# Impatient service in a G-network 

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#### Abstract

An open exponential queueing network with signals and impatient service is considered. Upon completion of service at a node, a positive customer passes to another node with fixed probabilities either as a positive customer or as a signal, or quits the network. Every signal is activated during a random exponentially distributed amount of time. Activated signals with fixed probabilities either move a customer from the node they arrive to another node or kill a positive customer. Each customer can be served in a node at most a random time ("patient" time) distributed exponentially. When the patient service is


finished, the customer with fixed probabilities either goes to another node or quits the network. The stationary state probabilities for such a G-network in which positive customers are processed in each node by a single server is derived in product form. The solution for an analogous symmetrical G-network in which service rate of a positive customer at each node depends on the number of positive customers in this node is expressed in product form too.

Keywords: G-networks, positive customers, impatient service and product form solution.

## 1 Introduction

In the last years queueing networks with "negative and positive customers, triggers, signals", called Gnetworks have been studied and solution in product form have been obtained [Gelenbe and Pujolle, 1998]. Positive customers are ordinary customers and they are served by the server in the normal way, instead a negative customer deletes (kills) a positive customer, or a trigger, which moves a positive customer from a node to another one; a signal combines these two kinds of customers and can act either as a negative costomer or as a trigger. The analysis of this new class of queueing networks was first inspired by the study of neural networks. G-networks can be applied too in performance evaluation of computer networks, to model, for example, the effect of flow control [Gelenbe and Pujolle, 1998]. A vast review of papers concerning with G-networks is done in [Artalejo 2000].

We consider queueing networks with positive customers and signals. External arrival flows of positive customers and signals are independent Poisson processes. The service times of positive customers at each node are exponentially distributed. Upon completion of service at a node, a positive customer passes to another node with fixed probabilities either as a positive customer or as a signal, or quits the network. Every signal is activated during a random exponentially distributed amount of time. Ac-
tivated signals with fixed probabilities either move a customer from the node they arrive to another node or kill a positive customer. We assume additionally that each customer can be served in a node at most a random time ("patient" time) distributed exponentially. When the patient service is finished, the customer with fixed probabilities either goes to another node or quits the network.

A G-network with instantaneous signal activation is studied in [Gelenbe and Pujolle, 1998]. An analogous G-network with random signal activation period without impatient time is considered in [Bocharov 2002]. The stationary state distributions for the networks considered in [Gelenbe and Pujolle, 1998], [Bocharov 2002], were derived in product form. In this paper, the stationary state distribution for a Gnetwork with random delay of signals and impatient service in which positive customers are processed by a single server is derived in product form. Moreover, the solution for an analogous symmetrical G-network in which service rate of a positive customer at each node depends on the number of positive customers in this node is expressed in product form too.

## 2 Mathematical model

We deal with G-networks with $M$ nodes, positive customers and signals. Positive customers and signals arrive from outside (from node 0 ) according to inde-
pendent Poisson processes. We denote, respectively, with $\lambda_{0 i}^{+}$and $\lambda_{0 i}^{-}$the arrival rate of external positive customers and external signals at node $i$.

The service of a positive customer is completed at node $i$ with probability $\mu_{i}^{+}(k) \Delta+o(\Delta)$ in a time interval $(t, t+\Delta)$, provided that $k$ positive customers are present at this node at instant $t$.

Upon completion of service at node $i$, a positive customer goes from node $i$ to node $j$ with probability $p_{i j}^{+}$as a positive customer, and with probability $p_{i j}^{-}$ as a signal. He leaves the network with probability $p_{i 0}=1-\sum_{j=1}^{M}\left(p_{i j}^{+}+p_{i j}^{-}\right)$.

Every signal is activated during a random time. A signal arriving at node $i$ is activated in a time interval $(t, t+\Delta)$ with probability $\mu_{i}^{-}(n) \Delta+o(\Delta)$, provided that $n$ non activated signals are present at this node at instant $t$.

After the completion of the activation period a signal:

- with probability $q_{i j}^{+}$moves a positive customer from node $i$ to node $j$ and retains him as a positive customer (in this case, a signal acts as a trigger);
- with probability $q_{i j}^{-}$moves a positive customer from node $i$ to node $j$ and retains him as a signal;
- with probability $q_{i 0}$ kills a positive customer at node $i$ and he vanishes (in this case, the signal acts as a negative customer).

If there are not positive customers at node $i$ an activated signal at the node disappears.

