SIMULATING MULTI-DIMENSIONAL LATTICES WITH CORRELATION: A CASE STUDY

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ABSTRACT

We employ a new numerical approximation technique for valuing multiple assets that are correlated. For any number of underlying variables, this procedure can simulate a set of correlated assets that follow geometric Brownian processes. In addition to the joint approximation of multiple series based on binomial lattices and Monte Carlo simulations, we introduce a straightforward extension of a structural lead-lag approach to capture the linear relationship between assets in the bid to improve computational accuracy on the proposed algorithm. We show that when applied to a series of correlated assets, the proposed algorithm is consistent, efficient and stable.

INTRODUCTION

The need for evaluating contingent claims on more than one state variable is not uncommon in most financial investment decisions. Though many numerical methods (most approaches can be classified into three broad classes: Lattice methods of Cox, Ross and Rubinstein, 1979, Monte Carlo simulations of Boyle, 1977, and finite difference methods of Brennan and Schwartz, 1977) have been proposed for pricing options in the financial literature, an outstanding challenge in computational finance is the simulation of portfolios with more than two correlated assets.

Lattices or trees were first introduced by Cox, Ross and Rubinstein (1979) to complement the Black and Scholes (1973) closed-form solution model where asset prices follow stochastic processes with multiplicative sequences of variable drifts (i.e., geometric Brownian motion). To obtain the terminal value for an option, the expected option payoff is weighted by its probability via the lattice and the expectation is discounted back at the risk-free rate. Using discrete binomial distributions to approximate the continuous lognormal distributions in the Black-Scholes model, it offers an efficient way of generating series of predetermined paths for pricing both American and European options (contingent claims on European options can only be exercised at maturity, while American options may be exercised at intermediated dates, not just at maturity. For a comprehensive exposition on derivatives, see Hull (2000)).

By and large, binomial trees can be configured in different ways according to a set of parameters for probabilities of ‘up’ and ‘down’ steps, and step size (see Jarrow and Rudd, 1983 and Leisen and Reimer, 1996 for example). In this paper, we employ the Cox, Ross and Rubinstein (1979) procedure where asset prices are modeled such that the product of up and down price multipliers equals to unity in a risk-neutral valuation environment. Clearly, as the number of replications in the lattice increases, the simulated value converges to that of the Black-Scholes pricing solution.

Existing papers on lattices are usually built upon a single-asset problem with time-dependent volatility, or multiple assets with constant correlation and volatility. Boyle, Evnine and Gibbs (1989) extend the Cox, Ross and Rubinstein approach to multiple assets but admit their technique is vulnerable to negative martingale probabilities when the correlation between assets is too large or if the volatilities are time-dependent. These negative martingale probabilities can lead to unstable pricing structures and if the number of state variables is large, the exponential complexity of the multi-dimensional binomial approach can prove to be a computational burden. Other extensions of the lattice procedure that can be computationally exhaustive include that of Boyle (1988) and Kamrad and Ritchken (1991) where trinomial algorithms are used to estimate multiple state variables.

Monte Carlo simulation on the other hand, entails the generation of asset price paths with a stochastic process. It is known to be ill equipped to handle problems involving early exercises (i.e., American options) but it enjoys computational advantages over pure lattice approximations for multiple-asset scenarios. With a new algorithm that is

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computationally efficient and stable, Otamendi and Hon (2004) demonstrate that it is possible to construct a combination of binomial lattice approximation with Monte Carlo simulation where probabilities stay positive even if the multiple state variables are highly correlated geometric Brownian processes.

In this paper, we simulate time paths for a portfolio with multiple correlated assets using a new joint lead-lag binomial lattice-Monte Carlo approximation proposed by Otamendi and Hon on four assets that are arbitrarily chosen from the New York Stock Exchange (NYSE) and the National Association of Securities Dealers (NASDAQ): Boeing Company (NYSE: BA), International Business Machines (NYSE: IBM), Hewlett-Packard (NYSE: HPQ) and Microsoft Corporation (NASDAQ: MSFT).

This article is organized as follows: We first present a numerical exposition of existing two-asset simulation models. The new algorithm for a two-asset case follows. The proposed algorithm is further extended to cover a three-asset example before we present a generalized *k* series elucidation of the methodology with a built in lead-lag structure. Finally, we apply the algorithm to a portfolio of four assets to show that the model is consistent, efficient and stable.

EXISTING TWO-ASSET MODELS

Most existing two-asset models are either based on Monte Carlo or lattice approximations.

Monte Carlo Simulation

This particular type of simulation is based on the continuous evolution of the series where the movement process follows a Brownian motion, and the variability a Wiener process. If the time period *D* is divided by small intervals *T* (t=0,…,*T*), the recursive equation to estimate V_t is (see Haug 1998):

$$V_t = V_{t-1}\, exp\left[\left(r - \frac{1}{2}\nu^2\right)\frac{D}{T} + \nu\,\varepsilon_t\sqrt{\frac{D}{T}}\right]$$

where:

V_t = value of the series at time *t*

r = annual interest rate
ν = volatility of the series
D = time horizon (expressed in years)
T = number of intervals per time period *D*
$\varepsilon_t \sim N(0,1)$

Note that the consecutive independent samples follow a normal distribution. To estimate V_T and *T,* independent samples must be generated for each time period *t*.

