{aligned}$$

where $\lfloor u \rfloor$ denotes the largest integer less than or equal to the real number u and

$$\gamma \;=\; (p_0 - p)(r/c - 1) \;\in [0,1]\,.$$

Proof. Let z be such that $|z| < \min\{1/2, c/(qr + c), 1/(\nu(G) + \nu(U))\}$. Since the first row of the matrix $WX(z)$ has all its entries equal to zero, we have from (9), for every $j \in \mathbb{N}$,

$$\begin{aligned}(\Phi_G(z))_{0,j} &= \eta(z)Y_{0,j}(z)\\ &+ \eta(z)\frac{qz\eta(z)(1+\rho-\rho qz\eta(z)) + \dfrac{zu(z)}{1-zu(z)}}{(1-qz\eta(z))(1-\rho qz\eta(z))}X_{0,j}(z)\\ &+ \left(\frac{c}{r-c}\right)\frac{zu(z)}{(1-zu(z))(1-qz\eta(z))}(X(z)V)_{0,j}\end{aligned}$$

By definition of $X(z)$, $Y(z)$ and V, we can easily verify that

$$Y_{0,j}(z) = X_{0,j}(z) = (pz\eta(z))^j \ , \ (X(z)V)_{0,0} = 0$$

and

$$(X(z)V)_{0,j} = (pz\eta(z))^{j-1} .$$

So, we obtain

$$(\Phi_G(z))_{0,0} = \frac{\eta(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))(1-zu(z))} \tag{11}$$

and for $j \geq 1$

$$\begin{aligned}(\Phi_G(z))_{0,j} &= \frac{p^j z^j \eta^{j+1}(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))(1-zu(z))}\\ &+ \left(\frac{c}{r-c}\right)\frac{p^{j-1}z^j\eta^{j-1}(z)u(z)}{(1-qz\eta(z))(1-zu(z))} .\end{aligned} \tag{12}$$

Before inverting the expressions (11) and (12), it must be remembered that for $|x| < 1$ and $n \in \mathbb{N}$

$$(1-x)^{-n-1} = \sum_{l=0}^{\infty}\frac{(n+l)!}{l!}x^l .$$

For $|z| < 1/2$, we have $|u(z)| < 1$ and $|qz\eta(z)| < 1$ and therefore, using the Cauchy product of two series, we obtain

$$\begin{aligned}(\Phi_G(z))_{0,0} &= \frac{\eta(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))}\sum_{n=0}^{\infty}(zu(z))^n\\ &= \frac{\eta(z)}{1-qz\eta(z)}\sum_{n=0}^{\infty}(\gamma z\eta(z))^n(1-\rho qz\eta(z))^{-n-1}\\ &= \frac{\eta(z)}{1-qz\eta(z)}\sum_{n,l=0}^{\infty}(\gamma z\eta(z))^n\frac{(n+l)!}{l!}(\rho qz\eta(z))^l\\ &= \eta(z)\sum_{n,l=0}^{\infty}(\gamma z\eta(z))^n\sum_{h=0}^{l}\frac{(n+h)!}{h!}(\rho qz\eta(z))^h\\ &\quad \times (qz\eta(z))^{l-h}\\ &= \sum_{n,l=0}^{\infty} z^{n+l}\eta^{n+l+1}(z)\gamma^n q^l\sum_{h=0}^{l}\frac{(n+h)!}{h!}\rho^h .\end{aligned}$$

From Lemma 10, we have

$$\begin{aligned}\eta^{n+l+1}(z) &= C^{n+l+1}(pqz^2)\\ &= \sum_{k=0}^{\infty}s(n+l+1,k)p^kq^kz^{2k}\end{aligned}$$

which leads, by changing the order of summations, to

$$\begin{aligned}(\Phi_G(z))_{0,0} &= \sum_{n,l,k=0}^{\infty} z^{n+l+2k}s(n+l+1,k)\gamma^np^kq^{l+k}\\ &\quad \times \sum_{h=0}^{l}\frac{(n+h)!}{h!}\rho^h\\ &= \sum_{n=0}^{\infty}z^n\sum_{m=0}^{n}\gamma^m\sum_{k=0}^{\lfloor\frac{n-m}{2}\rfloor}s(n-2k+1,k)p^k\\ &\quad \times q^{n-m-k}\sum_{h=0}^{n-m-2k}\frac{(m+h)!}{h!}\rho^h .\end{aligned}$$

Then, we have for every $n \in \mathbb{N}$,

$$\begin{aligned}G^n_{0,0} &= \sum_{m=0}^{n}\gamma^m\sum_{k=0}^{\lfloor\frac{n-m}{2}\rfloor}s(n-2k+1,k)p^k\\ &\quad \times q^{n-m-k}\sum_{h=0}^{n-m-2k}\frac{(m+h)!}{h!}\rho^h .\end{aligned} \tag{13}$$

Similary, for $j \geq 1$, we obtain

$$\begin{aligned}(\Phi_G(z))_{0,j} &= z^jp^j\sum_{n=0}^{\infty}z^n\sum_{m=0}^{n}\gamma^m\sum_{k=0}^{\lfloor\frac{n-m}{2}\rfloor}p^kq^{n-m-k}\\ &\times \ s(n-2k+j+1,k)\sum_{h=0}^{n-m-2k}\frac{(m+h)!}{h!}\rho^h\\ &+ \ z^jp^{j-1}(p_0-p)\sum_{n=0}^{\infty}z^n\sum_{m=0}^{n}\gamma^m\sum_{k=0}^{\lfloor\frac{n-m}{2}\rfloor}p^kq^{n-m-k}\\ &\times \ s(n-2k+j,k)\sum_{h=0}^{n-m-2k}\frac{(m+h)!}{h!}\rho^h .\end{aligned}$$