The impatience time of a customer in the node $i$ is completed in a time interval $(t, t+\Delta)$ with probability $\gamma_{i}(k) \Delta+o(\Delta)$, provided that $k$ positive customers are present at this node at instant $t$. Then a positive customer with probability $r_{i j}^{+}$goes from node $i$ to node $j$ as a positive customer, with probability $r_{i j}^{-}$as a signal, and with probability $r_{i 0}=1-\sum_{j=1}^{M}\left(r_{i j}^{+}+r_{i j}^{-}\right)$ he leaves the network.

## 3 Equilibrium equations

Let us denote with $P^{+}, P^{-}, Q^{+}, Q^{-}, R^{+}, R^{-}$the matrices with elements $p_{i j}^{+}, p_{i j}^{-}, q_{i j}^{+}, q_{i j}^{-}, r_{i j}^{+}, r_{i j}^{-}$, respectively, $i, j=\overline{1, M}$, and let us set $P=P^{+}+P^{-}$, $Q=Q^{+}+Q^{-}$, and $R=R^{+}+R^{-}$.

The stochastic behaviour of the queueing network under consideration can be described by an homogeneous Markov process $\{X(t), \quad t \geq 0\}$ with the following state space:
$\mathcal{X}=\left\{\left(\left(k_{1}, n_{1}\right), \ldots,\left(k_{M}, n_{M}\right)\right), k_{i} \geq 0, n_{i} \geq 0, i=\overline{1, M}\right\}$.

The state $\left(\left(k_{1}, n_{1}\right),\left(k_{2}, n_{2}\right), \ldots,\left(k_{M}, n_{M}\right)\right)$ means that at any instant there are $k_{1}$ positive customers and $n_{1}$ non-activated signals at node $1, k_{2}$ customers and $n_{2}$ signals at node $2, \ldots$, and finally, $k_{M}$ customers and $n_{M}$ signals at node $M$.

Introducing vectors $\vec{k}=\left(k_{1}, k_{2}, \ldots, k_{M}\right)$ and $\vec{n}=\left(n_{1}, n_{2}, \ldots, n_{M}\right)$, let us take $(\vec{k}, \vec{n})=$ $\left(\left(k_{1}, n_{1}\right),\left(k_{2}, n_{2}\right), \ldots,\left(k_{M}, n_{M}\right)\right)$. We also introduce
the vector $\vec{e}_{i}$ with $i$-th component equal to 1 and other components equal to 0 . We also use the notation $\lambda_{0}^{+}=\sum_{i=1}^{M} \lambda_{0 i}^{+}$and $\lambda_{0}^{-}=\sum_{i=1}^{M} \lambda_{0 i}^{-}$.

Let $p(\vec{k}, \vec{n})$ denote the stationary probability of the state $(\vec{k}, \vec{n})$. If the stationary distribution $\{p(\vec{k}, \vec{n}), \quad \vec{k}, \vec{n} \geq \overrightarrow{0}\}$ of the process $\{X(t), \quad t \geq 0\}$ exists, then the following system of equilibrium equations holds:
$p(\vec{k}, \vec{n})\left(\lambda_{0}^{+}+\lambda_{0}^{-}+\sum_{i=1}^{M} \mu_{i}^{+}\left(k_{i}\right)\left(1-p_{i i}^{+}\right)+\sum_{i=1}^{M} \mu_{i}^{-}\left(n_{i}\right)\right.$
$\left.+\sum_{i=1}^{M} \gamma_{i}\left(k_{i}\right)\left(1-\gamma_{i i}^{+}\right)\right)=\sum_{i=1}^{M} p\left(\vec{k}-\vec{e}_{i}, \vec{n}\right) \lambda_{0 i}^{+} u\left(k_{i}\right)+$
$\sum_{i=1}^{M} p\left(\vec{k}, \vec{n}-\vec{e}_{i}\right) \lambda_{0 i}^{-} u\left(n_{i}\right)+$
$\sum_{i=1}^{M} p\left(\vec{k}+\vec{e}_{i}, \vec{n}\right) \mu_{i}^{+}\left(k_{i}+1\right) p_{i 0}+$
$\sum_{i=1}^{M} p\left(\vec{k}+\vec{e}_{i}, \vec{n}+\vec{e}_{i}\right) \mu_{i}^{-}\left(n_{i}+1\right) q_{i 0}+$
$\sum_{i=1}^{M} p\left(\vec{k}+\vec{e}_{i}, \vec{n}\right) \gamma_{i}\left(k_{i}+1\right) r_{i 0}+$
$\sum_{i=1}^{M} p\left(\vec{k}, \vec{n}+\vec{e}_{i}\right) \mu_{i}^{-}\left(n_{i}+1\right)\left(1-u\left(k_{i}\right)\right)+$
$\sum_{i=1}^{M} \sum_{j=1, j \neq i}^{M} p\left(\vec{k}+\vec{e}_{i}-\vec{e}_{j}, \vec{n}\right) \mu_{i}^{+}\left(k_{i}+1\right) p_{i j}^{+} u\left(k_{j}\right)+$
$\sum_{i=1}^{M} \sum_{j=1}^{M} p\left(\vec{k}+\vec{e}_{i}, \vec{n}-\vec{e}_{j}\right) \mu_{i}^{+}\left(k_{i}+1\right) p_{i j}^{-} u\left(n_{j}\right)+$
$\sum_{i=1}^{M} \sum_{j=1}^{M} p\left(\vec{k}+\vec{e}_{i}-\vec{e}_{j}, \vec{n}+\vec{e}_{i}\right) \mu_{i}^{-}\left(n_{i}+1\right) q_{i j}^{+} u\left(k_{j}\right)+$
$\sum_{i=1}^{M} \sum_{j=1, j \neq i}^{M} p\left(\vec{k}+\vec{e}_{i}, \vec{n}+\vec{e}_{i}-\vec{e}_{j}\right) \mu_{i}^{-}\left(n_{i}+1\right) q_{i j}^{-} u\left(n_{j}\right)+$
$\sum_{i=1}^{M} p\left(\vec{k}+\vec{e}_{i}, \vec{n}\right) \mu_{i}^{-}\left(n_{i}\right) q_{i i}^{-}+$
$\sum_{i=1}^{M} \sum_{j=1, j \neq i}^{M} p\left(\vec{k}+\vec{e}_{i}-\vec{e}_{j}, \vec{n}\right) \gamma_{i}\left(k_{i}+1\right) r_{i j}^{+} u\left(k_{j}\right)+$
$\sum_{i=1}^{M} \sum_{j=1}^{M} p\left(\vec{k}+\vec{e}_{i}, \vec{n}-\vec{e}_{j}\right) \gamma_{i}\left(k_{i}+1\right) r_{i j}^{-} u\left(n_{j}\right)$,