For a correlated two-asset series, the recursive equations (Haug 1998) are:

$$V_{1,t} = V_{1,t-1}\, exp\left[\left(r - \frac{1}{2}\nu_1^2\right)\frac{D}{T} + \nu_1\,\alpha_{1t}\sqrt{\frac{D}{T}}\right]$$

$$V_{2,t} = V_{2,t-1}\, exp\left[\left(r - \frac{1}{2}\nu_2^2\right)\frac{D}{T} + \nu_2\,\alpha_{2t}\sqrt{\frac{D}{T}}\right]$$

where:

$V_{k,t}$ = value of the asset *k* at period *t*
$\alpha_{1t} = \varepsilon_{1t}$
$\alpha_{2t} = \rho\,\varepsilon_{1t} + \varepsilon_{2t}\sqrt{1-\rho^2}$
ρ = correlation between the two assets

Binomial Tree

With a binomial tree, the path taken by the asset at each time step is either a positive upward (*u)* jump with a probability *p*, or a negative downward move *d*, with a probability (1-*p*). At time *t*=0 with an initial value V_0, the successive values V_t are obtained with the following recursive equations:

$$V_t = \begin{cases} V_{t-1}\ u & \forall U[0,1] \le p \\ V_{t-1}\ d & \forall U[0,1] > p \end{cases}$$

where:

$u = e^{\nu\sqrt{D/T}}$
$d = 1/u$
$p = (f-d)/(u-d)$
$f = 1+(rD/T)$

The pseudo-code for simulating V_T is as follows:

```
Obtain V0, D, T
Calculate p, u, d
t←0
While t<T
    Generate a random sample τ=U[0,1]
    If τ≤p
        Vt+1=Vt u
    Else
        Vt+1=Vt d
    t←t+1
```

The binomial tree model can be modified (Rubinstein, 1994) to simulate two assets with correlation ρ. For asset 1, *u* and *d* are redefined:

$$u_1 = exp\left[\left(r - \frac{1}{2}\nu_1^2\right)\frac{D}{T} + \nu_1\sqrt{\frac{D}{T}}\right]$$

$$d_1 = exp\left[\left(r - \frac{1}{2}\nu_1^2\right)\frac{D}{T} - \nu_1\sqrt{\frac{D}{T}}\right]$$

$p_1 = (f-d_1)/(u_1-d_1)$

For asset 2, A and B are defined as the time-step jumps u_2 and d_2 if asset 1 moved to u_1; C and D are the time-step jumps u_2 and d_2 if asset 1 moved to d_1:

$$A=exp\left[\left(r-\frac{1}{2}\nu_2{}^2\right)\frac{D}{T}+\nu_2\sqrt{\frac{D}{T}}\left(\rho+\sqrt{1-\rho^2}\right)\right]$$

$$B=exp\left[\left(r-\frac{1}{2}\nu_2{}^2\right)\frac{D}{T}+\nu_2\sqrt{\frac{D}{T}}\left(\rho-\sqrt{1-\rho^2}\right)\right]$$

$$C=exp\left[\left(r-\frac{1}{2}\nu_2{}^2\right)\frac{D}{T}-\nu_2\sqrt{\frac{D}{T}}\left(\rho-\sqrt{1-\rho^2}\right)\right]$$

$$D=exp\left[\left(r-\frac{1}{2}\nu_2{}^2\right)\frac{D}{T}-\nu_2\sqrt{\frac{D}{T}}\left(\rho+\sqrt{1-\rho^2}\right)\right]$$

$$p_2=0.5$$

The pseudo-code to simulate the two assets is:

```
Obtain V1,0, V2,0, ρ, D, T
Calculate p1, u1, d1, A, B, C, D
t←0
While t<T
    Generate a sample τ1=U[0,1]
        If τ1≤p1
            V1,n+1=V1,n u
            Generate a sample τ2=U[0,1]
            If τ2≤0.5
                V2,n+1=V2,n A
            Else
                V2,n+1=V2,n B
        Else
            V1n+1=V1n d
            Generate a sample τ2=U[0,1]
                If τ2≤0.5
                    V2,n+1=V2,n C
                Else
                    V2,n+1=V2,n D
    t←t+1
```

PROPOSED MODEL

The models introduced in the previous section are useful for two assets, but they cannot be generalized for three or more variables. We now introduce the proposed model that can be generalized for two or more assets.

Proposed Two-Asset Model

Because asset returns often exhibit a high degree of correlation in practice, we have to simulate correlated normal random variables to proxy for the correlated returns.

Let *Xi* and *Xj* be two random variables. The covariance of *Xi* and *Xj* is defined to be:

$$Cov(X_i, X_j)=E[X_iX_j]-E[X_i]E[X_j]$$

and the correlation of X_i and X_j is then defined to be

$$Corr(X_i, X_j)=\rho(X_i, X_j)=\frac{Cov(X_i,X_j)}{\sqrt{Var(X_i)Var(X_j)}}$$

If X_i and X_j are independent, then $\rho=0$. The proposed algorithm is formulated in part with a correlation-based lattice. If we have a correlation of ρ=1 between the two assets, both assets will have identical co-movements (e.g., if asset *i* moves to the upper bifurcation in the binomial tree, asset *j* will move in the same direction).