Therefore, $G^n_{0,j} = 0$ if $n < j$, and for $n \geq j$

$$\begin{aligned}G^n_{0,j} &= \left(\frac{p}{q}\right)^j\sum_{m=0}^{n-j}\gamma^m\sum_{k=0}^{\lfloor\frac{n-j-m}{2}\rfloor}p^kq^{n-m-k}\\ &\times \ s(n-2k+1,k)\sum_{h=0}^{n-j-m-2k}\frac{(m+h)!}{h!}\rho^h\\ &+ \ \frac{p_0-p}{q}\left(\frac{p}{q}\right)^{j-1}\sum_{m=0}^{n-j}\gamma^m\sum_{k=0}^{\lfloor\frac{n-j-m}{2}\rfloor}p^kq^{n-m-k}\\ &\times \ s(n-2k,k)\sum_{h=0}^{n-j-m-2k}\frac{(m+h)!}{h!}\rho^h .\end{aligned} \tag{14}$$

Putting Relations (13) and (14) in (2), we obtain the result by summing over j and changing the order of summations.

4 THE FLUID CONTENT OF THE $(i+1)$TH BUFFER

We suppose that the stability condition of the tandem fluid queues is satisfied, that is, $\rho_M < 1$. For $1 \le i \le M-1$, we derive the distribution of the stationary level Q_{i+1} of the $(i+1)$th buffer.

Theorem 12 *For every $x \ge 0$ and $1 \le i \le M-1$*

$$\begin{aligned}
\Pr\{Q_{i+1} \le x\} \;=\; & (1-\rho_i)\sum_{n=0}^{\infty} e^{-\theta_i x}\frac{(\theta_i x)^n}{n!} \\
\times \; & \left(1+\frac{\lambda x}{(n+1)c_0(1-c_{i+1}/c_i)}\right)\sum_{j=0}^{n}\left(\frac{p_i}{q_i}\right)^j \\
\times \; & \sum_{m=0}^{n-j}\gamma_i^m \sum_{k=0}^{\lfloor\frac{n-j-m}{2}\rfloor} s(n-2k+1,k)p_i^k q_i^{n-m-k} \\
\times \; & \sum_{h=0}^{n-j-m-2k}\frac{(m+h)!}{h!}\rho_i'^h
\end{aligned}$$

where

$$\begin{aligned}
\lambda_i &= \lambda(N-c_i/c_0)\,, \;\; \mu_i \;=\; \mu c_i/c_0\,, \\
p_i &= \lambda_i/(\lambda_i+\mu_i)\,, \quad q_i \;=\; 1-p_i\,, \\
\theta_i &= (\lambda_i+\mu_i)/(c_i-c_{i+1})\,, \\
\rho_i' &= c_i\lambda_i/(c_{i+1}\mu_i)\;, \\
\gamma_i &= (c_i/c_0)(c_i/c_{i+1}-1)\lambda/(\lambda_i+\mu_i)\,.
\end{aligned}$$

Proof. We saw that the stationary level of the $(i+1)$th buffer is equivalent to the stationary level of an infinite buffer with service rate c_{i+1} and fed by an MM1($N\lambda$,λ_i,μ_i,c_i). We then apply Theorem 11 to this fluid model and set $\beta = N\lambda$, $a = \lambda_i$, $b = \mu_i$, $r = c_i$ and $c = c_{i+1}$.

REFERENCES

Aalto S. 1998, "Output of a multiplexer loaded by heterogeneous on-off sources". *Commun. Statist.-Stochastic Models*, Vol. 14(4). Pp993–1005.

Aalto S. and Scheinhardt W.R.W. 2000, "Tandem fluid queues fed by homogeneous on-off sources". *Op. Res. Letters*, Vol. 27. Pp73–82.

Anick D., Mitra D. and Sondhi M. M. 1982, "Stochastic Theory of a Data-Handling System with Multiple Sources". *Bell Syst. Tech. J.*, Vol. 61(8). Pp1871–1894.

Barbot N. and Sericola B. 2002, "Stationary solution to the fluid queue fed by an M/M/1 queue". *J. Appl. Prob.* Vol. 39. Pp359–369.

Boxma O.J. and Dumas V. 1998, "The busy period in the fluid queue". *Perf. Eval. Rew.* Vol. 26. Pp100–110.

Leguesdron P., Pellaumail J., Rubino G. and Sericola B. 1993, "Transient analysis of the M/M/1 queue". *Adv. Appl. Prob.* Vol. 25. Pp702–713.

Mitra D. 1988, "Stochastic theory of a fluid model of producers and consumers coupled by a buffer". *Adv. Appl. Prob.* Vol. 20. Pp646–676.

Riordan J. 1968, *Combinatorial identities*, John Wiley, New York.

Sericola B. and Tuffin B. 1999, "A fluid queue driven by a Markovian queue". *Queueing Systems* Vol. 31. Pp253–264.

BIOGRAPHY

NELLY BARBOT is a temporary associate professor in applied mathematics at ENSSAT, an engineering school (École Nationale Supérieure de Sciences Appliquées et Technologie), University of Rennes I (France). She received the Ph.D. degree in applied mathematics from the Uni versity of Rennes I in 2002. She is also part of the ARMOR (Networks ARchitectures and Modeling) group at IRISA-INRIA laboratory. Her research subject is communication systems performance evaluation with stochastic models.

BRUNO SERICOLA received the Ph.D. degree in computer science from the University of Rennes I in 1988. He has been with INRIA (Institut National de Recherche en Informatique et Automatique, a public research French laboratory) sin ce 1989. His main research activity is in computer and communication systems performance evaluation, dependability and performability analysis of fault-tolerant architectures and applied stochastic processes.