$$
(\vec{k}, \vec{n}) \in \mathcal{X}
$$

where $\mu_{i}^{+}(0)=0, \mu_{i}^{-}(0)=0, \gamma_{i}(0)=0$ and $u(x)$ is a unit Heavyside function.

## 4 Solution in product form

We can not possible find the general product-form of the system of equations (1). Nevertheless, solutions for two important cases are given.

### 4.1 Service of positive customers by a single server

Consider a network in which positive customers are served at every node by a single server and the service time at node $i$ is exponentially distributed with parameter $\mu_{i}^{+}$. Therefore

$$
\begin{equation*}
\mu_{i}^{+}\left(k_{i}\right)=u\left(k_{i}\right) \mu_{i}^{+}, \quad i=\overline{1, M} \tag{2}
\end{equation*}
$$

We also assume that

$$
\begin{equation*}
\gamma_{i}\left(k_{i}\right)=u\left(k_{i}\right) \gamma_{i}, \quad i=\overline{1, M} \tag{3}
\end{equation*}
$$

Let us introduce the following notations:

$$
\begin{align*}
q_{i} & =\frac{\lambda_{i}^{+}}{\lambda_{i}^{-}+\mu_{i}^{+}+\gamma_{i}}, \quad \rho_{i}^{-}(j)=\frac{\lambda_{i}^{-}}{\mu_{i}^{-}(j)}, \quad i, j=\overline{1, M} \\
\lambda_{i}^{+} & =\lambda_{0 i}^{+}+\sum_{j=1}^{M} q_{j}\left(\mu_{j}^{+} p_{j i}^{+}+\lambda_{j}^{-} q_{j i}^{+}+\gamma_{j} r_{j i}^{+}\right), i=\overline{1, M}  \tag{4}\\
\lambda_{i}^{-} & =\lambda_{0 i}^{-}+\sum_{j=1}^{M} q_{j}\left(\mu_{j}^{+} p_{j i}^{-}+\lambda_{j}^{-} q_{j i}^{-}+\gamma_{j} r_{j i}^{-}\right), i=\overline{1, M} . \tag{5}
\end{align*}
$$

As in [1] we can prove that there exists a unique positive solution $\lambda_{i}^{+}, \lambda_{i}^{-}, i=\overline{1, M}$ of the system of equa(1) tions (5).