If ρ=-1, the movements per time step will be exactly the opposite for both series. And when ρ=0, there is no relationship between the two series (i.e., they are independent).

Simulating the lattice for a two-variable (asset 1 and asset 2) case based on the above exposition is straightforward: In time period *t*, we first simulate the path for asset 1. If it experiences an upward movement, asset 2 must follow a downward movement if ρ=-1, a random movement if ρ=0 and a corresponding upward movement if ρ=1. In other words, the probability of matching co-movements is β=0% in the first case, β=50% in the second, and β=100% in the third case.

The structural relationship between the coefficient of correlation ρ and the probability of co-movement (β), might then be set over the entire range of possible values for ρ ($-1\leq\rho\leq1$) with a simple linear transformation:

$$\beta = 0.50 + 0.50\ \rho \qquad \forall\ -1\leq\rho\leq 1$$

The pseudo-code based on the proposed co-movement probability model is as follows:

```
Obtain V1,0, V2,0, ρ12, D, T
Calculate u1, d1, p1, u2, d2, β1,2
t←0
While t<T
    Generate a random sample τ1=U[0,1]
    If τ1≤p1
        V1,t+1= V1,t u1
        Generate a random sample τ2=U[0,1]
        If τ2≤ β1,2
            V2,t+1= V2,t u2
        Else
            V2,t+1= V2,t d2
    Else
        V1,t+1= V1,t d1
        Generate a random sample τ2=U[0,1]
        If τ2≤(1-β1,2)
            V2,t+1= V2,t u2
        Else
            V2,t+1= V2,t d2
    t←t+1
```

where $\beta_{1,2}$ is the probability of co-movement between the assets 1 and 2 - obtained from the structural relationship between ρ and β specified above ($\beta = 0.50 + 0.50\,\rho \quad \forall -1 \leq \rho \leq 1$).

Validation of the Proposed Model

To demonstrate that the asset value at maturity for the proposed model follows a lognormal distribution (and that the correlation between the two assets is ρ), we perform two separate simulations: A Monte Carlo Brownian process and the proposed model based on the following parameters:

	Asset 1	Asset 2
Initial Value (V_0)	9	14
Correlation (ρ)	0.70	0.70
Annual Interest (r)	4%	4%
Interval (T)	63	63
Time (D)	0.25	0.25
Volatility (ν)	39%	25%

The frequency distributions for assets 1 and 2 are shown in Figure 1 below:

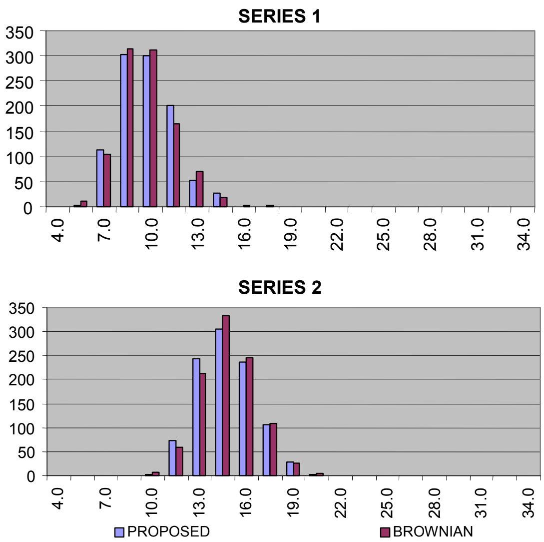

Figure 1. Frequency Distribution for Asset 1 (Series 1) and Asset 2 (Series 2)

A χ^2 goodness-of-fit test (see Law, 1991 for more on statistical tests for simulations) is used to test the null hypothesis that both models yield identical results. Even with intervals T as small as 63, we fail to reject the null with a confidence level of 99.73% (a total of 2000 simulation runs were performed).

We also test to see if the correlation at maturity is what it should be (see Ruiz-Maya and Martín Pliego, 1995 for example):

H_0: ρ=0.70
H_1: ρ<>0.70

The results are displayed in Table 1 (with a 99.73% confidence level).

Table 1. Confidence Intervals for the Correlation Coefficients

	BROWNIAN MODEL 1-2	PROPOSED MODEL 1-2
Minimum	-0.6751	-0.6976
Confidence Interval (Inferior)	0.5966	0.5934
Mean	0.6542	0.6514
Confidence Interval (Superior)	0.7052	0.7027
Maximum	0.9898	0.9929
	PASS	PASS

As both confidence intervals contain the theoretical values 0.70, the null hypothesis cannot be rejected.

PROPOSED MODEL FOR THREE ASSETS

Formulation

The generalization of a three-asset bifurcation is shown in Figure 2. Note that as a function of co-movements between asset 1 (C_1=2) and asset 2 (C_2=4), asset 3 contains eight possible outcomes at each time period (C_3=8):

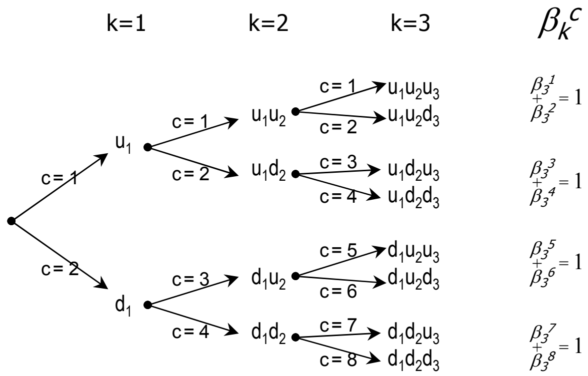

Figure 2. Probability Generalization for k Assets at Time t (β_k^c is the probability of co-movement for asset k and the bifurcation (c))