Besides let us denote

$$
\begin{equation*}
\Lambda_{0}=\sum_{j=1}^{M} q_{j} \mu_{j}^{+} p_{j 0}+\sum_{j=1}^{M} q_{j} \lambda_{j}^{-} q_{j 0}+\sum_{j=1}^{M} q_{j} \gamma_{j} r_{j 0} . \tag{6}
\end{equation*}
$$

From (4) - (6) we obtain

$$
\begin{gathered}
\Lambda_{0}+\sum_{j=1}^{M}\left(\lambda_{j}^{+}+\lambda_{j}^{-}\right)=\lambda_{0}^{+}+\lambda_{0}^{-}+\sum_{j=1}^{M} q_{j}\left(\mu_{j}^{+}+\lambda_{j}^{-}+\gamma_{j}\right)= \\
\lambda_{0}^{+}+\lambda_{0}^{-}+\sum_{j=1}^{M} \lambda_{j}^{+}
\end{gathered}
$$

Therefore

$$
\begin{equation*}
\Lambda_{0}+\sum_{j=1}^{M} \lambda_{j}^{-}=\lambda_{0}^{+}+\lambda_{0}^{-} \tag{7}
\end{equation*}
$$

The following theorem holds.

Theorem 1 If the matrices $P, Q$, and $R$ are irreducible, conditions (2) and (3) hold, and a unique positive solution of equations (5) exists such that

$$
\begin{gathered}
\lambda_{i}^{+}<\lambda_{i}^{-}+\mu_{i}^{+}+\gamma_{i}, \quad i=\overline{1, M}, \\
G_{i}=\sum_{n_{i}=0}^{\infty} \prod_{j=1}^{n_{i}} \rho_{i}^{-}(j)<\infty, \quad i=\overline{1, M}
\end{gathered}
$$

then the Markov process $\{X(t), \quad t \geq 0\}$ is ergodic and its stationary distribution is represented in a product form as

$$
\begin{equation*}
p(\vec{k}, \vec{n})=\prod_{i=1}^{M} p\left(k_{i}, n_{i}\right) \tag{8}
\end{equation*}
$$

where

$$
\begin{equation*}
p\left(k_{i}, n_{i}\right)=\left(1-q_{i}\right) q_{i}^{k_{i}} G_{i}^{-1} \prod_{j=1}^{n_{i}} \rho_{i}^{-}(j), \quad k_{i}, n_{i} \geq 0 \tag{9}
\end{equation*}
$$

and $\prod_{j=1}^{0} \equiv 1$.

Proof. The substitution of expressions (8), (9),
(4) for the stationary distribution of the process $\{X(t), \quad t \geq 0\}$ into the equilibrium system of equations (1) leads to the following equalities:

$$
\begin{align*}
& \lambda_{0}^{+}+\lambda_{0}^{-}+\sum_{i=1}^{M} \mu_{i}^{+} u\left(k_{i}\right)+\sum_{i=1}^{M} \mu_{i}^{-}\left(n_{i}\right)+\sum_{i=1}^{M} \gamma_{i} u\left(k_{i}\right)= \\
& \sum_{i=1}^{M} \frac{\lambda_{0 i}^{+}}{q_{i}} u\left(k_{i}\right)+\sum_{i=1}^{M} \frac{\mu_{i}^{-}\left(n_{i}\right)}{\lambda_{i}^{-}} \lambda_{0 i}^{-}+\sum_{i=1}^{M} q_{i} \mu_{i}^{+} p_{i 0}+ \\
& \sum_{i=1}^{M} q_{i} \lambda_{i}^{-} q_{i 0}+\sum_{i=1}^{M} q_{i} \gamma_{i} r_{i 0}+\sum_{i=1}^{M} \lambda_{i}^{-}\left(1-u\left(k_{i}\right)\right)+ \\
& \sum_{i=1}^{M} \sum_{j=1}^{M} \frac{q_{i}}{q_{j}} \mu_{i}^{+} p_{i j}^{+} u\left(k_{j}\right)+ \\
& \sum_{i=1}^{M} \sum_{j=1}^{M} q_{i} \frac{\mu_{j}^{-}\left(n_{j}\right)}{\lambda_{j}^{-}} \mu_{i}^{+} p_{i j}^{-}+\sum_{i=1}^{M} \sum_{j=1}^{M} \frac{q_{i}}{q_{j}} \lambda_{i}^{-} q_{i j}^{+} u\left(k_{j}\right)+ \\
& \sum_{i=1}^{M} \sum_{j=1}^{M} q_{i} \frac{\mu_{j}^{-}\left(n_{j}\right)}{\lambda_{j}^{-}} \lambda_{i}^{-} q_{i j}^{-}+\sum_{i=1}^{M} \sum_{j=1}^{M} \frac{q_{i}}{q_{j}} \gamma_{i} r_{i j}^{+} u\left(k_{j}\right)+ \\
& \sum_{i=1}^{M} \sum_{j=1}^{M} q_{i} \frac{\mu_{j}^{-}\left(n_{j}\right)}{\lambda_{j}^{-}} \gamma_{i} r_{i j}^{-} . \tag{10}
\end{align*}
$$