Because the eight outcomes are incompatible in general, the sum of the probabilities for each outcome is 1. However, with regard to the probability of co-movement between assets (β), we have pair-wise incompatibility. To generate the value of asset 3, a starting point is required (after generating asset 2). If for example, the starting bifurcation (c) is at the first branch for asset 2 (u_1u_2), we can generate bifurcation c=1 ($u_1u_2u_3$) or bifurcation c=2 ($u_1u_2d_3$) for asset 3. As a result, the co-movement probabilities β_3^1 and β_3^2 sum up to unity (1). Likewise, the same transpires in pairs: β_3^3 and β_3^4, β_3^5 and β_3^6, β_3^7 and β_3^8.

The probabilities β_3^1 and β_3^8 are the identical since they represent the occurrence of bifurcation c=1 and bifurcation c=8, which happens to be the case where the co-movements for all three assets match. Based on the same argument, we observe that $\beta_3^3 = \beta_3^6$ since there is co-movement between asset 1 and asset 3, but not with asset 2.

Consequently, before simulating asset 3, we know *a priori* that:

$\beta_3^1=\beta_3^1$ $\beta_3^3=\beta_3^3$
$\beta_3^5=\beta_3^4=1-\beta_3^3$ $\beta_3^7=\beta_3^2=1-\beta_3^1$
$\beta_3^2=1-\beta_3^1$ $\beta_3^4=1-\beta_3^3$
$\beta_3^6=\beta_3^3$ $\beta_3^8=\beta_3^1$

The crucial values left are β_3^1 and β_3^3. They can be obtained using simple probability theory (with the definition of classical probability).

β_3^1 is the probability of asset 3 moving to u_3 divided by the probability of moving to the upper bifurcation or lower bifurcation (u_3 or d_3); if assets 1 and 2 have moved to u_1 and u_2 respectively. The jump to u_3 is equivalent to the condition that the correlations of ρ_{13} and ρ_{23} exist and that it has been converted to the probability of co-movement using the structural relationship between β and ρ ($\beta_{1,3}$ and $\beta_{2,3}$). The movement to d_3 can also be derived from ($1-\beta_{1,3}$) and ($1-\beta_{2,3}$):

$$\beta_3^1=\frac{\beta_{1,3}\beta_{2,3}}{(1-\beta_{1,3})(1-\beta_{2,3})+\beta_{1,3}\beta_{2,3}}$$

In the case of β_3^3, the outcome for $u_1d_2u_3$ is:

$$\beta_3^3=\frac{\beta_{1,3}(1-\beta_{2,3})}{(1-\beta_{1,3})\beta_{2,3}+\beta_{1,3}(1-\beta_{2,3})}$$

The pseudo-code for a three-asset approximation described above is:

```
Obtain V_{1,0}, V_{2,0}, V_{3,0}, ρ12, ρ13, ρ23, D, T
Calculate u1, d1, p1, u2, d2, u3, d3
Calculate β_{1,2}, β_{1,3}, β_{2,3}, β3^1, …, β3^8
t←0
While t<T
    Generate a random sample τ1=U[0,1]
    If τ1≤p1
        V_{1,t+1}=V_{1,t} u1
        Generate a random sample τ2=U[0,1]
        If τ2≤ β_{1,2}
            V_{2,t+1}=V_{2,t} u2
            Generate a random sample τ3=U[0,1]
            If τ3≤ β3^1
                V_{3,t+1}=V_{3,t} u3
            Else
                V_{3,t+1}=V_{3,t} d3
        Else
            V_{2,t+1}=V_{2,t} d2
            Generate a random sample τ3=U[0,1]
            If τ3≤ β3^3
                V_{3,t+1}=V_{3,t} u3
            Else
                V_{3,t+1}=V_{3,t} d3
    Else
        V_{1,t+1}=V_{1,t} d1
        Generate a random sample τ2=U[0,1]
        If τ2≤(1- β_{1,2})
            V_{2,t+1}=V_{2,t} u2
            Generate a random sample τ3=U[0,1]
            If τ3≤ β3^5
                V_{3,t+1}=V_{3,t} u3
            Else
                V_{3,t+1}=V_{3,t} d3
        Else
            V_{2,t+1}=V_{2,t} d2
            Generate a random sample τ3=U[0,1]
            If τ3≤ β3^7
                V_{3,t+1}=V_{3,t} u3
            Else
                V_{3,t+1}=V_{3,t} d3
    t←t+1
```

Note that the correlation matrix may not always be valid: A congruent matrix is required (a positive definite or positive semi-definite correlation matrix - Morrison, 1990). For example, it is unlikely that we will obtain a perfect correlation between asset 1 and asset 2 (ρ=1). If the correlation between asset 1 and 2 is perfect, the correlation between assets 1 and 3, and assets 2 and 3 will be zero.