The latter equality takes place for all $(\vec{k}, \vec{n}) \in \mathcal{X}$.
Let us denote by

$$
\begin{aligned}
& A=\sum_{i=1}^{M} \frac{\mu_{i}^{-}\left(n_{i}\right)}{\lambda_{i}^{-}} \lambda_{0 i}^{-}+\sum_{i=1}^{M} \sum_{j=1}^{M} q_{i} \frac{\mu_{j}^{-}\left(n_{j}\right)}{\lambda_{j}^{-}} \mu_{i}^{+} p_{i j}^{-}+ \\
& \sum_{i=1}^{M} \sum_{j=1}^{M} q_{i} \frac{\mu_{j}^{-}\left(n_{j}\right)}{\lambda_{j}^{-}} \lambda_{i}^{-} q_{i j}^{-}+\sum_{i=1}^{M} \sum_{j=1}^{M} q_{i} \frac{\mu_{j}^{-}\left(n_{j}\right)}{\lambda_{j}^{-}} \gamma_{i} r_{i j}^{-} .
\end{aligned}
$$

Taking into account (5) we obtain

$$
\begin{equation*}
A=\sum_{j=1}^{M} \mu_{j}^{-}\left(n_{j}\right) \tag{11}
\end{equation*}
$$

Further let us denote by

$$
\begin{aligned}
& B=\sum_{i=1}^{M} \frac{\lambda_{0 i}^{+}}{q_{i}} u\left(k_{i}\right)+\sum_{i=1}^{M} \sum_{j=1}^{M} \frac{q_{i}}{q_{j}} \mu_{i}^{+} p_{i j}^{+} u\left(k_{j}\right)+ \\
& \sum_{i=1}^{M} \sum_{j=1}^{M} \frac{q_{i}}{q_{j}} \lambda_{i}^{-} q_{i j}^{+} u\left(k_{j}\right)+\sum_{i=1}^{M} \sum_{j=1}^{M} \frac{q_{i}}{q_{j}} \gamma_{i} r_{i j}^{+} u\left(k_{j}\right) .
\end{aligned}
$$

After some transformations of the right part with
combination with (5) we obtain

$$
\begin{align*}
& B=\sum_{j=1}^{M} \frac{\lambda_{j}^{-}+\mu_{j}^{+}+\gamma_{j}}{\lambda_{j}^{+}}\left[\lambda_{0 j}^{+}+\right. \\
& \left.\sum_{i=1}^{M} q_{i}\left(\mu_{i}^{+} p_{i j}^{+}+\lambda_{i}^{-} q_{i j}^{+}+\gamma_{i} r_{i j}^{+}\right)\right] u\left(k_{j}\right)=  \tag{12}\\
& \sum_{j=1}^{M}\left(\lambda_{j}^{-}+\mu_{j}^{+}+\gamma_{j}\right) u\left(k_{j}\right) .
\end{align*}
$$

Finally let us introduce

$$
\begin{equation*}
C=\Lambda_{0}+\sum_{i=1}^{M} \lambda_{i}^{-}\left(1-u\left(k_{i}\right)\right) . \tag{13}
\end{equation*}
$$

Then the right part of equalities (10) can be represented as $A+B+C$. Then we have

$$
\begin{aligned}
& A+B+C= \\
& \sum_{j=1}^{M} \mu_{j}^{-}\left(n_{j}\right)+\sum_{j=1}^{M}\left(\lambda_{j}^{-}+\mu_{j}^{+}+\gamma_{j}\right) u\left(k_{j}\right)+\Lambda_{0}+ \\
& \sum_{i=1}^{M} \lambda_{i}^{-}-\sum_{i=1}^{M} \lambda_{i}^{-} u\left(k_{i}\right)= \\
& \sum_{j=1}^{M} \mu_{j}^{-}\left(n_{j}\right)+\sum_{j=1}^{M}\left(\mu_{j}^{+}+\gamma_{j}\right) u\left(k_{j}\right)+\Lambda_{0}+\sum_{i=1}^{M} \lambda_{i}^{-}= \\
& \lambda_{0}^{+}+\lambda_{0}^{-}+\sum_{j=1}^{M} \mu_{j}^{-}\left(n_{j}\right)+\sum_{j=1}^{M}\left(\mu_{j}^{+}+\gamma_{j}\right) u\left(k_{j}\right) .
\end{aligned}
$$

This coincides with the left part of equalities (10). Thus the substitution of (8), (9) into the system of equations (1)-(3) leads to a system of identities for all $(\vec{k}, \vec{n}) \in \mathcal{X}$. Under the assumptions of the theorem the expressions (8), (9) determine a positive solution of the equilibrium system of equations (1) - (3) and this solution is bounded. Moreover under theorem assumptions the process $\{X(t), \quad t \geq 0\}$ is irreducible. Therefore, according to Foster's theorem the process is ergodic and the relations (8), (9) give us its unique stationary distribution. Thus, the theorem is proved.

### 4.2 Symmetrical network

We consider the network described in the section 2 with $(i, j=\overline{1, M})$

$$
\begin{equation*}
p_{i j}^{+}=q_{i j}^{+}=r_{i j}^{+}, p_{i j}^{-}=q_{i j}^{-}=r_{i j}^{-}, p_{i 0}=q_{i 0}=r_{i 0} \tag{14}
\end{equation*}
$$

It is convenient to call a queueing network under these conditions as a symmetrical network.

Let us introduce the following notations:

$$
\begin{gather*}
q_{i}(j)=\frac{\lambda_{i}^{+}}{\lambda_{i}^{-}+\mu_{i}^{+}(j)+\gamma_{i}(j)}, \rho^{-}(j)=\frac{\lambda_{i}^{-}}{\mu_{i}^{-}(j)}, \\
i, j=\overline{1, M} .  \tag{15}\\
\lambda_{i}^{+}=\lambda_{0 i}^{+}+\sum_{j=1}^{M} \lambda_{j}^{+} p_{j i}^{+}, \quad i=\overline{1, M},  \tag{16}\\
\lambda_{i}^{-}=\lambda_{0 i}^{-}+\sum_{j=1}^{M} \lambda_{j}^{+} p_{j i}^{-}, \quad i=\overline{1, M} .
\end{gather*}
$$

If the matrix $P$ is irreducible, the system (16) has a unique positive solution for $\lambda_{i}^{+}, \lambda_{i}^{-}, i=\overline{1, M}$.

Let us denote

$$
\begin{equation*}
\Lambda_{0}=\sum_{i=1}^{M} \lambda_{i}^{+} p_{i 0} \tag{17}
\end{equation*}
$$

From (16) and (17) we obtain

$$
\Lambda_{0}+\sum_{j=1}^{M}\left(\lambda_{j}^{+}+\lambda_{j}^{-}\right)=\lambda_{0}^{+}+\lambda_{0}^{-}+\sum_{j=1}^{M} \lambda_{j}^{+} .
$$

This yields

$$
\begin{equation*}
\Lambda_{0}+\sum_{j=1}^{M} \lambda_{j}^{+}=\lambda_{0}^{+}+\lambda_{0}^{-} \tag{18}
\end{equation*}
$$

The relation (18) formally coincides with relation (7) obtained for the case of single-server processing of positive customers but the values of $\lambda_{i}^{+}$and $\lambda_{i}^{-}$for
the symmetrical network are determined from another system of equations which is a linear one.

Theorem 2 If matrix $P$ is irreducible and the following conditions hold $(i=\overline{1, M})$ :

$$
F_{i}=\sum_{k_{i}=0}^{\infty} \prod_{j=1}^{k_{i}} q_{i}(j)<\infty, \quad G_{i}=\sum_{n_{i}=0}^{\infty} \prod_{j=1}^{n_{i}} \rho^{-}(j)<\infty
$$

then the Markov process $\{X(t), \quad t \geq 0\}$ is ergodic and its stationary distribution is represented in a product form as

$$
\begin{equation*}
p(\vec{k}, \vec{n})=\prod_{i=1}^{M} p\left(k_{i}, n_{i}\right) \tag{19}
\end{equation*}
$$

where

$$
\begin{equation*}
p\left(k_{i}, n_{i}\right)=F_{i}^{-1} G_{i}^{-1} \prod_{j=1}^{k_{i}} q_{i}(j) \prod_{l=1}^{n_{i}} \rho_{l}^{-}(j), \quad k_{i}, n_{i} \geq 0 \tag{20}
\end{equation*}
$$

Proof. We make the substitution of (19), (20) into the system of equations (1), for which the assumptions (14) take place.