Proposed Model Validation

A third asset is added to the two-asset example that was used previously for validation of a two-asset case (the data is the same from the previous section for the first two assets - the correlation matrix is positive definite (Schmidt (1981)):

Initial value (V_0)= 18
Volatility (v) = 28%
Correlation (ρ) between asset 1&3 = 0%

Correlation (ρ) between asset 2&3 = -5%

The frequency histogram for asset 3 is depicted in Figure 3.

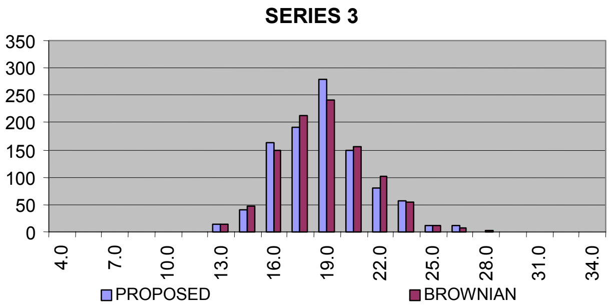

Figure 3. Frequency Distribution for Asset 3 (Series 3)

The results with a 99.73% confidence level show that the proposed model follows a lognormal process. Table 2 confirms that the correlations between assets are valid.

Table 2. Asset Correlation

	BROWNIAN MODEL	PROPOSED MODEL		
	1-2	1-2	1-3	2-3
Minimum	-0.6751	-0.6976	-0.9396	-0.9571
Confidence Interval (Inferior)	0.5966	0.5934	-0.1408	-0.1419
Mean	0.6542	0.6514	-0.0468	-0.0480
Confidence Interval (Superior)	0.7052	0.7027	0.0481	0.0469
Maximum	0.9898	0.9929	0.9377	0.9378
	PASS	PASS	PASS	PASS

It is clear at this point that the proposed model can be generalized for two or more assets.

PROPOSED MODEL FOR *K* SERIES

In this section, we present the generalization for three assets or more. If K is the total number of assets, an algorithm can be developed to simulate asset k as a function of the k-1 series previously simulated. As in the two- and three-asset models, the bifurcations are generated for each of the asset and the probabilities of co-movements obtained. The assets are further simulated for each time period till the maturity period.

Incorporated Lead-Lag Structure

To capture the lead-lag (i.e., the order of k asset simulation) transmission between assets in the proposed model, we first define the changes in daily returns:

$$\text{Let } R_t^k = Log\left(\frac{P_t^k}{P_{t-1}^k}\right)$$

where P_t^k (P_{t-1}^k) represents the closing price of asset k on day t (t-1).

For distinguishing the specific lead-lag effects (the number of possibilities is $P_k = k!$), we use the following system of equations (for conciseness, we use a two-asset example):

$$R_t^i = a_i + b_{ij} R_{t-1}^j + b_{ii} R_{t-1}^i + \varepsilon_t^i$$

$$R_t^j = a_j + b_{ji} R_{t-1}^i + b_{jj} R_{t-1}^j + \varepsilon_t^j$$

where coefficients a_k are intercepts, b_k are slope coefficients and ε_k are error terms. To capture the serial correlation of returns on each asset, the preceding returns at time t-1 are included as explanatory variables (the inclusion of this term will not affect the estimation process in any substantial way).

The coefficient b_{ij} (b_{ji}) measures the effect of the daily returns of asset i (j) on the following returns of asset j (i). If the returns on asset i (j) influence the subsequent returns on asset j (i), b_{ji} (bij) should be positive.

If the signal quality of asset i (j) is better (in terms of its ability to signal a "lead" on the returns of asset j (i)) than that of asset j (i), we will have $b_{ji} > b_{ij}$.

The proposed model uses iterated seemingly unrelated regressions (ITSUR) to estimate the above system of equations for all k assets simultaneously. By allowing cross correlations between error terms of all k assets, this procedure delivers more efficient estimates of coefficients than ordinary least squares (OLS) regressions in large samples. If the error terms are assumed to be normal the ITSUR technique will yield equivalent maximum likelihood estimators.

Proposed Model Formulation

The number of bifurcations for individual asset k is $C_k = 2^k$. They are generated as follows:

```
Ii←1
While I1≤2
    I2←1
    While I2≤2
        …
        IK←1
        While Ik≤2
            mI1, mI2, …, mIK
        Ik← Ik+1
    …
    I2← I2+1
I1← I1+1
```

where $m_{Ii}=u_i$ if $I_i=1$ and $m_{Ii}=d_i$ if $I_i=2$.

The β_k^c variables are equal in pairs:

$$\beta_k^c = \beta_k^{Ck-c+1} \quad \forall\ c = 1, \ldots, C_k \text{ cases}$$

The pairs β_s^c and β_s^{c+1} (when c is odd) are calculated as follows:

$$\beta_k^c = \frac{\prod_{i=1}^{k-1} H_{i,k}}{\prod_{i=1}^{k-1} H_{i,k} + \prod_{i=1}^{k-1}\left(1 - H_{i,k}\right)}$$

$$\beta_k^c = 1 - \beta_k^{c+1}$$

where:

$$H_{i,k} = \beta_{i,k} \qquad \forall\ u_i$$
$$H_{i,k} = 1 - \beta_{i,k} \qquad \forall\ d_i$$