After some algebraic transformations we obtain the
equality

$$
\begin{align*}
& \lambda_{0}^{+}+\lambda_{0}^{-}+\sum_{i=1}^{M} \mu_{i}^{+}\left(k_{i}\right)+\sum_{i=1}^{M} \mu_{i}^{-}\left(n_{i}\right)+\sum_{i=1}^{M} \gamma_{i}\left(k_{i}\right)= \\
& \sum_{i=1}^{M} \mu_{i}^{+}\left(k_{i}\right) p_{i i}^{+}+\sum_{i=1}^{M} \gamma_{i}\left(k_{i}\right) p_{i i}^{+}+\sum_{i=1}^{M} \frac{\lambda_{0 i}^{+}}{q_{i}\left(k_{i}\right)} u\left(k_{i}\right)+ \\
& \sum_{i=1}^{M} \frac{\mu_{i}^{-}\left(n_{i}\right)}{\lambda_{i}^{-}} \lambda_{0 i}^{-}+ \\
& \sum_{i=1}^{M} q_{i}\left(k_{i}+1\right) \mu_{i}^{+}\left(k_{i}+1\right) p_{i 0}+\sum_{i=1}^{M} q_{i}\left(k_{i}+1\right) \lambda_{i}^{-} p_{i 0}+ \\
& \sum_{i=1}^{M} q_{i}\left(k_{i}+1\right) \gamma_{i}^{+}\left(k_{i}+1\right) p_{i 0}+\sum_{i=1}^{M} \lambda_{i}^{-}\left(1-u\left(k_{i}\right)\right)+ \\
& \sum_{i=1}^{M} \sum_{j=1, j \neq i}^{M}{ }_{q}^{\frac{q_{i}\left(k_{i}+1\right)}{q_{j}\left(k_{j}\right)} \mu_{i}^{+}\left(k_{i}+1\right) p_{i j}^{+} u\left(k_{j}\right)+} \\
& \sum_{i=1}^{M} \sum_{j=1}^{M} \frac{\mu_{j}^{-}\left(n_{j}\right)}{\lambda_{j}^{-}} q_{i}\left(k_{i}+1\right) \mu_{i}^{+}\left(k_{i}+1\right) p_{i j}^{-}+ \\
& \sum_{i=1}^{M} \sum_{j=1, j \neq i}^{M}{ }_{\frac{q_{i}\left(k_{i}+1\right)}{q_{j}\left(k_{j}\right)} \lambda_{i}^{-} p_{i j}^{+} u\left(k_{j}\right)+}^{\sum_{i=1}^{M} \lambda_{i}^{-} p_{i i}^{+} u\left(k_{i}\right)+\sum_{i=1}^{M} \sum_{j=1}^{M} \frac{\mu_{j}^{-}\left(n_{j}\right)}{\lambda_{j}^{-}} q_{i}\left(k_{i}+1\right) \lambda_{i}^{-} p_{i j}^{-}+} \\
& \sum_{i=1}^{M} \sum_{j=1, j \neq i}^{M}{ }_{q_{i}\left(k_{i}+1\right)}^{q_{j}\left(k_{j}\right)} \gamma_{i}\left(k_{i}+1\right) p_{i j}^{+} u\left(k_{j}\right)+ \\
& \sum_{i=1}^{M} \sum_{j=1}^{M} \frac{\mu_{j}^{-}\left(n_{j}\right)}{\lambda_{j}^{-}} q_{i}\left(k_{i}+1\right) \gamma_{i}\left(k_{i}+1\right) p_{i j}^{-} .
\end{align*}
$$

This equality is true for all $(\vec{k}, \vec{n}) \in \mathcal{X}$.
Similarly to the proof of the theorem of the previous case we transform the right part of the equality (21). Let us denote by

$$
\begin{gathered}
A=\sum_{i=1}^{M} \frac{\mu_{i}^{-}\left(n_{i}\right)}{\lambda_{i}^{-}} \lambda_{0 i}^{-}+ \\
\sum_{i=1}^{M} \sum_{j=1}^{M} \frac{\mu_{j}^{-}\left(n_{j}\right)}{\lambda_{j}^{-}} q_{i}\left(k_{i}+1\right) \mu_{i}^{+}\left(k_{i}+1\right) p_{i j}^{-}+ \\
\sum_{i=1}^{M} \sum_{j=1}^{M} \frac{\mu_{j}^{-}\left(n_{j}\right)}{\lambda_{j}^{-}} q_{i}\left(k_{i}+1\right) \lambda_{i}^{-} p_{i j}^{-}+ \\
\sum_{i=1}^{M} \sum_{j=1}^{M} \frac{\mu_{j}^{-}\left(n_{j}\right)}{\lambda_{j}^{-}} q_{i}\left(k_{i}+1\right) \gamma_{i}\left(k_{i}+1\right) p_{i j}^{-}
\end{gathered}
$$

Taking into account the relation

$$
q_{i}\left(k_{i}+1\right)\left[\lambda_{i}^{-}+\mu_{i}^{+}\left(k_{i}+1\right)+\gamma_{i}\left(k_{i}+1\right)\right]=\lambda_{i}^{+}
$$

we obtain

$$
\begin{equation*}
A=\sum_{j=1}^{M} \mu_{j}^{-}\left(n_{j}\right) . \tag{22}
\end{equation*}
$$

Further let us denote by

$$
\begin{aligned}
& B=\sum_{i=1}^{M} \mu_{i}^{+}\left(k_{i}\right) p_{i i}^{+}+\sum_{i=1}^{M} \frac{\lambda_{0 i}^{+}}{q_{i}\left(k_{i}\right)} u\left(k_{i}\right)+ \\
& \sum_{i=1}^{M} \sum_{j=1, j \neq i}^{M} \frac{q_{i}\left(k_{i}+1\right)}{q_{j}\left(k_{j}\right)} \mu_{i}^{+}\left(k_{i}+1\right) p_{i j}^{+} u\left(k_{j}\right)+ \\
& \sum_{i=1}^{M} \sum_{j=1, j \neq i}^{M} \frac{q_{i}\left(k_{i}+1\right)}{q_{j}\left(k_{j}\right)} \lambda_{i}^{-} p_{i j}^{+} u\left(k_{j}\right)+\sum_{i=1}^{M} \lambda_{i}^{-} p_{i i}^{+} u\left(k_{i}\right)+ \\
& \sum_{i=1}^{M} \gamma_{i}\left(k_{i}\right) p_{i i}^{+}+\sum_{i=1}^{M} \sum_{j=1,, j \neq i}^{M} \frac{q_{i}\left(k_{i}+1\right)}{q_{j}\left(k_{j}\right)} \gamma_{i}\left(k_{i}+1\right) p_{i j}^{+} u\left(k_{j}\right) .
\end{aligned}
$$

After some transformations of the right part with combination with (15) and (16) we obtain

$$
\begin{equation*}
B=\sum_{j=1}^{M}\left(\lambda_{j}^{-} u\left(k_{j}\right)+\mu_{j}^{+}\left(k_{j}\right)+\gamma_{j}\left(k_{j}\right)\right) . \tag{23}
\end{equation*}
$$

Finally, introducing

$$
\begin{equation*}
C=\Lambda_{0}+\sum_{i=1}^{M} \lambda_{i}^{-}\left(1-u\left(k_{i}\right)\right) \tag{24}
\end{equation*}
$$

we represent the right part of equalities (21) as $A+$ $B+C$.

Using (22) - (24), where $\Lambda_{0}, \lambda_{i}^{+}$and $\lambda_{i}^{-}$are determined by relations (16) and (17), we represent the right part of the equality (21) in the following form:

$$
\begin{aligned}
& A+B+C= \\
& \sum_{j=1}^{M} \mu_{j}^{-}\left(n_{j}\right)+\sum_{j=1}^{M}\left(\lambda_{j}^{-} u\left(k_{j}\right)+\mu_{j}^{+}\left(k_{j}\right)+\gamma_{j}\left(k_{j}\right)\right)+ \\
& \Lambda_{0}+\sum_{i=1}^{M} \lambda_{i}^{-}-\sum_{i=1}^{M} \lambda_{i}^{-} u\left(k_{i}\right)= \\
& \sum_{j=1}^{M} \mu_{j}^{-}\left(n_{j}\right)+\sum_{j=1}^{M}\left(\mu_{j}^{+}\left(k_{j}\right)+\gamma_{j}\left(k_{j}\right)\right)+\Lambda_{0}+\sum_{i=1}^{M} \lambda_{i}^{-}= \\
& \lambda_{0}^{+}+\lambda_{0}^{-}+\sum_{j=1}^{M} \mu_{j}^{-}\left(n_{j}\right)+\sum_{j=1}^{M}\left(\mu_{j}^{+}\left(k_{j}\right)+\gamma_{j}\left(k_{j}\right)\right) .
\end{aligned}
$$

Thus the substitution of (19), (20) into the system of equations (1), (14), for all $(\vec{k}, \vec{n}) \in \mathcal{X}$, leads to a system of identities. Therefore, the expressions (19), (20) give a solution of the equilibrium system of equations (1), (14) which under the assumptions of the theorem is positive and bounded. As a consequence of this result the process $\{X(t), \quad t \geq 0\}$ is ergodic, thus the theorem is proved.

### 4.3 Conclusion

G-networks provide a versatile class to model complex systems in various applications fields such as computer network and telecommunication systems. In this paper we extended the results of G-netwowks with product form solution introducing the impatient service. We provided a proof of the product form results for a network in which positive customers are processed by a single server at every node and for a simmetrical networks.

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