For $K = 4$ assets, the $2^4=16$ bifurcations are:

c=1	$u_1u_2u_3u_4$
c=2	$u_1u_2u_3d_4$
c=3	$u_1u_2d_3u_4$
c=4	$u_1u_2d_3d_4$
c=5	$u_1d_2u_3u_4$
c=6	$u_1d_2u_3d_4$
c=7	$u_1d_2d_3u_4$
c=8	$u_1d_2d_3d_4$
c=9	$d_1u_2u_3u_4$
c=10	$d_1u_2u_3d_4$
c=11	$d_1u_2d_3u_4$
c=12	$d_1u_2d_3d_4$
c=13	$d_1d_2u_3u_4$
c=14	$d_1d_2u_3d_4$
c=15	$d_1d_2d_3u_4$
c=16	$d_1d_2d_3d_4$

To obtain β_4^5 which corresponds to $u_1d_2u_3u_4$, we have to assess $H_{1,4}$, $H_{2,4}$ and $H_{3,4}$:

$$\beta_4^5=\{H_{1,4}(1-H_{2,4})H_{3,4}\}/\{[H_{1,4}(1-H_{2,4})H_{3,4}]+[(1-H_{1,4})H_{2,4}(1-H_{3,4})]\}$$
$$\beta_4^6=1-\beta_4^5$$
$$\beta_4^{11}=\beta_4^6$$
$$\beta_4^{12}=\beta_4^5$$

Once the co-movement probabilities β_k^c are calculated, a general algorithm can be applied:

```
Obtain V_{k,0}, ρ, D, T
Calculate u_k, d_k, p_1
Calculate β_k^c
t←0
While t<T
    k←2
    Generate a random sample τ_1=U[0,1]
    If τ_1≤p_1
        c←1
        V_{1,t+1}=V_{1,t} u_1
        While k≤K
        Generate a random sample τ_k=U[0,1]
        If τ_s≤β_s^{2c-1}
            V_{k,t+1}=V_{k,t} u_k
            c←2c-1
        Else
            V_{k,t+1}=V_{k,t} d_k
            c←2c
        k←k+1
    Else
        c←2
        V_{1,t+1}=V_{1,t} d_1
        While k≤K
            Generate a random sample τ_k=U[0,1]
            If τ_k≤β_k^{2c-1}
                V_{k,t+1}=V_{k,t} u_k
                c←2c-1
            Else
                V_{k,t+1}=V_{k,t} d_k
                c←2c
        k←k+1
    t←t+1
```

CASE STUDY

In the sections to follow, we present the simulation results for the proposed joint lead-lag binomial lattice-Monte Carlo approximation. The real assets were selected arbitrarily from the New York Stock Exchange (NYSE) and the National Association of Securities Dealers (NASDAQ): Boeing Company (NYSE: BA), International Business Machines (NYSE: IBM), Hewlett-Packard (NYSE: HPQ) and Microsoft Corporation (NASDAQ: MSFT). The assets are simulated independently but with an imbedded correlation and lead-lag algorithm. The terminal values V_{tk} are aggregated according to the weights wk to calculate the value of the portfolio V_T.

To benchmark the proposed model, we perform an independent Monte Carlo simulation with no correlation or lead-lag structure (i.e., each asset path is simulated independently) where the asset values at maturity V_{tk} are aggregated according to weights wk to obtain the value of the portfolio V_T.

The sample data spans from 23 August 2002 to 2 April 2003. The initial time t=0 is set at 2 January 2003. Before the actual approximation, we perform the lead-lag ITSUR analysis to obtain the order of simulation: MSFT -> IBM -> BA -> HPQ.

The summary statistics 90 days prior to simulation are as follows:

	S_0	v	Correlation Matrix			
MSFT	25.62	0.3931	1.00	0.91	-0.36	0.88
IBM	76.79	0.4853	0.91	1.00	-0.10	0.88
BA	32.18	0.4377	-0.36	-0.10	1.00	-0.10
HPQ	17.08	0.6115	0.88	0.88	-0.10	1.00

r = annual interest rate=0.04
D = time horizon (expressed in years) = 0.25
T = number of intervals per time period $D=63$
N=200 repetitions

The following parameters are used to compare the estimations from simulations: 1) The average value of assets $k = \overline{V_{Tk}} = \frac{1}{N}\sum_{n=1}^{N} V_{Tkn}$; 2) the average value of the portfolio = $\overline{V_T}$; 3) the Value-at-Risk (VaR) of each stock (VaR_k) = α-th percentile of the distribution of simulated values of V_{Tk}; and 4) VaR of the portfolio= α-th percentile of the distribution of simulated values of V_T. For added robustness, we employ three different measures of volatility: 1) constant volatility; 2) variable volatility (lag = 90 days); and 3) exponential weighted moving average (EWMA) (with decay factor λ = 0.94).

The graphical outputs are as follows:

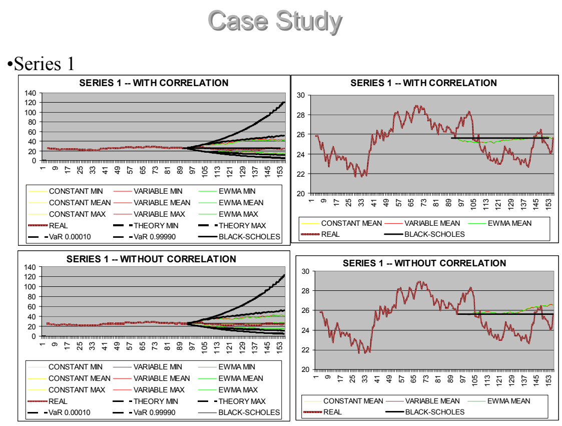

Figure 4. Microsoft (MSFT)

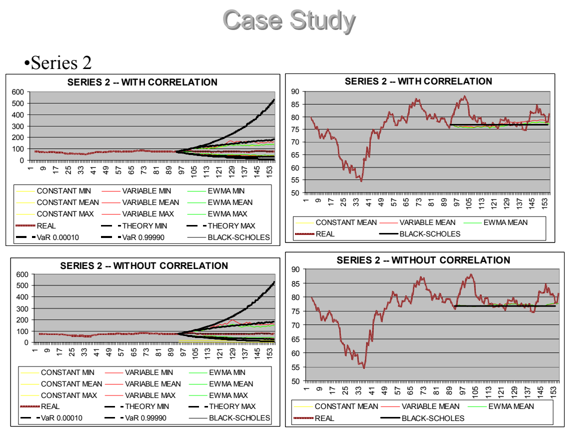

Figure 5. International Business Machines (IBM)

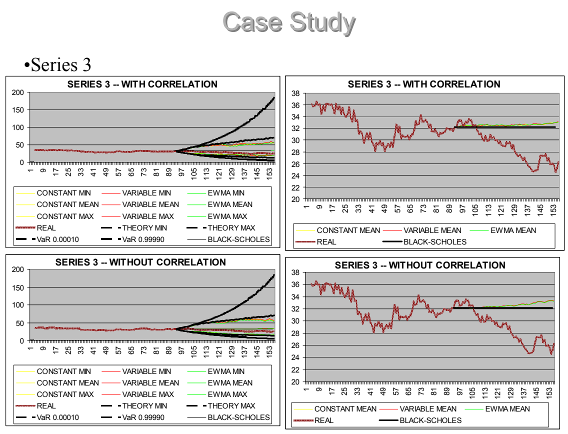

Figure 6. Boeing (BA)

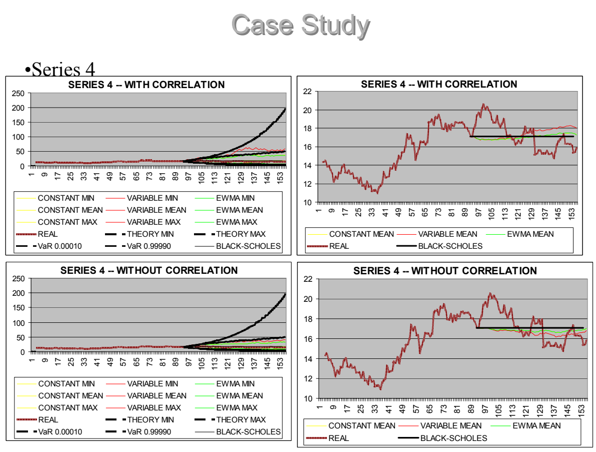

Figure 7. Hewlett-Packard (HPQ)

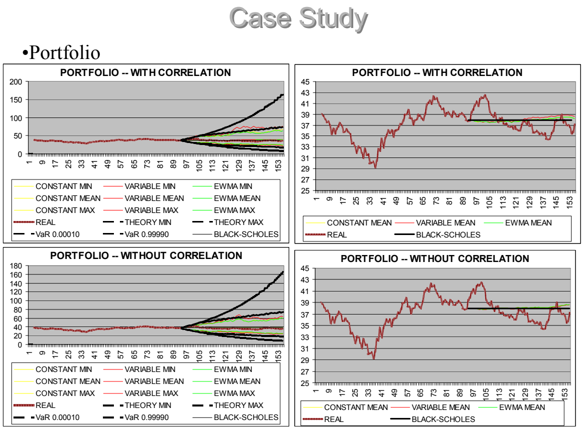

Figure 8. Portfolio

The results are summarized in Table 3 and Table 4:

Table 3. Simulation Results with Correlation and Lead-Lag (Proposed Model)

PROPOSED MODEL

Volatility (v)	V_T	MSFT	IBM	BA	HPQ	PORTFOLIO
CONSTANT	Di	38.44	-181.86	233.35	-9.68	20.06
VARIABLE	Di	38.51	-149.37	233.02	12.79	33.74
EWMA	Di	38.53	-186.42	234.83	-9.24	19.43
CONSTANT	AVERAGE Di	0.61	-2.89	3.70	-0.15	0.32
VARIABLE	AVERAGE Di	0.61	-2.37	3.70	0.20	0.54
EWMA	**AVERAGE Di**	**0.61**	**-2.96**	**3.73**	**-0.15**	**0.31**
CONSTANT	VARIANCE Di	2.39	11.61	8.13	3.12	4.78
VARIABLE	VARIANCE Di	2.40	12.74	8.33	4.03	5.33
EWMA	VARIANCE Di	2.39	11.58	8.03	3.10	4.77
CONSTANT	LOWER CI Di	0.09	-4.03	2.75	-0.74	-0.41
VARIABLE	LOWER CI Di	0.09	-3.57	2.73	-0.47	-0.24
EWMA	LOWER CI Di	0.09	-4.10	2.78	-0.74	-0.42
CONSTANT	UPPER CI Di	1.13	-1.75	4.66	0.44	1.05
VARIABLE	UPPER CI Di	1.13	-1.18	4.67	0.88	1.31
EWMA	UPPER CI Di	1.13	-1.82	4.68	0.44	1.04
CONSTANT	ABS(Di)	91.02	202.07	245.51	93.93	112.13
VARIABLE	ABS(Di)	91.27	189.65	247.65	107.98	123.95
EWMA	ABS(Di)	91.01	205.21	246.97	93.75	111.56
CONSTANT	AVERAGE ABS(Di)	1.44	3.21	3.90	1.49	1.78
VARIABLE	AVERAGE ABS(Di)	1.45	3.01	3.93	1.71	1.97
EWMA	AVERAGE ABS(Di)	1.44	3.26	3.92	1.49	1.77
CONSTANT	VARIANCE ABS(Di)	0.65	9.63	6.64	0.88	1.66
VARIABLE	VARIANCE ABS(Di)	0.65	9.25	6.53	1.09	1.68
EWMA	VARIANCE ABS(Di)	0.65	9.70	6.53	0.87	1.68
CONSTANT	LOWER CI ABS(Di)	1.18	2.17	3.03	1.18	1.35
VARIABLE	LOWER CI ABS(Di)	1.18	1.99	3.08	1.37	1.53
EWMA	LOWER CI ABS(Di)	1.18	2.21	3.06	1.18	1.34
CONSTANT	UPPER CI ABS(Di)	1.71	4.25	4.76	1.81	2.21
VARIABLE	UPPER CI ABS(Di)	1.72	4.03	4.79	2.06	2.40
EWMA	UPPER CI ABS(Di)	1.71	4.30	4.78	1.80	2.20

Table 4. Simulation Results with No Correlation and No Lead-Lag (Independent Model)

INDEPENDENT MODEL

Volatility (v)	V_T	MSFT	IBM	BA	HPQ	PORTFOLIO
CONSTANT	Di	72.34	-156.84	231.78	-23.67	30.90
VARIABLE	Di	70.87	-156.56	233.30	-36.35	27.82
EWMA	Di	71.98	-155.51	230.58	-23.39	30.91
CONSTANT	AVERAGE Di	1.15	-2.49	3.68	-0.38	0.49
VARIABLE	AVERAGE Di	1.12	-2.49	3.70	-0.58	0.44
EWMA	**AVERAGE Di**	**1.14**	**-2.47**	**3.66**	**-0.37**	**0.49**
CONSTANT	VARIANCE Di	2.60	11.62	9.36	2.26	4.55
VARIABLE	VARIANCE Di	2.57	11.75	9.46	1.98	4.44
EWMA	VARIANCE Di	2.60	11.66	9.28	2.27	4.55
CONSTANT	LOWER CI Di	0.61	-3.63	2.65	-0.88	-0.22
VARIABLE	LOWER CI Di	0.59	-3.63	2.67	-1.05	-0.26
EWMA	LOWER CI Di	0.60	-3.61	2.64	-0.88	-0.22
CONSTANT	UPPER CI Di	1.69	-1.35	4.70	0.13	1.20
VARIABLE	UPPER CI Di	1.66	-1.34	4.73	-0.11	1.15
EWMA	UPPER CI Di	1.68	-1.32	4.68	0.13	1.20
CONSTANT	ABS(Di)	112.37	186.93	250.69	80.17	113.88
VARIABLE	ABS(Di)	110.91	187.85	252.16	76.46	111.42
EWMA	ABS(Di)	112.02	186.38	249.53	80.37	113.94
CONSTANT	AVERAGE ABS(Di)	1.78	2.97	3.98	1.27	1.81
VARIABLE	AVERAGE ABS(Di)	1.76	2.98	4.00	1.21	1.77
EWMA	AVERAGE ABS(Di)	1.78	2.96	3.96	1.28	1.81
CONSTANT	VARIANCE ABS(Di)	0.71	8.97	7.02	0.76	1.47
VARIABLE	VARIANCE ABS(Di)	0.71	8.99	7.12	0.82	1.46
EWMA	VARIANCE ABS(Di)	0.71	8.96	6.95	0.76	1.47
CONSTANT	LOWER CI ABS(Di)	1.50	1.96	3.09	0.98	1.40
VARIABLE	LOWER CI ABS(Di)	1.48	1.98	3.11	0.91	1.36
EWMA	LOWER CI ABS(Di)	1.50	1.96	3.08	0.98	1.40
CONSTANT	UPPER CI ABS(Di)	2.07	3.97	4.87	1.56	2.21
VARIABLE	UPPER CI ABS(Di)	2.04	3.99	4.90	1.52	2.17
EWMA	UPPER CI ABS(Di)	2.06	3.96	4.84	1.57	2.21

From the absolute errors *Di* (real value at maturity less average simulated value), it is clear that the optimal procedure is the proposed joint lead-lag binomial lattice-Monte Carlo model approximated with a EWMA volatility.

SUMMARY AND CONCLUSIONS

In this article, we employ a new numerical approximation technique for valuing multiple assets that are correlated, using jointly binomial lattices and Monte Carlo simulation. We show that when applied to a series of correlated assets (BA, IBM, MSFT, HPQ), the proposed algorithm offers a higher level of consistent, efficient and stable results.

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